

Optimization of the concentration changes in a chemostat with one species under biomass retention

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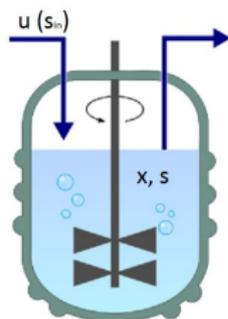
Outline

- 1 Introduction
- 2 Controllability
- 3 The Pontryagin's maximum principle (PMP)
- 4 Optimal synthesis for Monod growth function with $\alpha = 1$
- 5 Optimal synthesis for Monod growth function with $\alpha < 1$
- 6 Conclusions

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The chemostat



Let ξ_u the unique solution of:

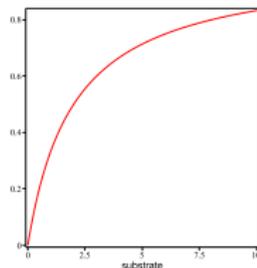
$$\begin{cases} \dot{x} = \mu(s)x - ux, \\ \dot{s} = -\mu(s)x + u(s_{in} - s), \end{cases} \quad (1)$$

s.t. $\xi_u(0) = \xi^0 = (x_0, s_0)$.

Growth functions:

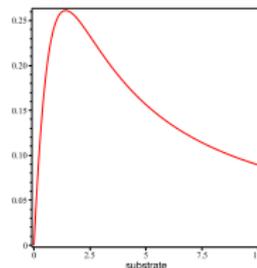
Monod:

$$\mu_m(s) := \frac{\bar{\mu}s}{k + s}$$



Haldane:

$$\mu_h(s) := \frac{\bar{\mu}s}{k_s + s + s^2/k_i}$$



Motivations and practical interest:

- Degrading a substrate: water treatment.
- Producing a biomass: culture of micro algae.
- Controlling the biomass/substrate concentration: wine fermentation.

Model with biomass retention and optimal control problem

Biomass retention parameter:

$$\begin{cases} \dot{x} = \mu(s)x - \alpha ux, \\ \dot{s} = -\mu(s)x + u(s_{in} - s), \end{cases} \quad \text{with } \alpha \in [0, 1].$$

Transformation of the system:

Setting $M = x + s$ we obtain an **equivalent system**:

$$\begin{cases} \dot{s} = -\mu(s)(M - s) + u(s_{in} - s), \\ \dot{M} = u(s_{in} - s - \alpha(M - s)). \end{cases} \quad (2)$$

Observation

The system (2) is easier to deal with when $\alpha = 1$, as M becomes independent of s .

The dilution rate u is taken within the set

$$\mathcal{U} := \{u : [0, +\infty) \rightarrow [0, 1] ; u \text{ meas.}\}.$$

Optimal control problem: Drive the system in *minimal time* from a given initial condition

$\xi_0 := (s_0, M_0)$ to a target point $\bar{\xi} := (\bar{s}, \bar{M})$:

$$\mathbf{v}(\xi_0) := \inf_{u \in \mathcal{U}} t(u) \quad \text{s.t.} \quad \xi_u(0) = \xi_0, \quad \xi_u(t(u)) = \bar{\xi}.$$

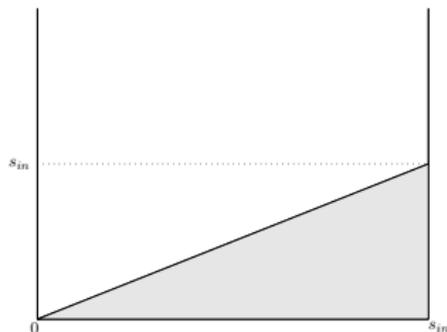
Hypothesis and invariant set

Hypothesis

$$\mu(s) < \alpha, \quad \forall s \in [0, s_{in}].$$

Proposition

The set $F := \{(s, M) : 0 < s < s_{in} \text{ and } M > s\}$ is flow invariant.



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Controllability Set

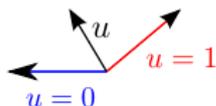
Question

From which initial conditions can we reach the target (\bar{s}, \bar{M}) ?

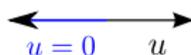
$$\begin{aligned}\dot{s} &= -\mu(s)(M-s) + u(s_{in} - s), \\ \dot{M} &= u \underbrace{(s_{in} - s - \alpha(M-s))}_{f_0^\alpha(s, M)}.\end{aligned}$$

Therefore

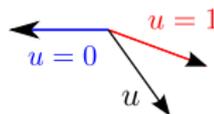
$$f_0^\alpha(s, M) > 0$$



$$f_0^\alpha(s, M) = 0$$



$$f_0^\alpha(s, M) < 0$$



Definition

We define the *collinearity curve*,

$$\Delta_0^\alpha := \{(s, M) \in F : f_0^\alpha(s, M) = 0\}.$$

Controllability Set

Definition

Given a target point $(\bar{s}, \bar{M}) \in F$, we define the *controllability set*

$$C(\bar{s}, \bar{M}) := \{(s_0, M_0) \in F : \exists u \in \mathcal{U}, \exists t_f > 0 \text{ s.t. } (s_u(t_f), M_u(t_f)) = (\bar{s}, \bar{M})\}.$$

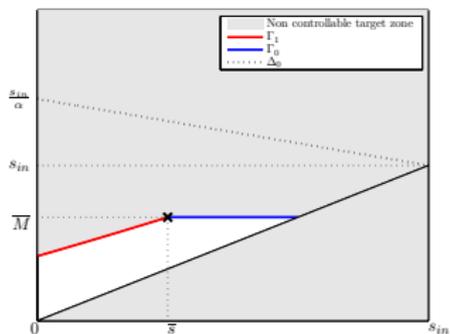
Definition

Let Γ_0 , resp. Γ_1 the graph of the trajectory backward in time from (\bar{s}, \bar{M}) with $u = 0$, resp. $u = 1$ and set $\Gamma = \Gamma_0 \cup \Gamma_1$.

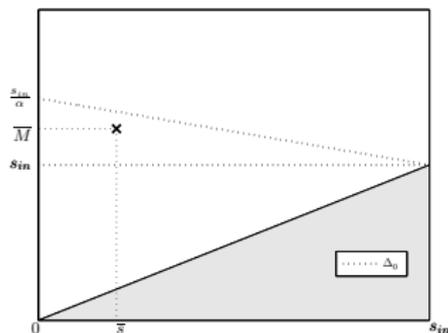
Proposition

- If $\bar{M} \leq s_{in}$, then the controllability set is the set **below** Γ .
- If $\bar{M} \geq \frac{s_{in}}{\alpha}$ and
 - (1) $\Gamma_1 \cap 0 \times \mathbb{R} \neq \emptyset$, then the controllability set is the set **above** Γ ,
 - (2) $\Gamma_1 \cap s_{in} \times \mathbb{R} \neq \emptyset$, then the controllability set is the set **between** Γ_0 and Γ_1 .
- If $s_{in} < \bar{M} < \frac{s_{in}}{\alpha}$, then $C(\bar{s}, \bar{M}) = F$.

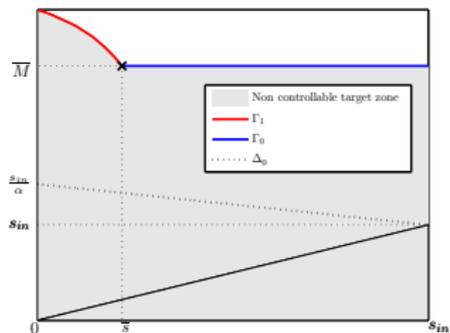
$$\bar{M} \leq s_{in}$$



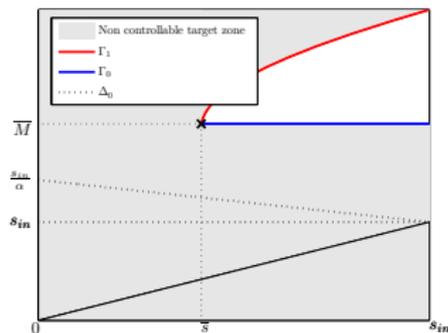
$$s_{in} < \bar{M} < \frac{s_{in}}{\alpha}$$



$$\bar{M} \geq s_{in}$$



$$\bar{M} \geq s_{in}$$



Difference between $\alpha = 1$ and $\alpha < 1$

Important observation

Retention of biomass allows a greater controllability set.

Indeed,

$$\begin{aligned}\alpha = 1 &\Rightarrow s_{in} = \frac{s_{in}}{\alpha}, \\ &\Rightarrow C(\bar{s}, \bar{M}) \neq F, \quad \forall (\bar{s}, \bar{M}) \in F.\end{aligned}$$

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Generalities on two-dimensional controlled affine systems

Consider the **two dimensional controlled system affine w.r.t. the control u** :

$$\dot{x} = f(x) + ug(x), \quad u \in [0, 1] \text{ and } f, g : \mathbb{R}^2 \rightarrow \mathbb{R}^2.$$

Hamiltonian:

$$H = H(x, \lambda, \lambda_0, u) := \lambda \cdot f(x) + \underbrace{u \lambda \cdot g(x)}_{\phi} + \lambda_0.$$

Pontryagin Maximum Principle

Let u an optimal control and $t(u)$ the terminal time. Then, $\exists \lambda : [0, t(u)] \rightarrow \mathbb{R}^2$, $\exists \lambda_0 \leq 0$ s.t. $(\lambda(\cdot), \lambda_0) \neq 0$ and

$$\dot{\lambda}(t) = -\lambda(t) \cdot \nabla f(x(t)) - u(t)\lambda(t) \cdot \nabla g(x(t)),$$

$$u(t) \in \operatorname{argmax}_{v \in [-1, 1]} H(x(t), \lambda(t), \lambda_0, v).$$

Control law:

$$\begin{cases} \phi > 0 & \implies u = 1, \\ \phi < 0 & \implies u = 0, \\ \phi(t) = 0 \text{ on } [t_1, t_2] & \implies \text{Singular Arc.} \end{cases}$$

Hamiltonian:

$$H(s, M, \lambda_s, \lambda_M, \lambda_0, u) := -\lambda_s \mu(s)(M - s) + u \underbrace{[(\lambda_s + \lambda_M)(s_{in} - s) - \alpha \lambda_M (M - s)]}_{\dot{\phi}} + \lambda_0$$

Suppose u is an optimal control, then there exists $\lambda_s, \lambda_M : [0, t(u)] \rightarrow \mathbb{R}$ and $\lambda_0 \leq 0$ s.t. $(\lambda_s(\cdot), \lambda_M(\cdot), \lambda_0) \neq 0$ and

$$\begin{cases} \dot{\lambda}_s = \lambda_s(\mu'(s)(M - s) - \mu(s) + u) + (1 - \alpha)\lambda_M u, \\ \dot{\lambda}_M = \lambda_s \mu(s) + \alpha \lambda_M u. \end{cases}$$

We deduce,

$$\dot{\phi} = (M - s)[\lambda_s \mu'(s)(s_{in} - s) + (1 - \alpha)(\lambda_M + \lambda_s)\mu(s)].$$

Singular arc:

$$\begin{aligned} \phi(t) = 0 \text{ on } [t_1, t_2] &\Rightarrow \dot{\phi}(t) = 0 \text{ on } (t_1, t_2) \\ &\Rightarrow \underbrace{(M - s)[\alpha(M - s)((1 - \alpha)\mu(s) + \mu'(s)(s_{in} - s)) - (s_{in} - s)^2 \mu'(s)]}_{f_{SA}^\alpha(s, M)} = 0. \end{aligned}$$

Definition

We define the *singular locus*,

$$\Delta_{SA}^\alpha := \{(s, M) \in F : f_{SA}^\alpha(s, M) = 0\}.$$

Key proposition

Proposition

If $(s(\cdot), M(\cdot), \lambda_s(\cdot), \lambda_M(\cdot), u(\cdot))$ is a normal extremal trajectory, then there exists a real valued function g_α defined on $\mathcal{U} \times (F \setminus \Delta_0^\alpha)$ such that

$$\dot{\phi}(t) = g_\alpha(u(t), s(t), M(t))\phi(t) + \frac{f_{SA}^\alpha(s(t), M(t))}{\mu(s)(M-s)f_0^\alpha(s(t), M(t))}, \quad \text{for a.e. } t.$$

Important observation

If t_s is a switching time with $(s(t_s), M(t_s)) \notin \Delta_0$ and $(s(t_s), M(t_s)) \notin \Delta_{SA}$, then we know what kind of switch it is.

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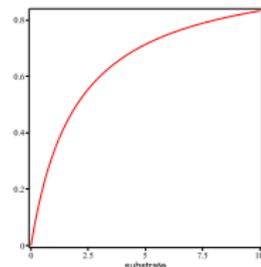
Optimal synthesis for Monod growth function

$$\dot{\phi} = \lambda_s \mu'(s)(M - s)(s_{in} - s).$$

Monod:

$$\mu_m(s) := \frac{\bar{\mu}s}{k + s} \quad \Rightarrow \quad \mu'_m(s) > 0, \quad \forall s \in [0, s_{in}],$$

$$\Rightarrow \quad \dot{\phi} < 0.$$



Important facts

- There is no singular arc.
- The only switch possible is from $u = 1$ to $u = 0$.

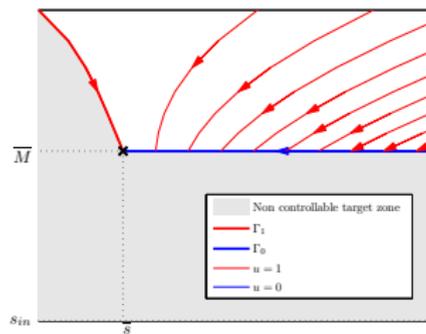
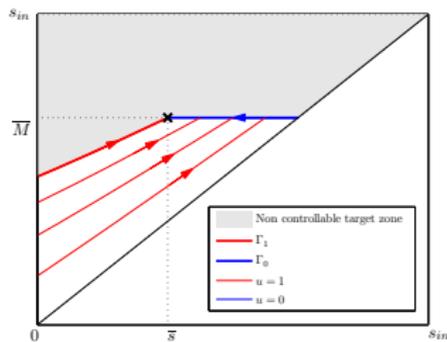
Optimal synthesis

Consider the **feedback control law**:

$$\begin{cases} u[s, M] = 0, & \text{if } (s, M) \in \Gamma_0, \\ u[s, M] = 1, & \text{if } (s, M) \in \Gamma_1, \\ u[s, M] = 1, & \text{elsewhere.} \end{cases} \quad (3)$$

Theorem (see G. D'Ans et al., '71)

In the case where $\mu = \mu_m$, then the optimal strategy is given by (3) in the controllability set.



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Singular locus

When $\alpha < 1$ and $\mu = \mu_M$ then the singular locus Δ_{SA}^α is the graph of the function:

$$\delta_{SA}^\alpha(s) = s + \psi_\alpha(s),$$

where

$$\psi_\alpha(s) = \frac{1}{\alpha} \frac{\mu'(s)(s_{in} - s)^2}{(s_{in} - s)\mu'(s) + (1 - \alpha)\mu(s)}.$$

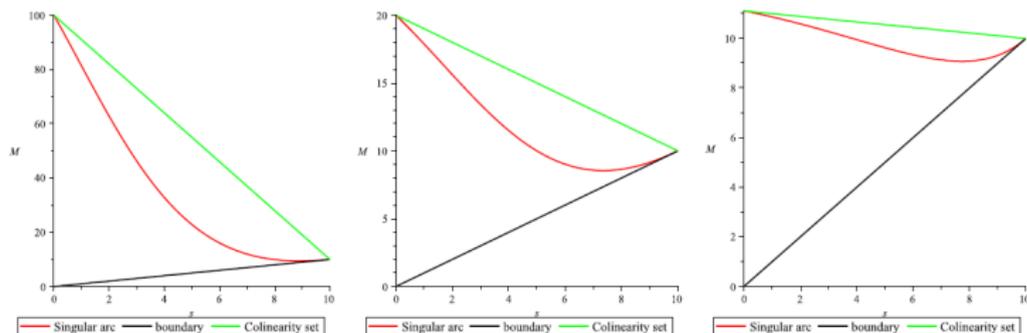


Figure: Plot of Δ_0^α and Δ_{SA}^α for $\alpha = 0.1, 0.5, 0.9$ with $\mu(s) = \frac{s}{5+s}$ and $s_{in} = 10$.

Singular control

The *singular control* which allows to stay on Δ_{SA}^α is given by

$$u_s^\alpha(s) := \frac{\mu(s)\psi_\alpha(s)(1 + \psi'_\alpha(s))}{\alpha\psi_\alpha(s) + \psi'_\alpha(s)(s_{in} - s)}. \quad (4)$$

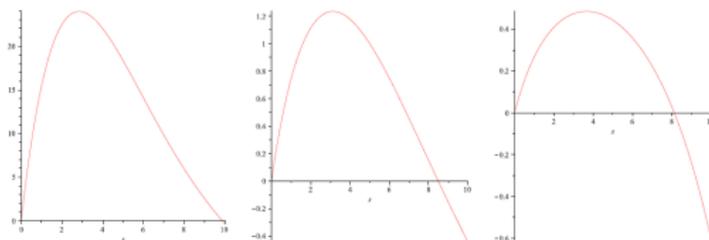


Figure: Plot of the singular control $s \mapsto u_s^\alpha(s)$ given by (4) with $\mu(s) = \frac{s}{5+s}$ and $s_{in} = 10$ for $\alpha = 0.1, 0.5, 0.9$.

Important observation

- There exists $s_m \in (0, s_{in})$ s.t. $u_s^\alpha(\cdot)$ is positive on $[0, s_m]$ and negative on $(s_m, s_{in}]$.
- For small values of α the singular control u_s can exceed the maximal value $u_{max} = 1$.

Optimal synthesis

Hypothesis 2

The singular control satisfies $u_s^\alpha(s) \leq 1$, for all $s \in [0, s_m]$.

Consider the **feedback control law**:

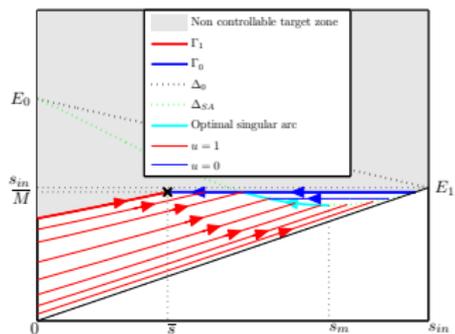
$$\begin{cases} u[s, M] = 1, & \text{if } M < \delta_{SA}^\alpha(s), \\ u[s, M] = 0, & \text{if } M < \delta_{SA}^\alpha(s) \text{ or } (M = \delta_{SA}^\alpha(s) \text{ and } s > s_m) \\ u[s, M] = u_s^\alpha(s), & \text{if } M = \delta_{SA}^\alpha(s) \text{ and } s \leq s_m. \end{cases} \quad (5)$$

Theorem

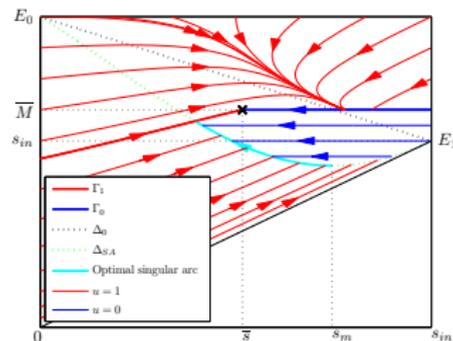
In the case where $\mu = \mu_m$ and $\alpha < 1$ such that Hypothesis 2 is fulfilled, the optimal strategy is given by (5) in the controllability set.

Sketch of proof: Deduction from the sign of $\frac{f_{SA}^\alpha(s(t), M(t))}{\mu(s)(M-s)f_0^\alpha(s(t), M(t))}$.

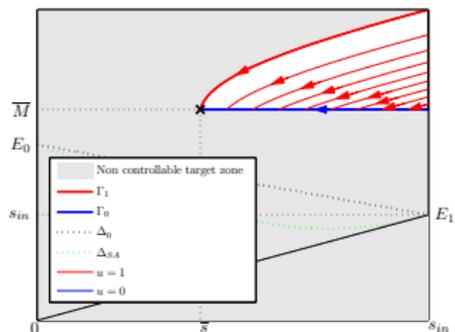
$$\bar{M} \leq s_{in}$$



$$s_{in} < \bar{M} < \frac{s_{in}}{\alpha}$$



$$\bar{M} \geq s_{in}$$



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Conclusions and perspectives

Conclusions:

- Biomass retention can give larger controllability set.
- Thanks to geometric control theory, we provide an optimal feedback control:
 - In the Monod case without biomass retention, the optimal strategy is bang-bang of the form 1-0.
 - Under biomass retention, there exists a **singular locus** which is **not controllable everywhere** and the optimal strategy is either bang-bang, bang-singular-bang or bang-bang-singular-bang.

The Haldane case: The synthesis is **more tricky**:

- Even without biomass retention there exists a singular locus not controllable everywhere.
- Under biomass retention the singular locus is not necessarily continuous.

Thank you for your attention.