Variational convergence of bifunctions and equilibrium points

Alejandro Jofré, Universidad de Chile

Roger J-B Wets, Univ. California, Davis

Non-Cooperative Games

- player: $a \in A$, payoff: $u_a(x_a, x_{-a}) : \mathbb{R}^N \to \overline{\mathbb{R}}$
- Nash equilibrium: $(\bar{x}_a, a \in A)$ such that

$$\bar{x}_a \in \arg\max_{X_a} u_a(x_a, \bar{x}_{-a}), \ \forall a \in A$$

- Function:

$$N(x,y) = \sum_{a \in A} u_a(x_a, x_{-a}) - \sum_{a \in A} u_a(y_a, x_{-a})$$

- $\bar{x} = (\bar{x}_a, a \in A)$ is a Nash equilibrium

$$\Leftrightarrow \bar{x} \in \arg\max\text{–}\min N \ \& \ N(\bar{x}, \bullet) \geq 0$$

The approach:

- Nash functions of approximating games (including strategies perturbations)

$$N^\nu(x,y) = \sum_{a\in A} u_a^\nu(x_a, x_{-a}) - \sum_{a\in A} u_a^\nu(y_a, x_{-a})$$

- $x^\nu \in \arg\max\!-\!\min N^\nu,\ \bar{x} \in$ cluster points $\{x^\nu\}$

- $$N^\nu \xrightarrow[\text{"...."}]{} N \text{ and } \ldots$$

$$\Rightarrow ?\ \bar{x} \in \arg\max\!-\!\inf N \sim \text{ equilibrium point}$$

Fixed Points (Brouwer)

C compact convex, $G : C \to C$ continuous

C^ν compact convex, $G^\nu : C^\nu \to C^\nu$ continuous

$\bar{x}^\nu \in C^\nu$ fixed points of G^ν on C^ν

and $\bar{x}^\nu \to \bar{x}$. When is $\bar{x}$ a fixed point of G on C?

The approach $K : C \times C \to \mathbb{R}$, $K^\nu : C^\nu \times C^\nu \to \mathbb{R}$,

set $K(x,y) = \langle G(x) - x, y \rangle$, $K^\nu(x,y) = \langle G^\nu(x) - x, y \rangle$

x fixed point $\Leftrightarrow x \in \arg\max_C (\min_{y \in C} K(\bullet, y))$

So, $K^\nu \underset{?}{\to} K$ yields convergence of max-inf points?

Variational Inequalities

- $C \subset \mathbb{R}^n$ non-empty, convex
- $G : C \to \mathbb{R}^n$ continuous
- find $\bar{u} \in C$ such that $-G(\bar{u}) \in N_C(\bar{u})$

  where $v \in N_C(\bar{u}) \Leftrightarrow \langle v, u - \bar{u} \rangle \leq 0, \forall u \in C$
- $C^\nu \to C, \quad G^\nu\!: C^\nu \to \mathbb{R}^n$ continuous
- approx. $u^\nu \in C^\nu$ such that $-G^\nu(u^\nu) \in N_{C^\nu}(u^\nu)$
- Question: $u^\nu \to \bar{u}$? when, how.

V.I.: The approach

- Let $\boxed{K(u,v) = \langle G(u), v-u \rangle}$ on $\operatorname{dom} K = C \times C$
- then $-G(\bar{u}) \in N_C(\bar{u})$ if and only if
- $\exists \hat{u} \in \arg\max\text{–}\inf K$ with $K(\hat{u},\bullet) \geq 0$
  - $\langle G(\bar{u}), v - \bar{u} \rangle = K(\bar{u}, v) \geq 0, \forall v \in C$

    $\Rightarrow 0 \leq K(\bar{u},\bullet) \leq \sup_{u \in C}\left(\inf_{v \in C} K(u,v)\right)$
  - $\hat{u} \in \arg\max\text{–}\inf K \;\&\; K(\hat{u},\bullet) \geq 0 \Rightarrow$

    $\langle -G(\hat{u}), v - \hat{u} \rangle \leq 0, \forall v \in C$ or $-G(\hat{u}) \in N_C(\hat{u})$

V.I.:The approach

- $K^\nu(u,v) := \left\langle G^\nu(u), v-u \right\rangle$, $\operatorname{dom} K^\nu = C^\nu \times C^\nu$
- $u^\nu \in \operatorname{arg\,max-inf} K^\nu$ with $K^\nu(u^\nu,\bullet) \le 0$
- $$\boxed{K^\nu \underset{"\ldots"}{\longrightarrow} K \text{ and } \ldots}$$
- $\bar{u} \in$ cluster points $\left\{u^\nu\right\} \Rightarrow ?\ \bar{u} \in \operatorname{arg\,min-sup} K$

Applications: Convergence and stability

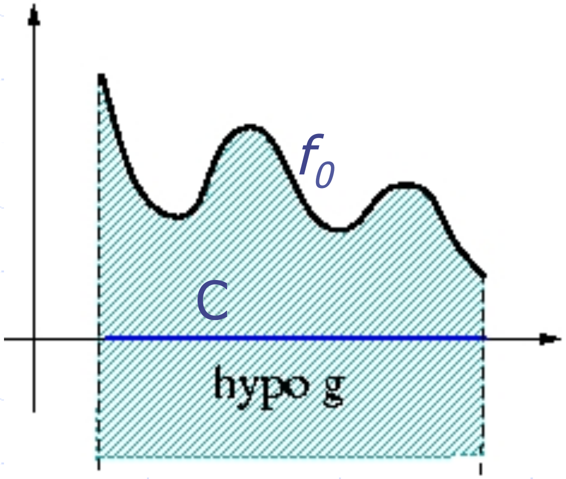
- Saddle-points: Lagrangians, Hamiltonians
- Fixed points
- Solutions of cooperative <u>and</u>
  non-cooperative games
- Economic-Equilibrium points (Walras)
- Generalized Nash Equilibrium Problems
- Solutions of set-valued inclusions
- Stability of mountain-pass paths

Optimization: Max-Framework

- $\max f(x),\ x \in \mathbb{R}^n$ with
  $$f : \mathbb{R}^n \to \overline{\mathbb{R}} = [-\infty, \infty]$$

- $$f(x) = \begin{cases} f_0(x) \text{ if } x \in C \subset \mathbb{R}^n \\ -\infty \qquad \text{if } x \notin C \end{cases}$$

$$C = \left\{ x \in X \,\middle|\, f_i(x) \leq 0, i \in I_1, f_i(x) = 0, i \in I_2 \right\}$$



Hypo-Convergence (max-framework)

- $\boxed{\text{argmax } f^a \to \text{argmax } f}$
- argmax $(f^\nu + g) \to$ argmax $(f+g)$,
  $\forall$ cont. g $\Rightarrow$ hypo-convergence
- 'new' definition $f : D \to \mathbb{R}$, $\left\{ f^\nu : D^\nu \to \mathbb{R} \right\}_{\nu \in \mathbb{N}}$

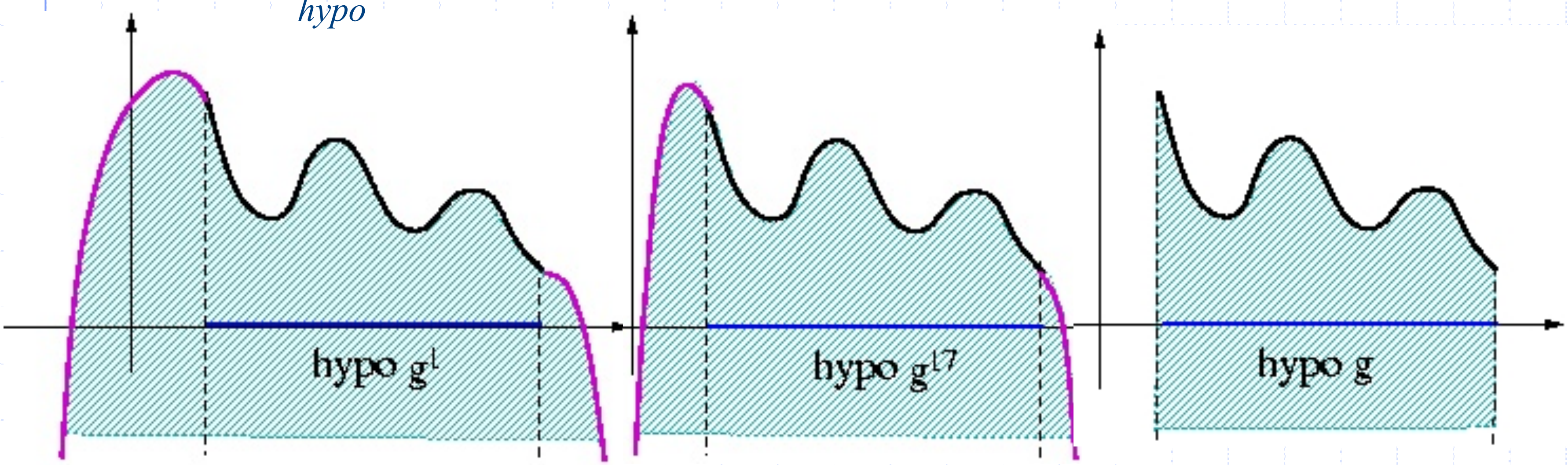
  (a) $\forall x^\nu \in D^\nu \to x \in D$, $\limsup f^\nu(x^\nu) \le f(x)$

  (a_∞) $\forall x^\nu \in D^\nu \to x \notin D$, $f^\nu(x^\nu) \to -\infty$

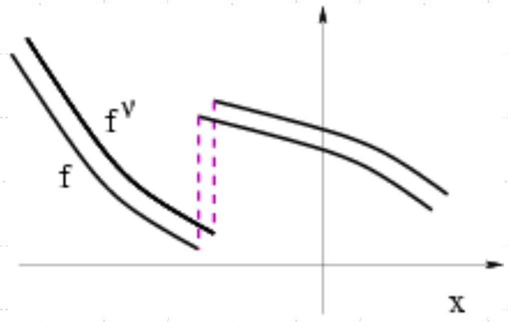
  (b) $\forall x \in D, \exists x^\nu \to x$, $\liminf f^\nu(x^\nu) \ge f(x)$

Hypo-Convergence

- $\boxed{\text{argmax } g^a \sim \text{argmax } g}$
- $g^\nu \xrightarrow[h]{} g \Leftrightarrow -g^\nu \xrightarrow[epi]{} -g$
- $g^\nu \xrightarrow[hypo]{} g \Leftrightarrow \text{hypo } g^\nu \to \text{hypo } g$



Hypo-convergence: Properties



- $f^\nu \to_h f \neq f^\nu \to_p f$
- $f^\nu \to_u f \Rightarrow f^\nu \to_h f$
- $\boxed{f^\nu \to_e f,\, x^\nu \in \arg\max_{D^\nu} f^\nu,\, x^{\nu_k} \to \bar{x} \in D \Rightarrow \bar{x} \in \arg\max_D f}$
- $\bar{x} \in \arg\max_D f \Rightarrow \exists \varepsilon^\nu \searrow 0,\ x^\nu \in \varepsilon^\nu\text{–}\arg\max_{D^\nu} f^\nu : x^\nu \to \bar{x}$
- $f^\nu \to_h f \Leftrightarrow dl(\text{hypo } f^\nu, \text{hypo } f) \to 0$

(Γ-convergence)

Tight Hypo-Convergence

- $$f^{\nu}_{D^{\nu}} \underset{h-tightly}{\longrightarrow} f_D : f^{\nu}_{D^{\nu}} \underset{h}{\longrightarrow} f_D \;\&\; \forall \varepsilon > 0, \exists B \text{ compact} :$$
$$\forall \nu > \nu_{\varepsilon} : \sup_{B \cap D^{\nu}} f^{\nu} \le \sup_{D^{\nu}} f^{\nu} - \varepsilon$$

- THM: $f^{\nu}_{D^{\nu}} \underset{h-tightly}{\longrightarrow} f_D, \sup_D f \in \mathbb{R} \Rightarrow \sup_{D^{\nu}} f^{\nu} \longrightarrow \sup_D f$

  also: $x^{\nu} \in \arg\max f^{\nu}_{D^{\nu}}, x^{\nu_k} \longrightarrow \bar{x} \in D \Rightarrow \bar{x} \in \arg\max f_D$

Lopsided convergence

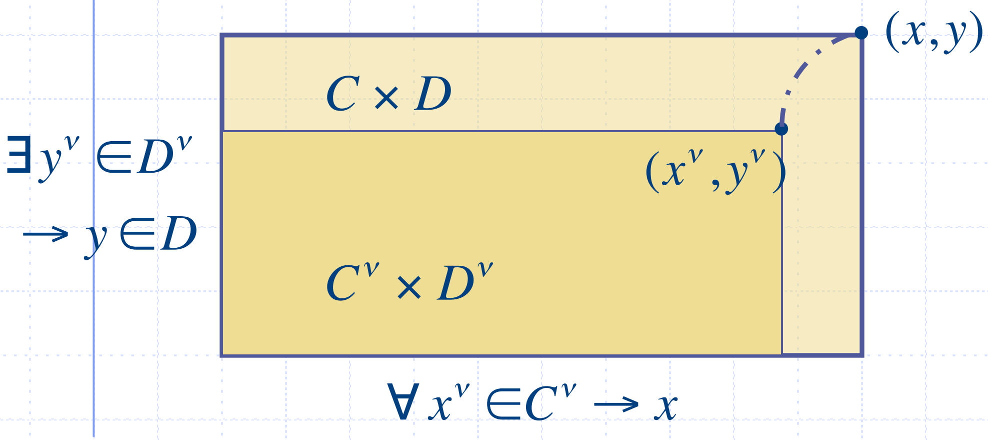
max-inf framework

$$K : C \times D \to \mathbb{R}, \ \left\{K^\nu : C^\nu \times D^\nu \to \mathbb{R}\right\}_{\nu\in\mathbb{N}}$$

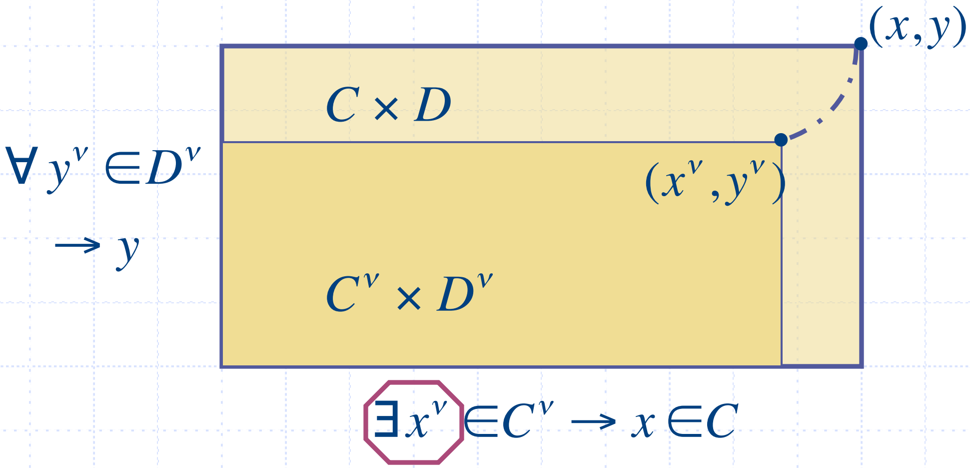
$$\boxed{\arg\max\text{-}\inf K^a \simeq \arg\max\text{-}\inf K}$$

$$\bar{x} \in \arg\ \max\text{-}\inf K \Leftrightarrow \bar{x} \in \arg\max(\inf_y K(x,y))$$

Lopsided convergence: definition



$$\limsup_\nu K^\nu(x^\nu, y^\nu) \le K(x,y)$$



$$\liminf_\nu K^\nu(x^\nu, y^\nu) \ge K(x,y) \text{ when } y \in D$$

$$K^\nu(x^\nu, y^\nu) \to \infty \text{ when } x \notin D$$

Lopsided: elementary properties

$$\boxed{\text{argmax-inf } K^a \sim \text{argmax-inf } K}$$

lopsided ⇨ epi-convergence if $K(x,y) = g(y)$

lopsided ⇨ hypo-convergence if $K(x,y) = f(x)$

Remarks: not a Γ-convergence, in ∞-dim. different topologies

Lopsided ancillary-tightly

- $K^\nu_{C^\nu \times D^\nu} \underset{lop\ ancillary-tight}{\longrightarrow} K_{C\times D}$ if $K^\nu_{C^\nu \times D^\nu} \underset{lop}{\longrightarrow} K_{C\times D}$ &

  (b) $\forall x \in C, \exists x^\nu \to x, \forall y^\nu \in D^\nu$ and $y^\nu \to y$:

  $$\liminf K^\nu(x^\nu, y^\nu) \geq K(x,y) \quad \text{if } y \in D$$

  $$K^\nu(x^\nu, y^\nu) \to \infty \quad \text{if } y \notin D$$

  but also $\forall \varepsilon > 0$, $\exists B_\varepsilon$ compact (depends on $x^\nu \to x$):

  $$\inf_{B_\varepsilon \cap D^\nu} K^\nu(x^\nu, \bullet) \leq \inf_{D^\nu} K^\nu(x^\nu, \bullet) + \varepsilon, \ \forall \nu \geq \nu_\varepsilon$$

- THM. $K^\nu_{C^\nu \times D^\nu} \to K_{C\times D}$ lop. ancillary-tight, $\bar{x}$ cluster point of $\{x^\nu \in \arg\max\text{–}\inf K^\nu_{C^\nu \times D^\nu}\}_{\nu\in\mathbb{N}} \Rightarrow \bar{x} \in \arg\max\text{–}\inf K_{C\times D}$

Proof ….

- $K^{\nu}_{C^{\nu}\times D^{\nu}} \underset{lop\ ancillary-tight}{\longrightarrow} K_{C\times D}$

  Let $g^{\nu} = \inf_{y\in D^{\nu}} K^{\nu}(\bullet, y), \quad g = \inf_{y\in D} K(\bullet, y).$

  $$\Rightarrow g^{\nu} \underset{hypo}{\longrightarrow} g \quad \text{when} \quad \begin{cases} C^{\nu}_{g} = \left\{ x \in C^{\nu} \,\middle|\, g^{\nu}(x) > -\infty \right\} \\ C_{g} = \left\{ x \in C \,\middle|\, g(x) > -\infty \right\} \end{cases} \neq \emptyset$$

- then apply

  $$g^{\nu}_{C^{\nu}} \underset{hypo}{\longrightarrow} g_{C},\ x^{\nu} \in \arg\max_{C^{\nu}} g^{\nu},\ x^{\nu_k} \to \bar{x} \in C \Rightarrow \bar{x} \in \arg\min_{C} g$$

… EVEN BETTER : CONVERGENCE

$K^v_{C^v \times D^v} \to K_{C\times D}$ lop. ancillary-tightly,

(i) $x^v \in \varepsilon\text{-maxinf}\, K^v_{C^v\times D^v}$, $\bar{x}$ cluster point of $\{x^v\}_{v\in\mathbb{N}}$

$\Rightarrow \bar{x} \in \varepsilon\text{-maxinf}\, K_{C\times D}$

(ii) $x^v \in \varepsilon_v\text{-maxinf}\, K^v_{C^v\times D^v}$, $\bar{x}$ cluster point of $\{x^v\}_{v\in\mathbb{N}}$

& $\varepsilon_v \searrow 0 \Rightarrow \bar{x} \in \text{maxinf}\, K_{C\times D}$ (special: unique)

(iii) $\bar{x} \in \text{maxinf}\, K_{C\times D} \Rightarrow \exists \varepsilon_v \searrow 0$ & $x^v \in \varepsilon_v\text{-maxinf}\, K^v_{C^v\times D^v}$

such that $x^v \to \bar{x}$,

Under tight-lop: convergence of the full ε_v-maxinf sets

and convergence of values

Ky Fan functions & inequality

- $K : C \times C \to \mathbb{R}$ generalized Ky Fan function if

  (a) $\forall$ y: $x \mapsto K(x, y)$ usc

  (b) $\forall$ x: $y \mapsto K(x, y)$ convex

- K Ky Fan fcn, dom $K = C \times C$, C compact

  $\Rightarrow \arg\max\text{–}\inf K \neq \emptyset$

  if $K(x,x) \geq 0$ on dom K, $\bar{x} \in \arg\max - \inf K$

  $\Rightarrow \inf_y K(\bar{x}, y) \geq 0.$

Extending Ky Fan's inequality

- $K^\nu \to K$ lopsided tightly with $C^\nu \to C$,

  K^ν Ky Fan $\Rightarrow K$ gen-Ky Fan

- & $\forall \nu : \arg\max\text{–}\inf K^\nu \neq \emptyset$

  if $\bar{x} \in$ cluster-pts $\{\arg\max\text{–}\inf K^\nu\}$

  $\Rightarrow \bar{x} \in \arg\max\text{–}\inf K$ & $K(\bar{x}, \bullet) \geq 0$

- gen-Ky Fan fcns closed under lopsided

  saddle fcns closed under h/e-convergence

  usc fcns closed under hypo-convergence

APPLICATIONS

Fixed Points (Brouwer)

C compact convex, $G: C \to C$ continuous

find $\bar{x} \in C: \ G(\bar{x}) = \bar{x}$, fixed point

set $K(x, y) = \langle G(x) - x, y \rangle$

Ky Fan Inequality applies (usc,convex, K(x,x) ≥ 0):

$$\exists \bar{x} \text{ such that } \langle G(\bar{x}) - \bar{x}, y \rangle \le 0, \ \forall y \in C$$

$$\Rightarrow \bar{x} \text{ is a fixed point of G}$$

Convergence of fixed points

$$G^{\nu} : C^{\nu} \to C^{\nu} \text{ continuous, } C^{\nu} \text{ convex}$$

$$C^{\nu} \to C \Rightarrow C^{\nu} \text{compact,} \quad G^{\nu} \underset{cont}{\to} G$$

$$\text{i.e., } \forall x^{\nu} \to x : G^{\nu}(x^{\nu}) \to G(x)$$

$$\Rightarrow K^{\nu} \underset{lop}{\to} K, \;\; K(x,y) = \langle G(x) - x, y \rangle$$

Hence $\exists x^{\nu}$ fixed points of G^{ν} on C^{ν}

$\to \bar{x}$ fixed point of G on C

Set-valued Inclusions

- $S^\nu, S : \mathbb{R}^d \to cl/cvx\text{–}sets(\mathbb{R}^n)$, non-empty
- Inclusions: $S^\nu(u) \ni a^\nu,\ S(u) \ni 0, \quad a^\nu \to 0$
- $\sigma^\nu(\bullet, u) =$ supp. fcn $S^\nu(u)$, $\sigma(\bullet, u)$ of $S(u)$
- $\forall u : \sigma^\nu(\bullet, u), \sigma(\bullet, u)$ convex
- S^ν, S continuous, compact-valued (sufficient)
- $\Leftrightarrow \sigma^\nu(\bullet, u^k) \to_{epi} \sigma^\nu(\bullet, \bar{u})$ as $u^k \to \bar{u}$; also σ
- $\Leftrightarrow \sigma^\nu(\bullet, u^k) \to_p \sigma^\nu(\bullet, \bar{u}) \Rightarrow \sigma^\nu(x, \bullet),\ \sigma(x, \bullet)$ usc
- $K^\nu = \sigma^\nu - \langle a^\nu, \bullet \rangle, K = \sigma$ Ky Fan fcns

Variational Inequalities

- $C \subset \mathrm{R}^n$ nonempty, convex, compact
- $G : C \rightarrow \mathrm{R}^m$ continuous, ($m=n$)
- find $\bar{u} \in C$ such that $-G(\bar{u}) \in N_C(\bar{u})$

  where $v \in N_C(\bar{u}) \Leftrightarrow \langle v, u - \bar{u} \rangle \le 0, \ \forall u \in C$
- with $K(u,v) = \langle G(u), v - u \rangle$ on dom $K = C \times C$

  $\Rightarrow$ K is a Ky Fan function, $K(u,u) \geq 0$.

Find

$$\boxed{\bar{u} \in \arg\max\text{–}\inf K(\bullet,\bullet) \text{ so that } K(\bar{u},\bullet) \geq 0}$$

Convergence: V.I. + extension

$$C^\nu \to C \Rightarrow C^\nu \text{ compact } \nu \geq \bar{\nu},\ G^\nu \text{ continuous}$$

$$G^\nu \underset{cont}{\longrightarrow} G : G^\nu(x^\nu) \to G(x),\ \forall x^\nu \in C^\nu \to x$$

$$K^\nu(u,v) = \left\langle G^\nu(u), v - u \right\rangle \text{ on dom } K^\nu = C^\nu \times C^\nu$$

i.e., Ky Fan functions

- lop-converges ancillary-tight to *K*, i.e. any cluster point of the solutions of the approximating V.I. is sol'n of limit V.I.

(sufficient conditions)

Nash Equilibrium points

- $\bar{x}_a \in \arg\max u_a(x_a, \bar{x}_{-a}),\ \forall a \in A$
- Nash function:

  $$N(x,y) = \sum_{a\in A} u_a(x_a, x_{-a}) - \sum_{a\in A} u_a(y_a, x_{-a})$$

- u_a usc & $u_a(x_a, \bullet)$ lsc; $u_a(\bullet, x_{-a})$ concave

  $\Rightarrow N$ a Ky Fan function & $N(x,x) \geq 0$

- $u_a^\nu \to_{cont} u_a$ & dom $u_a^\nu \to$ dom u_a compact

  $\Rightarrow N^\nu \to N$ lopsided ancillary-tight

  $\Rightarrow$ Nash equilibr.$^\nu$ $\to$ Nash equilibr. (cluster)

Walras Equilibrium points

- $\forall a \in A$: $d_a(p) \in \arg\max \left\{ u_a(x_a) \middle| \langle p, x_a \rangle \le \langle p, e_a \rangle \right\}$
- $s(p) = \sum_a (e_a - d_a(p))$ excess supply
- find $\bar{p} \in \Delta$ (unit simplex) so that $s(\bar{p}) \ge 0$
- Walrasian: $W(p,q) = \langle q, s(p) \rangle$ Ky Fan fcn
- $\boxed{\bar{p} \in \arg\max\text{–}\inf W \Leftrightarrow s(\bar{p}) \ge 0}$
- conditions: $e_a \in \text{int dom } u_a$, "globally compact"
- Convergence: $u_a^\nu \to_{hypo} u_a,\ e_a^\nu \to e_a \Rightarrow$

  W^ν lop-converges ancillary-tight to W

Numerical Schemes

Walras Equilibrium points

Augmented Walrasian

- A. Bagh & S. Lucero (2002, UC-Davis)

$$\bar{p} \in \text{argmax-inf W}$$

$$\cong \text{ saddle point } (\bar{p}, \bar{q}) \text{ of } \tilde{W}_r$$

$$\tilde{W}_r(p,q) = \inf_u \left\{ W(p,u) + r\|u\| - \langle q,u \rangle \right\}$$

$$= \sup_z \left\{ W(p,z) \,\middle|\, \|z-q\|^o \le r \right\}$$

with $\|\bullet\|$ and $\|\bullet\|^o$ dual norms.

Iterations

$$W(p,q) = \langle q, s(p) \rangle \text{ on } \Delta \times \Delta$$

$$\tilde{W}_r(p,q) = \sup_z \left\{ W(p,z) \middle| \, \|z - q\|^o \le r \right\}$$

$$q^{k+1} = \arg\max_{q \in \Delta} \left[\max_z \left\langle z, s(p^k) \right\rangle \middle| \, \|z - q\|^o \le r \right]$$

minimizing a linear form on a ball

reduces to finding the largest element of $s(p^k)$

$$p^{k+1} = \arg\min_{p \in \Delta} \left[\max_z \left\langle z, s(p) \middle| \, \|z - q^{k+1}\|^o \le r_{k+1} \right\rangle \right]$$

$$\text{as } r_k \nearrow \infty, p^k \to p \quad \text{(Walras equilibrium point)}$$

experiments: 10 agents, 150 goods (easy!)

Test: Demand functions

- Cobb-Douglas utility function:

$$u_a(c_a) = \gamma_a \prod_{l=1}^{n} c_{a,l}^{\beta_a^l} \quad \text{with} \quad \sum_{l=1}^{n} \beta_{a,l} = 1, \beta_{a,l} \geq 0$$

- budget constraint: $\sum_l p_l c_{a,l} \leq \sum_l p_l e_{a,l}$

- demand: $\bar{c}_{a,l}(p) = (\beta_{a,l} / p_l)\left(\sum_k p_l e_{a,k}\right), l = 1,\ldots,n$

(demand = supply)

SADDLE-POINTS

Lagrangians (& Hamiltonians)

- $$\max \quad f_0(x)$$
  $$\text{so that } f_i(x) \le 0, \quad i = 1,\dots,s$$
  $$f_i(x) = 0, \; i = s+1,\dots,m.$$
- Lagrangian:

$$L(x,y) = \begin{cases} f_0(x) + \sum_{i=1}^{m} y_i f_i(x) \text{ if } y_i \ge 0, i = 1,\dots,s \\ -\infty \quad \text{otherwise} \end{cases}$$

- concave-convex, max-inf framework

Hypo/Epi-Convergence

- saddle point L^a ~ saddle point L

- (90's) definition:

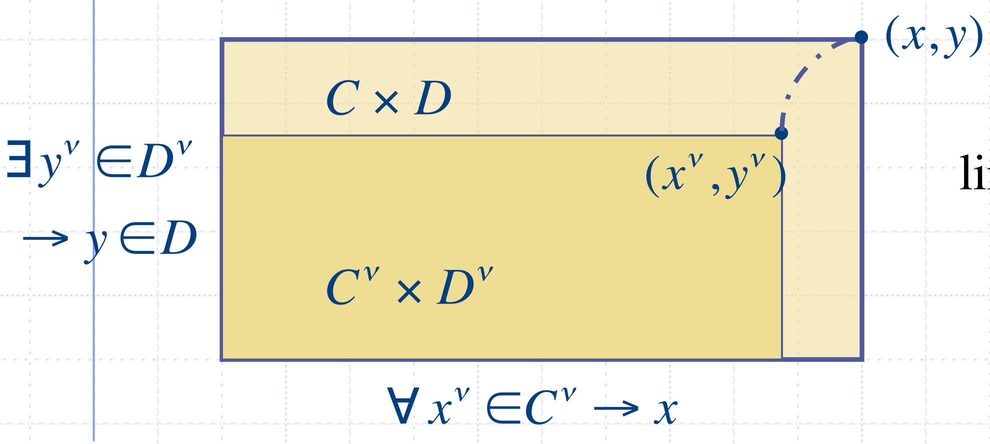
$\forall\,(x, y)$

$$\text{(a)}\ \forall x^\nu \to x, \exists\, y^\nu \to y: \quad \liminf L^\nu(x^\nu, y^\nu) \geq L(x, y)$$

$$\text{(b)}\ \forall y^\nu \to y, \exists\, x^\nu \to x: \quad \limsup L^\nu(x^\nu, y^\nu) \leq L(x, y)$$

- "problem": hypo/epi-topology not Hausdorff …. the pitfall "equivalence classes"

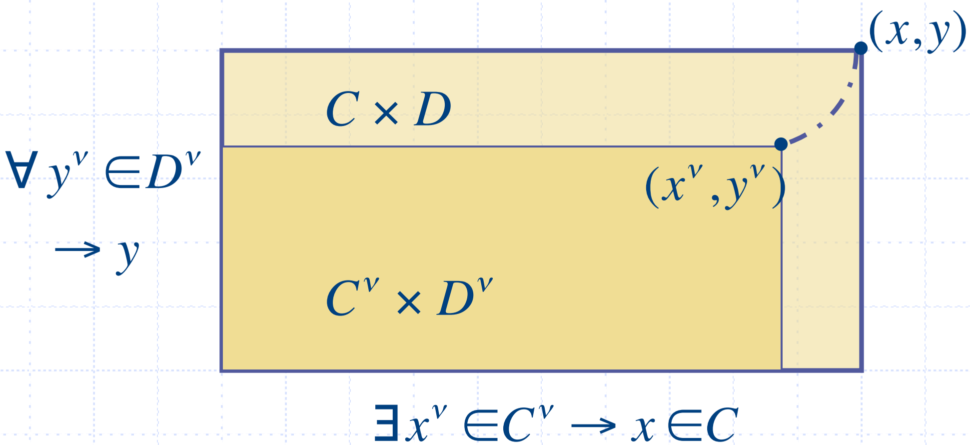
Hypo/Epi convergence



$\limsup_\nu K^\nu(x^\nu, y^\nu) \le K(x,y)$ when $x \in C$

$K^\nu(x^\nu, y^\nu) \to -\infty$ when $x \notin C$

like lopsided convergence



$\liminf_\nu K^\nu(x^\nu, y^\nu) \ge K(x,y)$ when $y \in D$

$K^\nu(x^\nu, y^\nu) \to \infty$ when $x \notin D$

not like lopsided convergence

Hypo/epi: elementary properties

saddle points L^a ~ saddle points L

hypo/epi ⇨ epi-convergence if $L(x,y) = g(y)$

hypo/epi ⇨ hypo-convergence if $L(x,y) = f(x)$

but not y-epi-convergence and x-hypo-convergence

Remarks: not a Γ-convergence, in ∞-dim. different topologies

Hypo/epi-convergence: Properties & Applications

$$L^\nu \xrightarrow[h/e]{} L, \quad (x^\nu, y^\nu) \text{ saddle-pts } L^\nu,$$

$$(x^{\nu_k}, y^{\nu_k}) \to (\bar{x}, \bar{y}) \quad \Rightarrow (\bar{x}, \bar{y}) \text{ saddle-pt of } L$$

Applications:

✓Convergence of solutions to
zero-sum games

✓Joint convergence of solutions to
dual convex programs

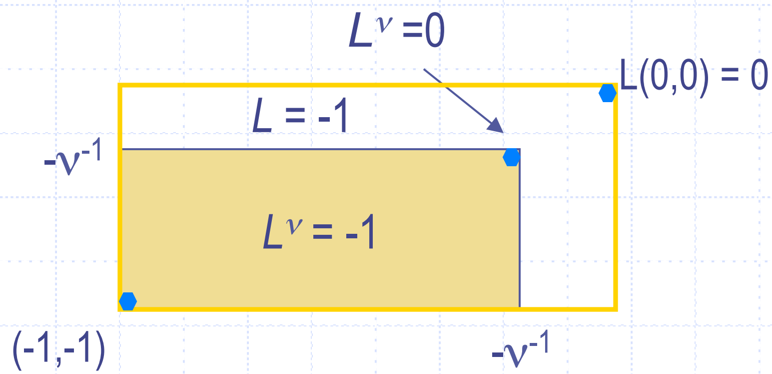
✓Mechanics

Lopsided & hypo/epi-convergence

- lopsided ⟹ hypo/epi-convergence
- not conversely

$$\forall x^\nu \to 0, \quad \exists y^\nu \to 0 \ldots$$

hypo/epi



$$\exists x^\nu \to 0, \quad \forall y^\nu \to 0 \ldots$$

lopsided

- $(x, y) = (0,0), y^\nu = -\nu^{-1} \Rightarrow x^\nu = -\nu^{-1} : \liminf K^\nu(x^\nu, y^\nu) \geq 0$

  but with $\widehat{y}^\nu = -\nu^{-1}/2 : \liminf K^\nu(x^\nu, \widehat{y}^\nu) = -1 < 0$!

Lopsided & hypo/epi-convergence

- $L^\nu \xrightarrow[h/e]{} L$ & convex-concave $\Rightarrow$ $L^\nu \xrightarrow[lop]{} L$

first condition … identical

for the second condition: given (x, y), let

$$u \in \partial_x L(x,y),\ -v \in \partial_y(-L(x,y))$$

$$L^\nu \xrightarrow[h/e]{} L \Rightarrow L^\nu - \langle u, \bullet \rangle + \langle v, \bullet \rangle \xrightarrow[h/e]{} L - \langle u, \bullet \rangle + \langle v, \bullet \rangle$$

(x, y) saddle point of $L - \langle u, \bullet \rangle + \langle v, \bullet \rangle$

$\exists (x^\nu, y^\nu)$ saddle-pts $\to (x, y)$ & $\left\{ x^\nu \right\}_{\nu \in \mathbb{N}}$ is lop-sequence $\to x$

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