Advanced Fixed Point Theory

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Equilibria in Games:
Existence, Selection, and Dynamics

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1. Existence and Algorithms

Introduction

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- Refinements of Nash equilibrium are often defined in terms of fixed points of perturbations of the best response correspondence.
- The mathematical concepts and tools used to study fixed points are useful throughout economics.
    - We will emphasize mathematical concepts and key results, with key ideas of proofs.

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- Dynamic stability and relation to index.

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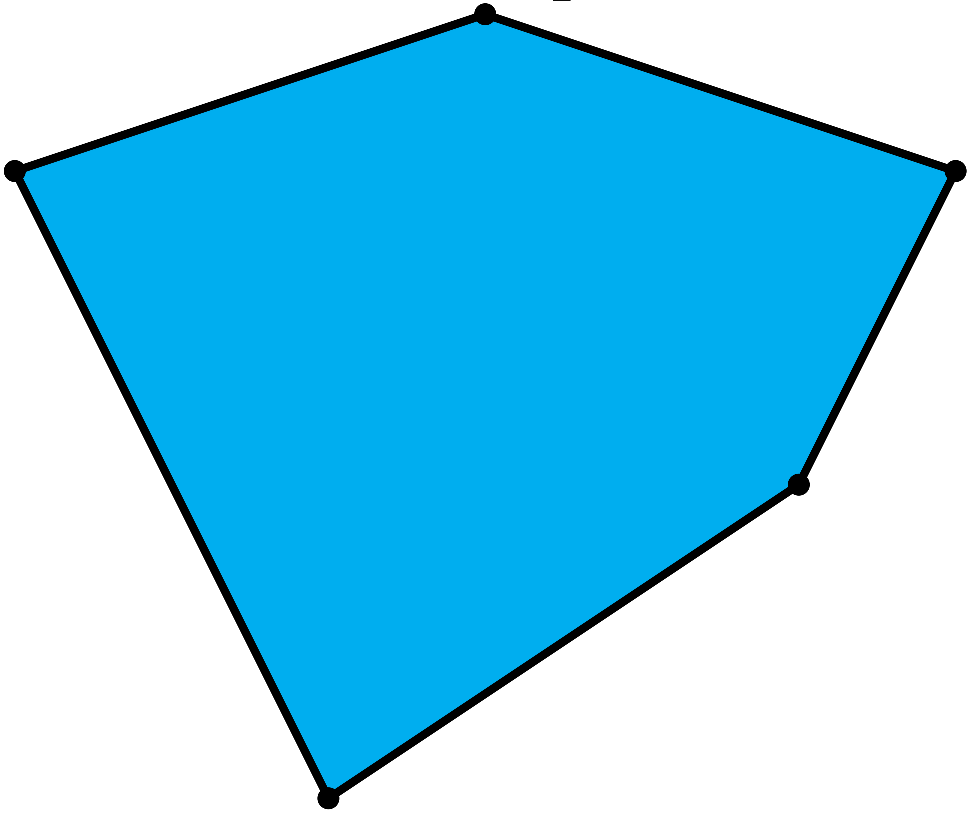

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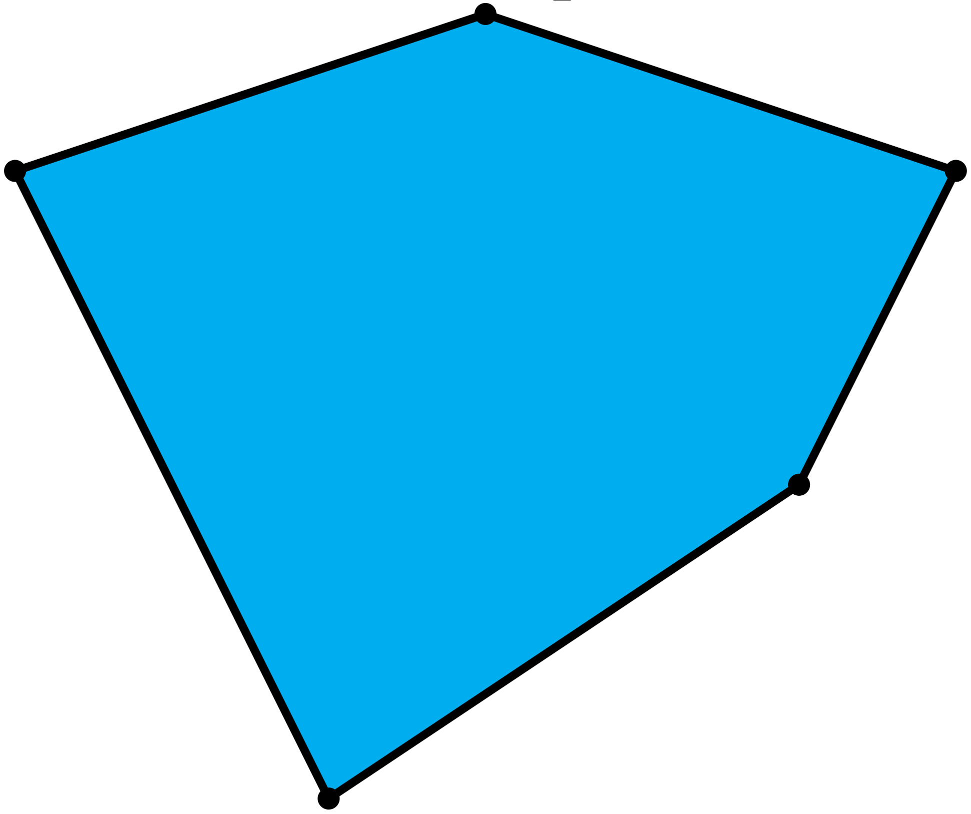

- *Exercise:* A polytope is the intersection of finitely many half spaces. An intersection of finitely many half spaces is a polytope if it is bounded.

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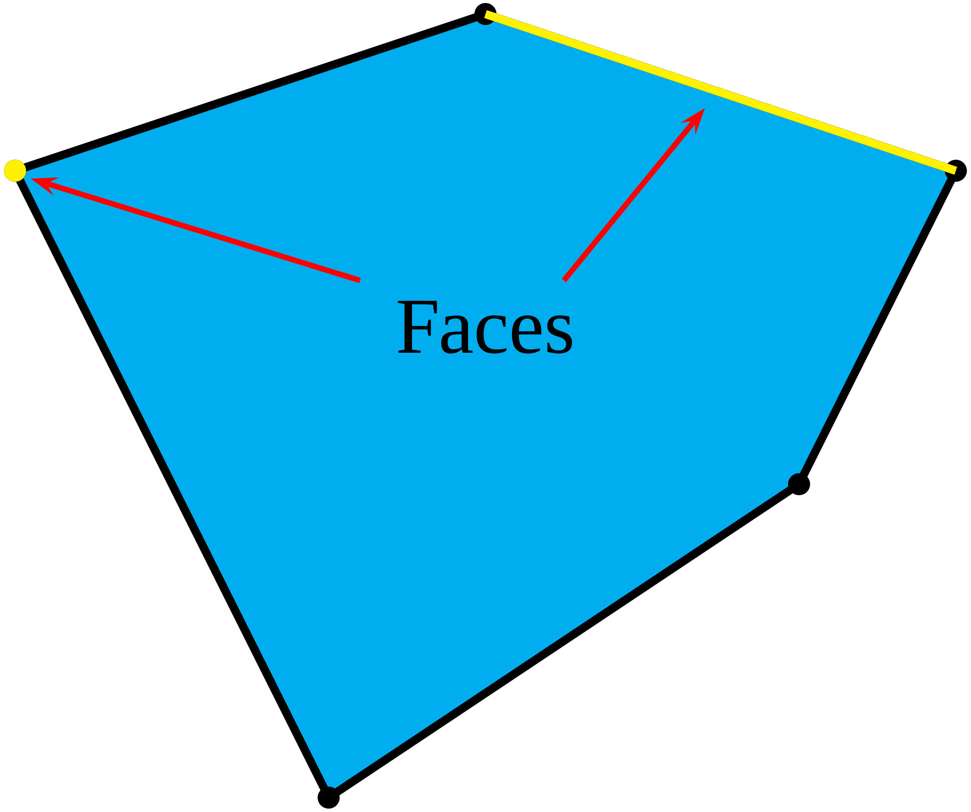

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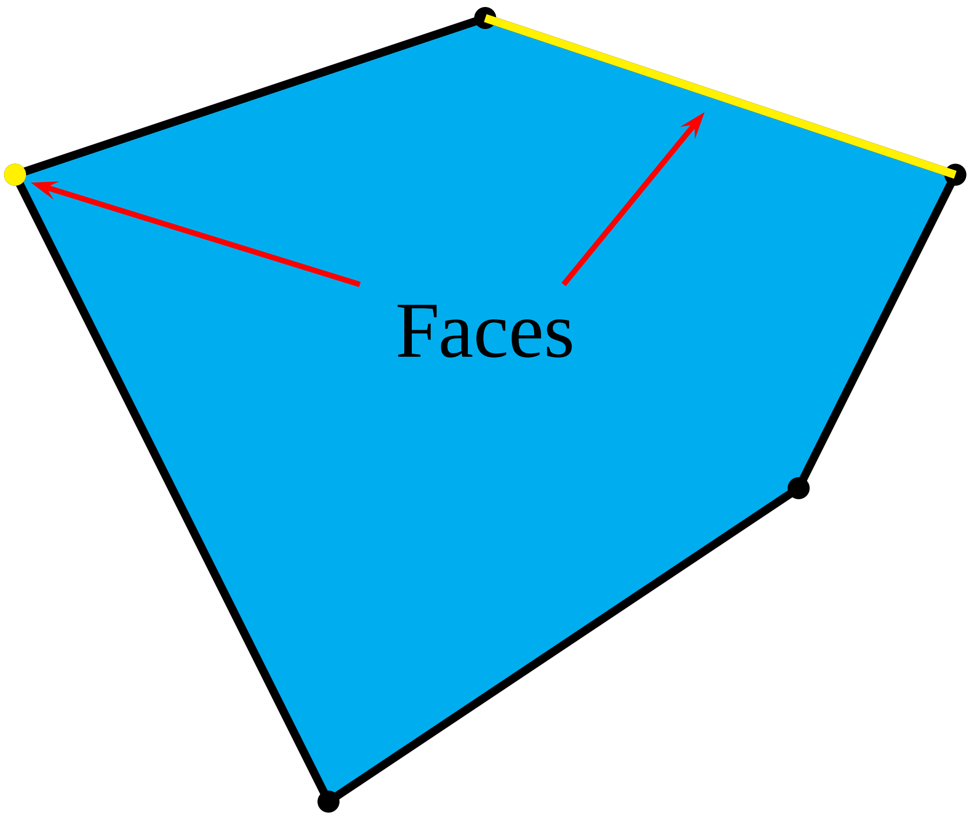

- A face is *proper* if it is not P itself.

Affine Independence

- An *affine combination* of points $p_0, \ldots, p_i$ is a sum $\alpha_0 p_0 + \cdots + \alpha_i p_i$ where $\alpha_0, \ldots, \alpha_i$ are real numbers with $\alpha_0 + \cdots + \alpha_i = 1$.

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- We say that $p_0, \ldots, p_i$ are *affinely independent* if it is not possible to write one of them as an affine combination of the others. (Equivalently, $p_1 - p_0, \ldots, p_i - p_0$ are linearly independent.)

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  - A *simplex* is the convex hull of an affinely independent collection of points.
- The *dimension* of a polytope is the maximal d such that the polytope has elements $p_0, \ldots, p_d$ that are affinely independent.

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  - The *interior* of P is the set of points that are not in any facet.

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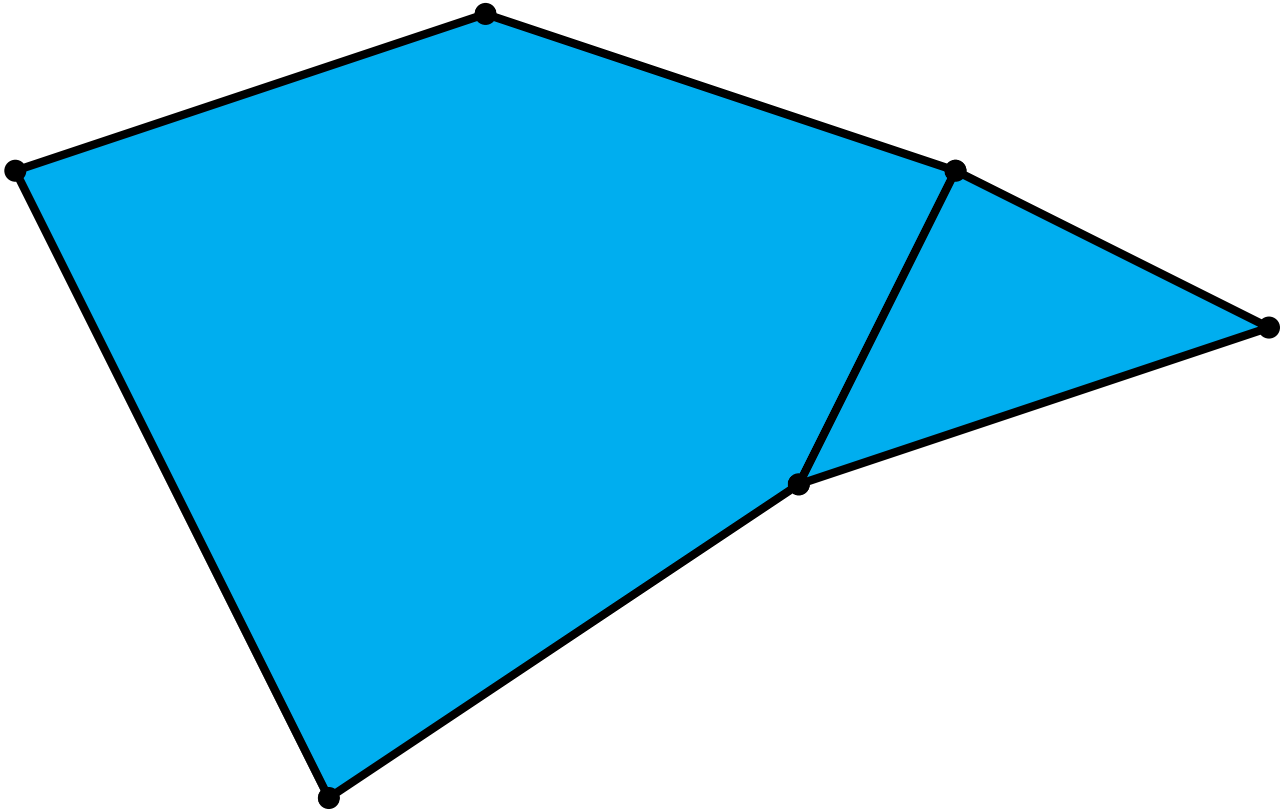
- If all of its elements are simplices, then $\mathcal{T}$ is a *simplicial complex*.

Barycentric Subdivision



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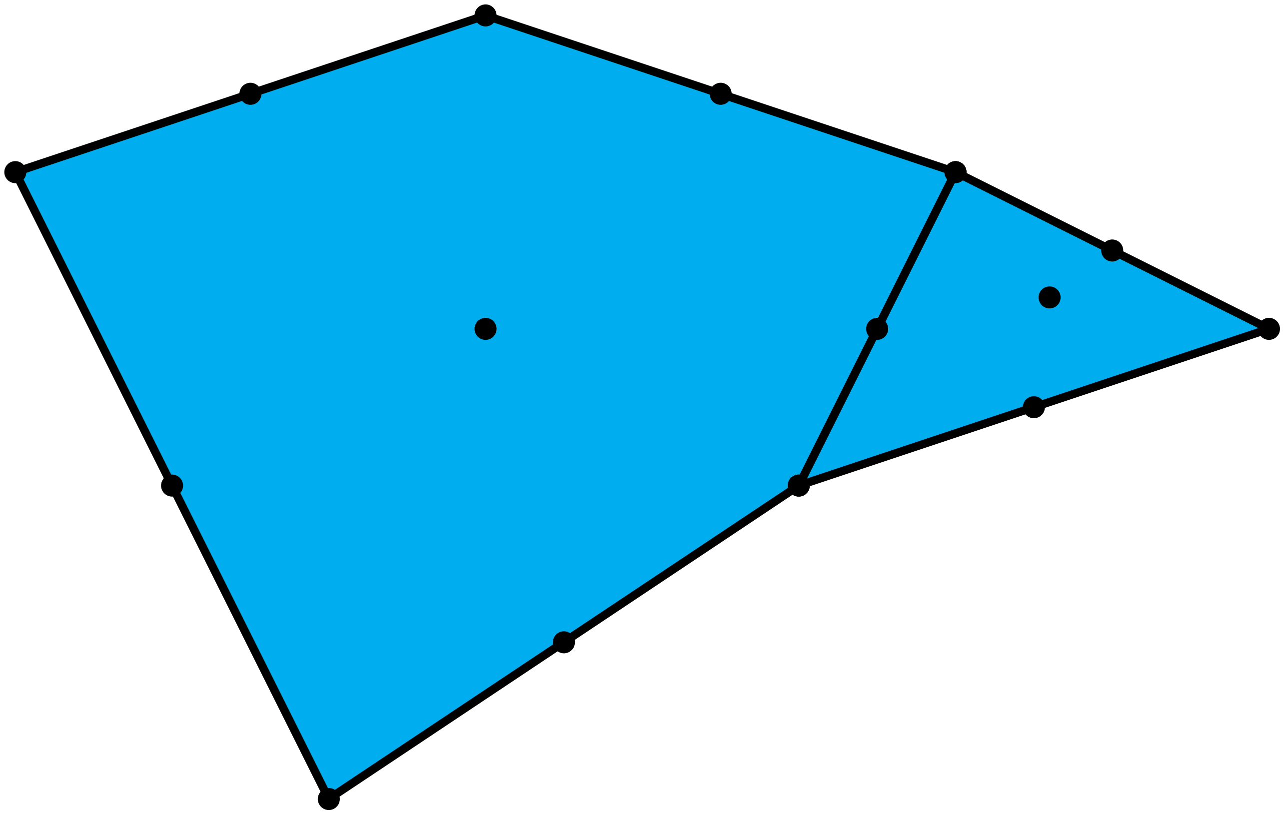

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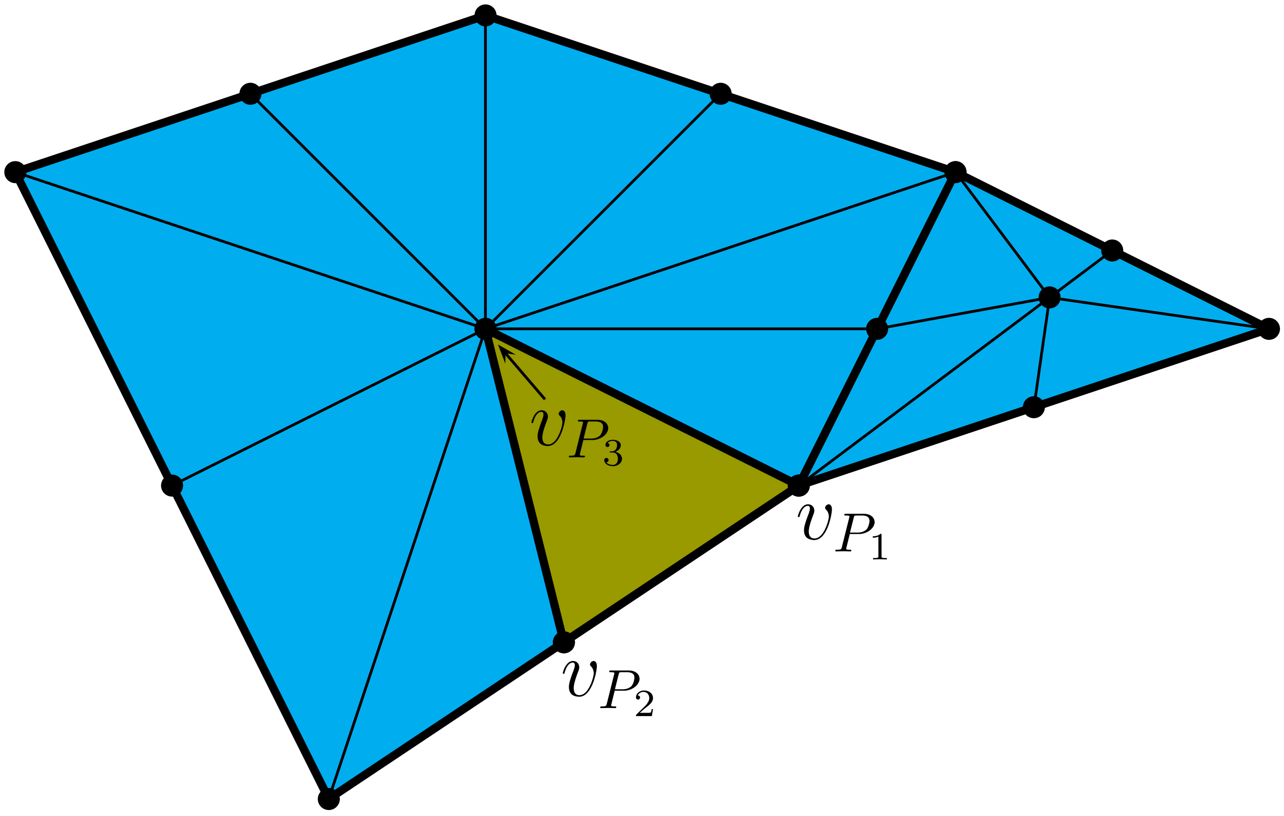

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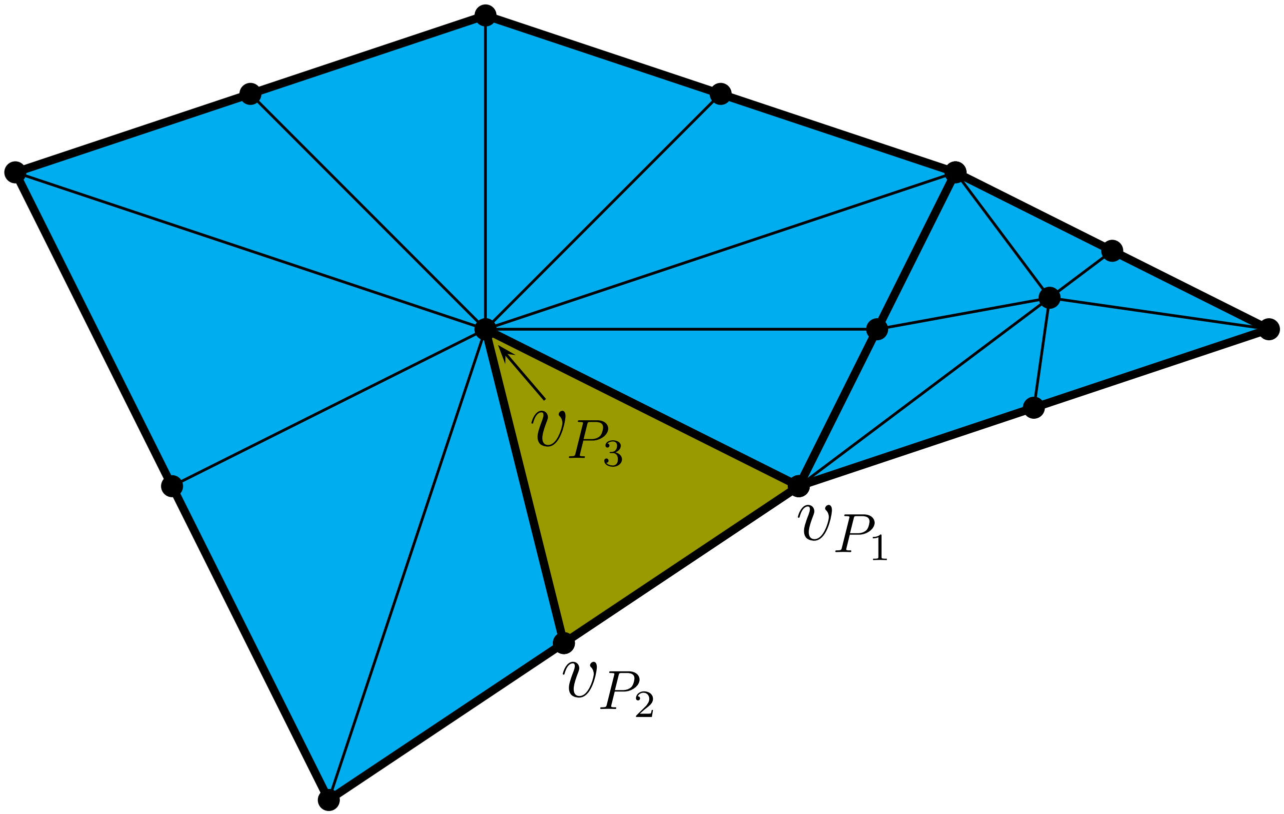

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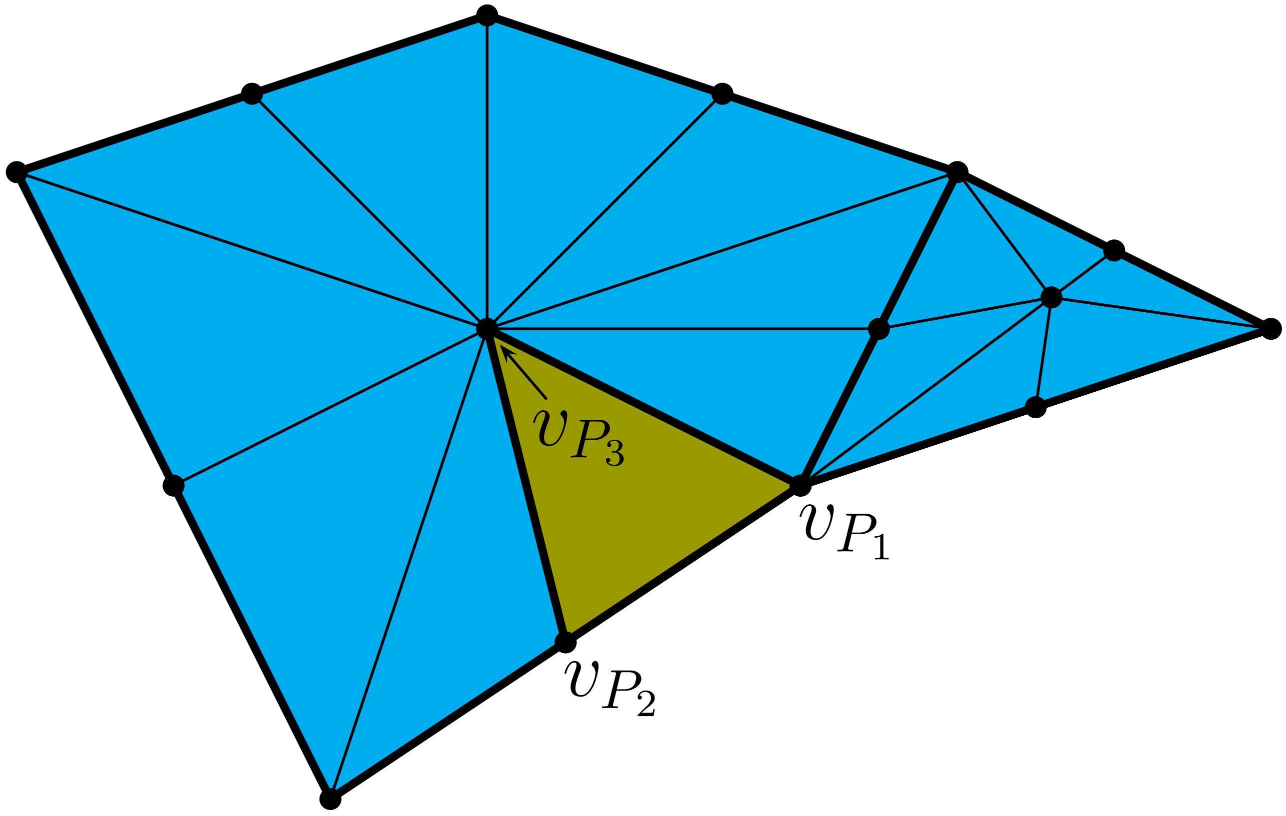

- For each $P \in \mathcal{T} \setminus \{\emptyset\}$ choose $v_P \in \operatorname{int} P$.
- Let $\mathcal{T}' = \{\emptyset\} \cup \{ \operatorname{conv}\{v_{P_1}, \ldots, v_{P_k}\} :$ $P_1, \ldots, P_k \in \mathcal{T} \setminus \{\emptyset\}, P_1 \subset \cdots \subset P_k \}$.

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  - *Ugly Calculation:* In this case

$$\frac{\mathrm{mesh}(\mathcal{T}')}{\mathrm{mesh}(\mathcal{T})} \leq \frac{\dim \mathcal{T}}{\dim \mathcal{T} + 1}.$$

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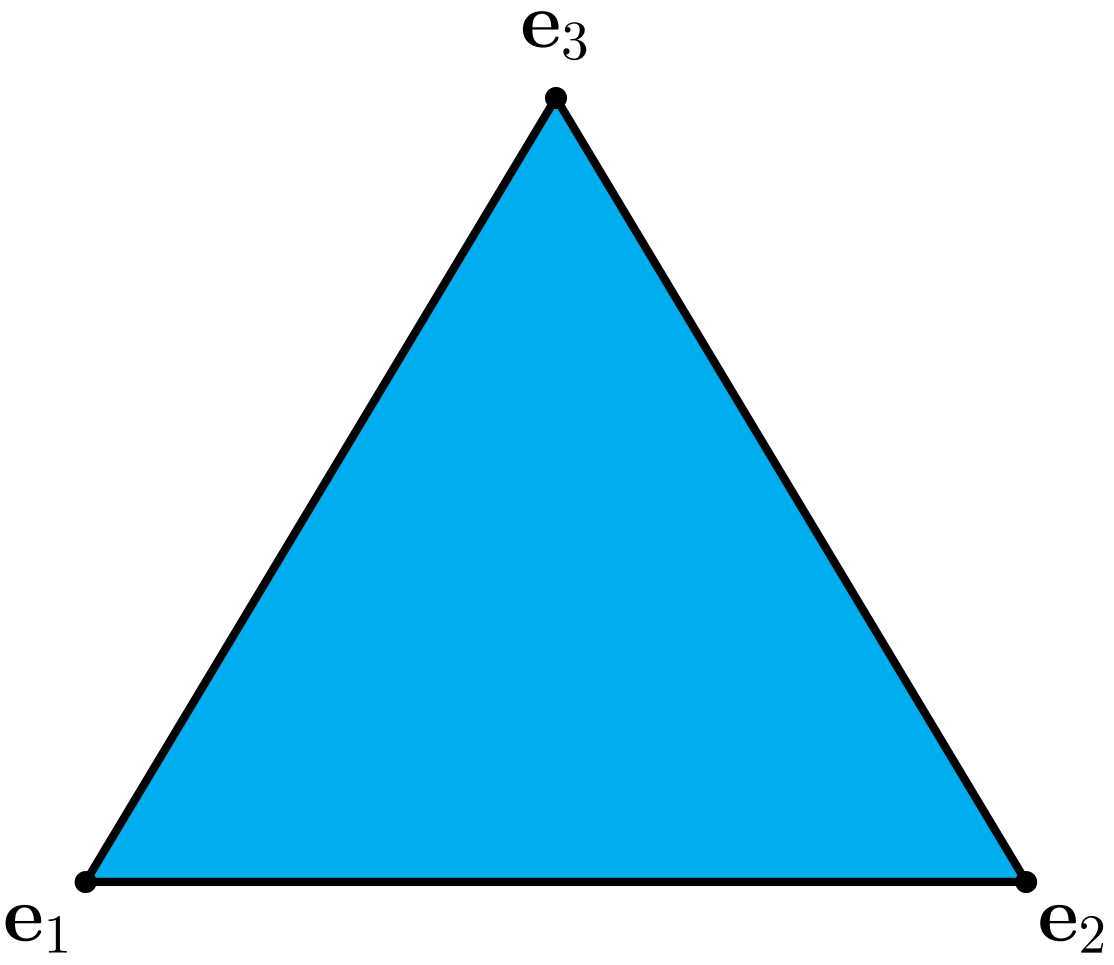
Sperner's Lemma: *A* Sperner labelling of a simplicial subdivision of Δ has a completely labelled simplex.

Sperner's Lemma Illustrated



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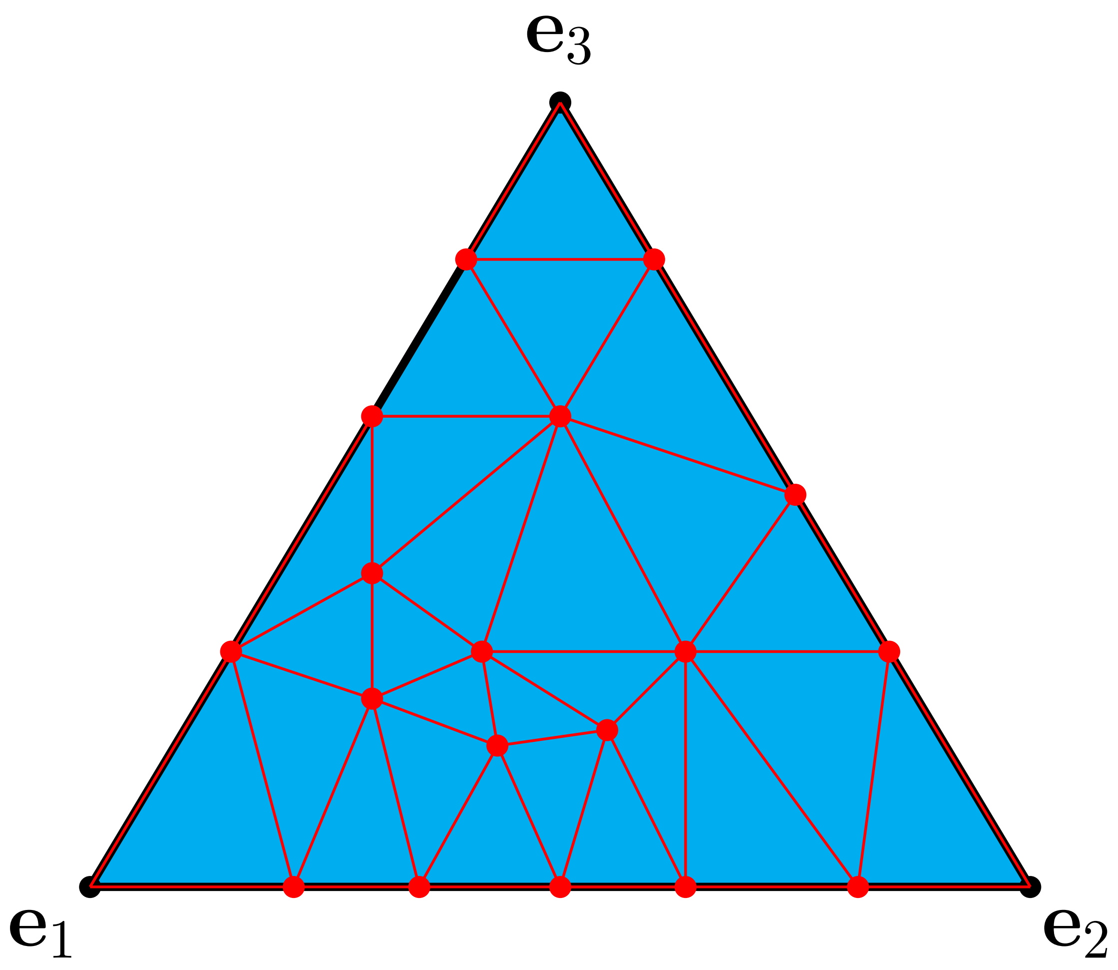

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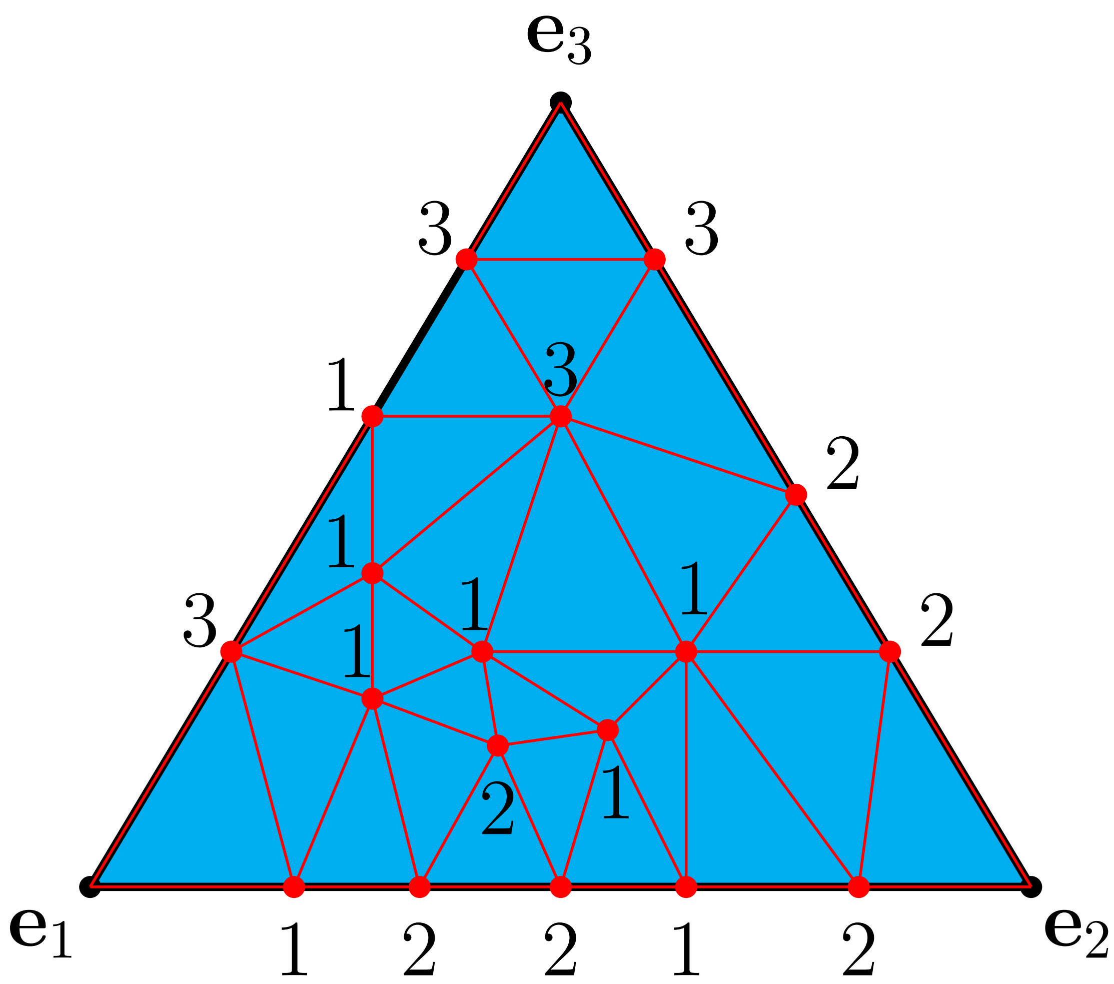

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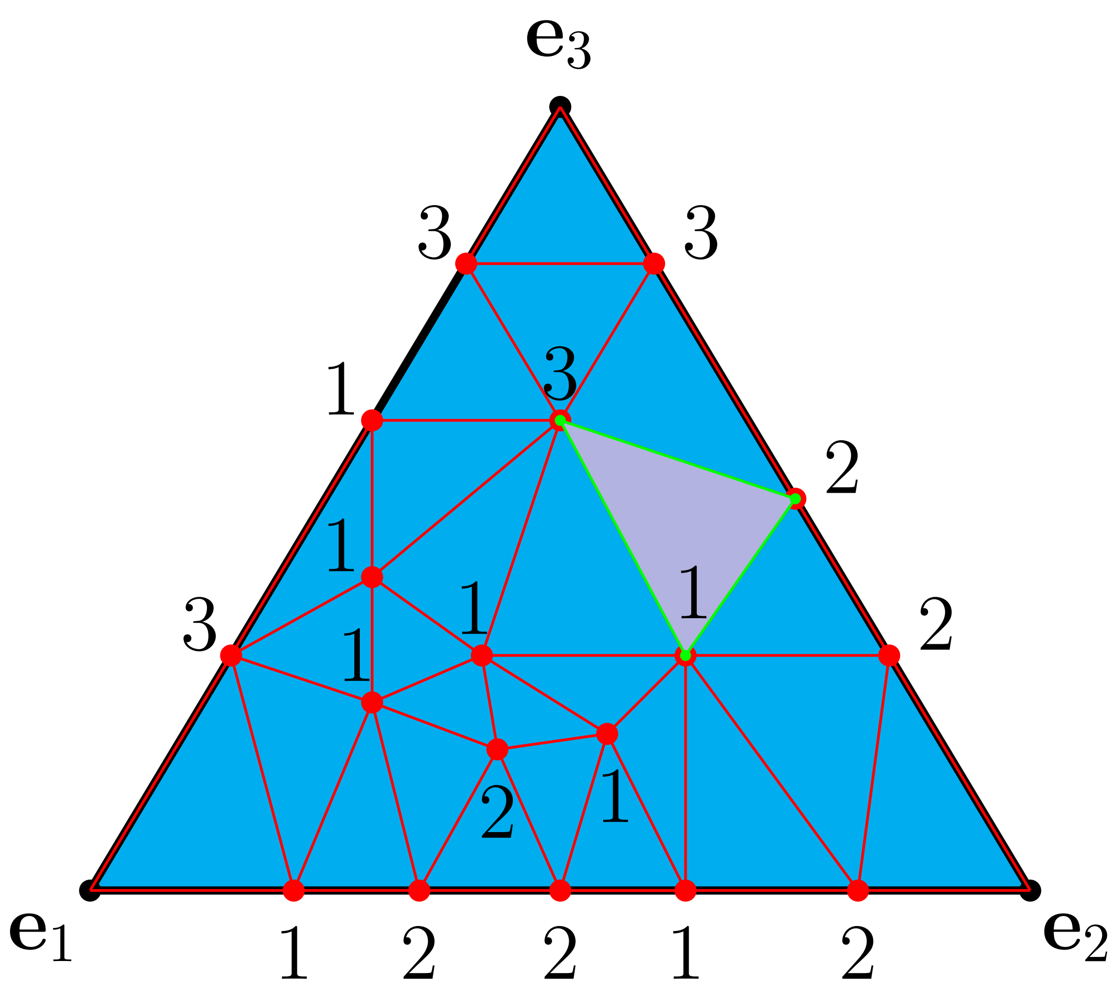

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- Pass to a convergent subsequence, so that $v_i^r \to x^*$ for some (hence all) i, so $x_i^* \geq f_i(x^*)$ for all i, but $\sum_i x_i^* = \sum_i f_i(x^*) = 1$.

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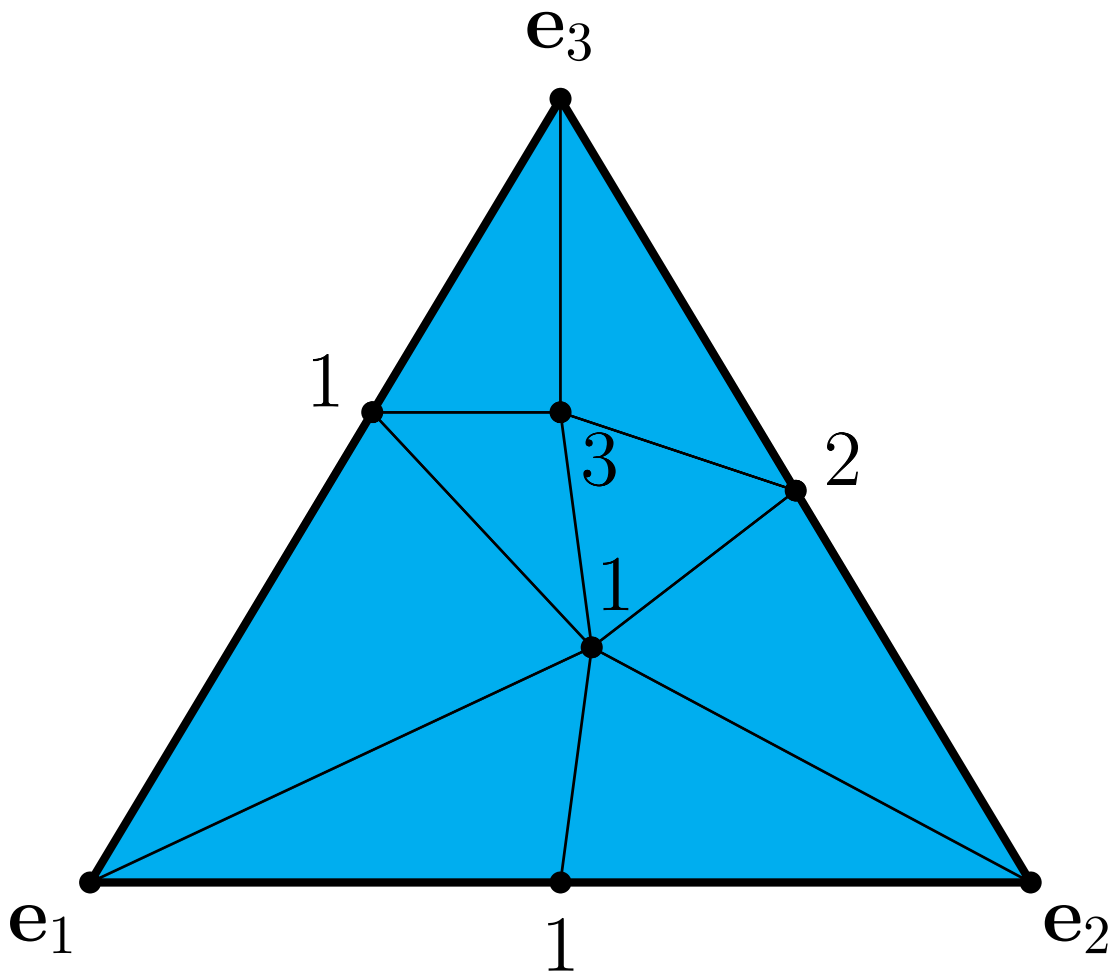

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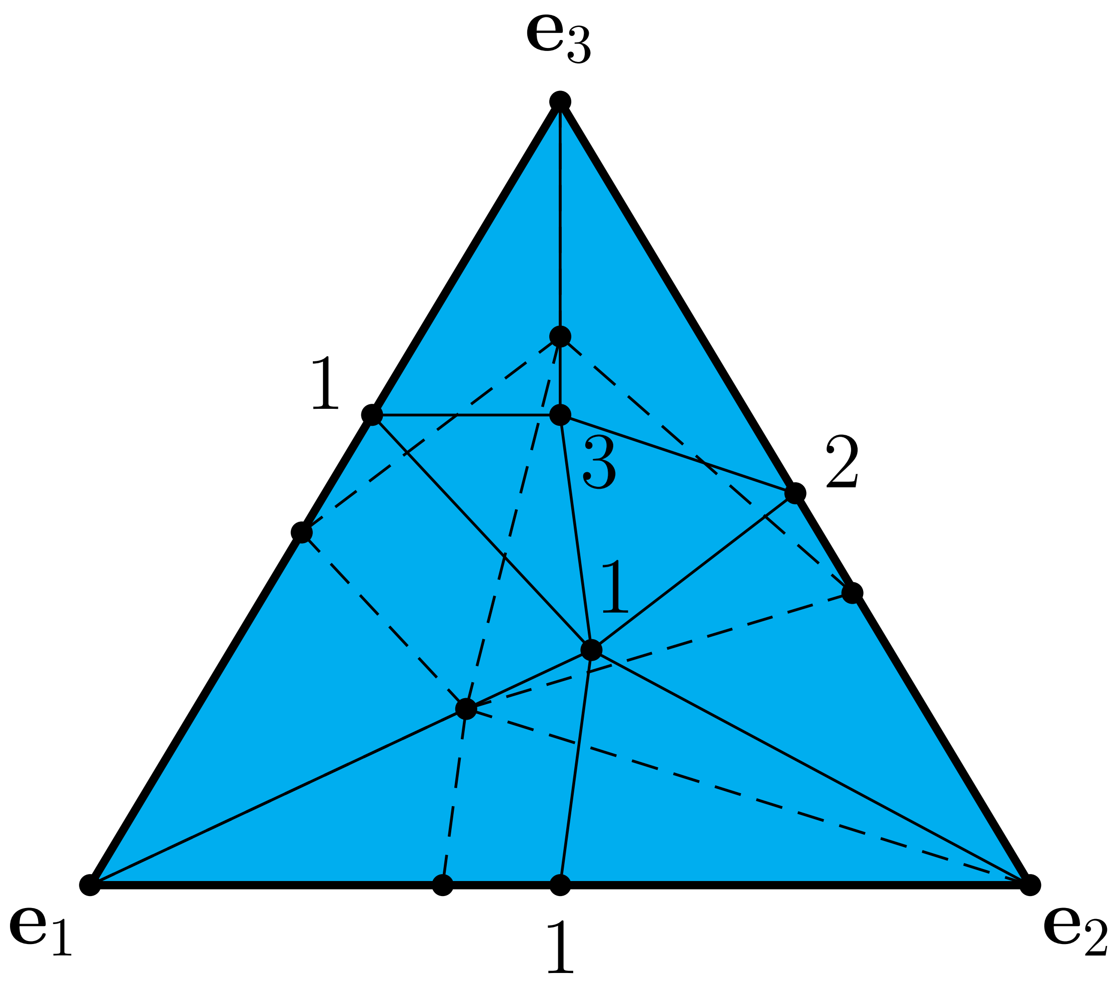

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Thus $\sum_{P \in \mathcal{T}_{d-1}} q_P(1) = Q(1) \neq 0$.

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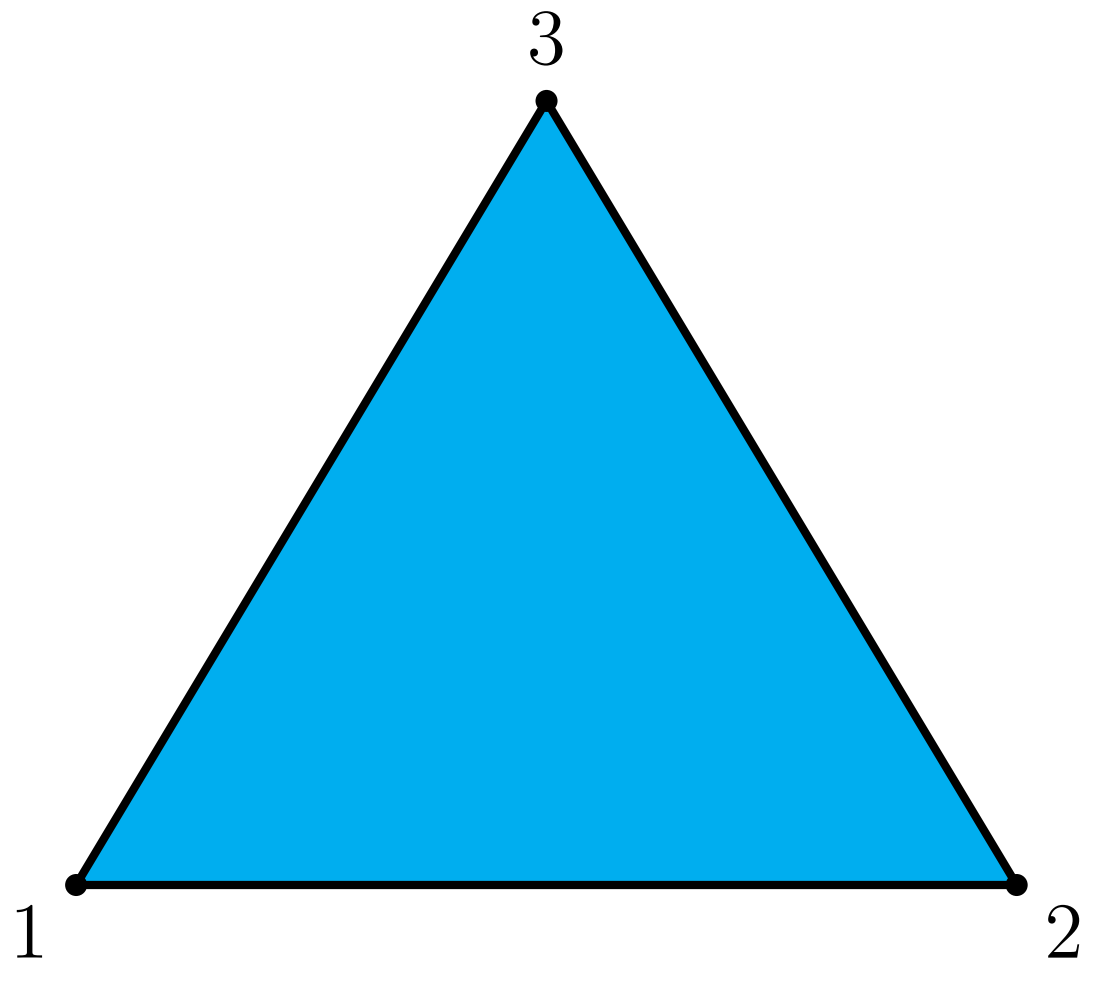
There is a proof of Sperner's lemma that provides an algorithm for computing a completely labelled simplex.



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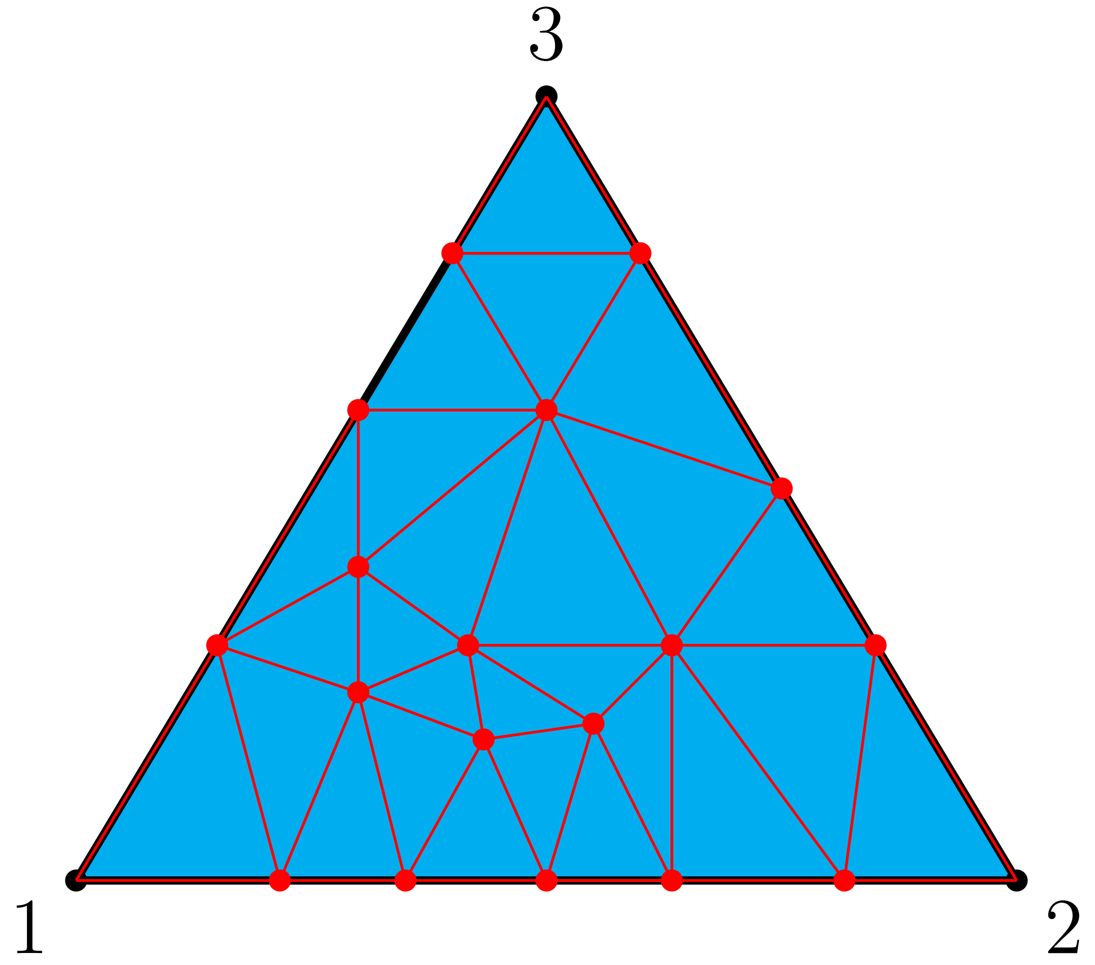

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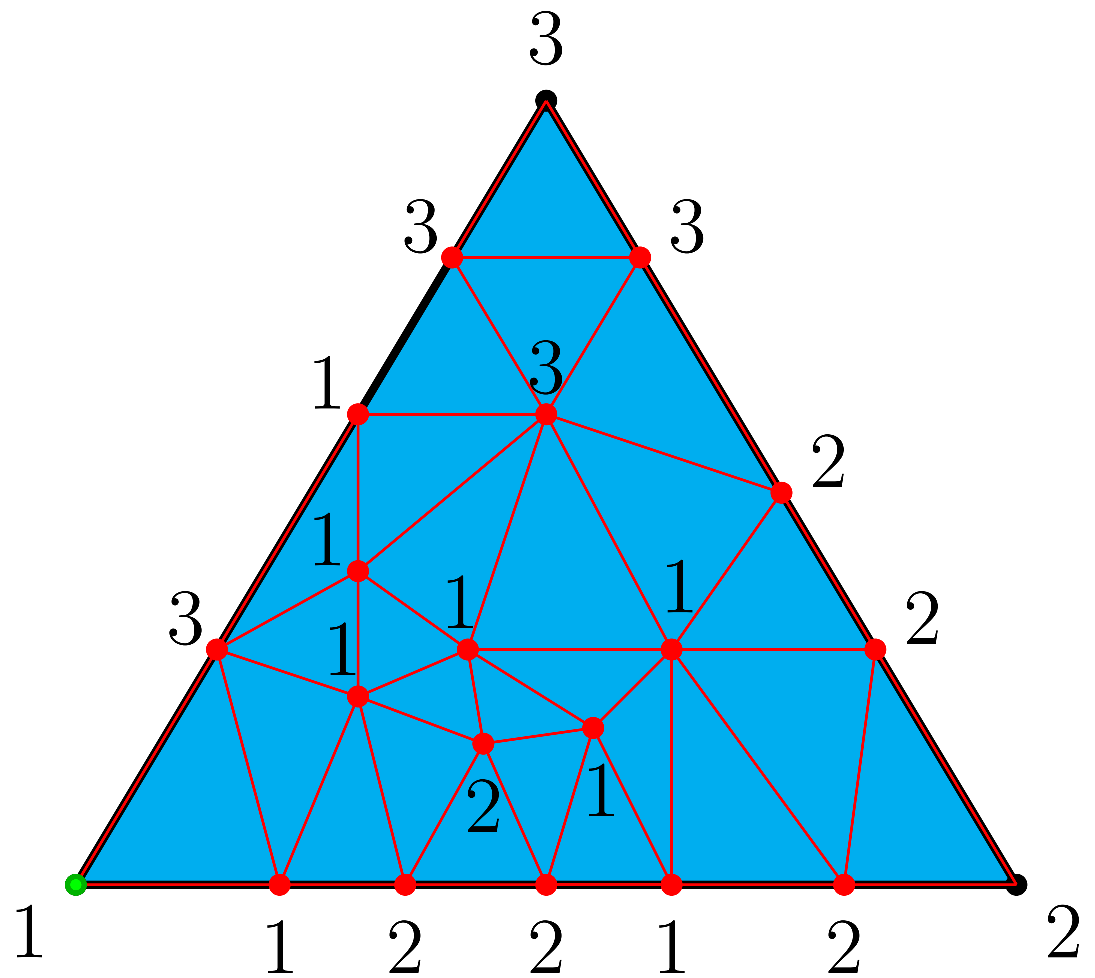

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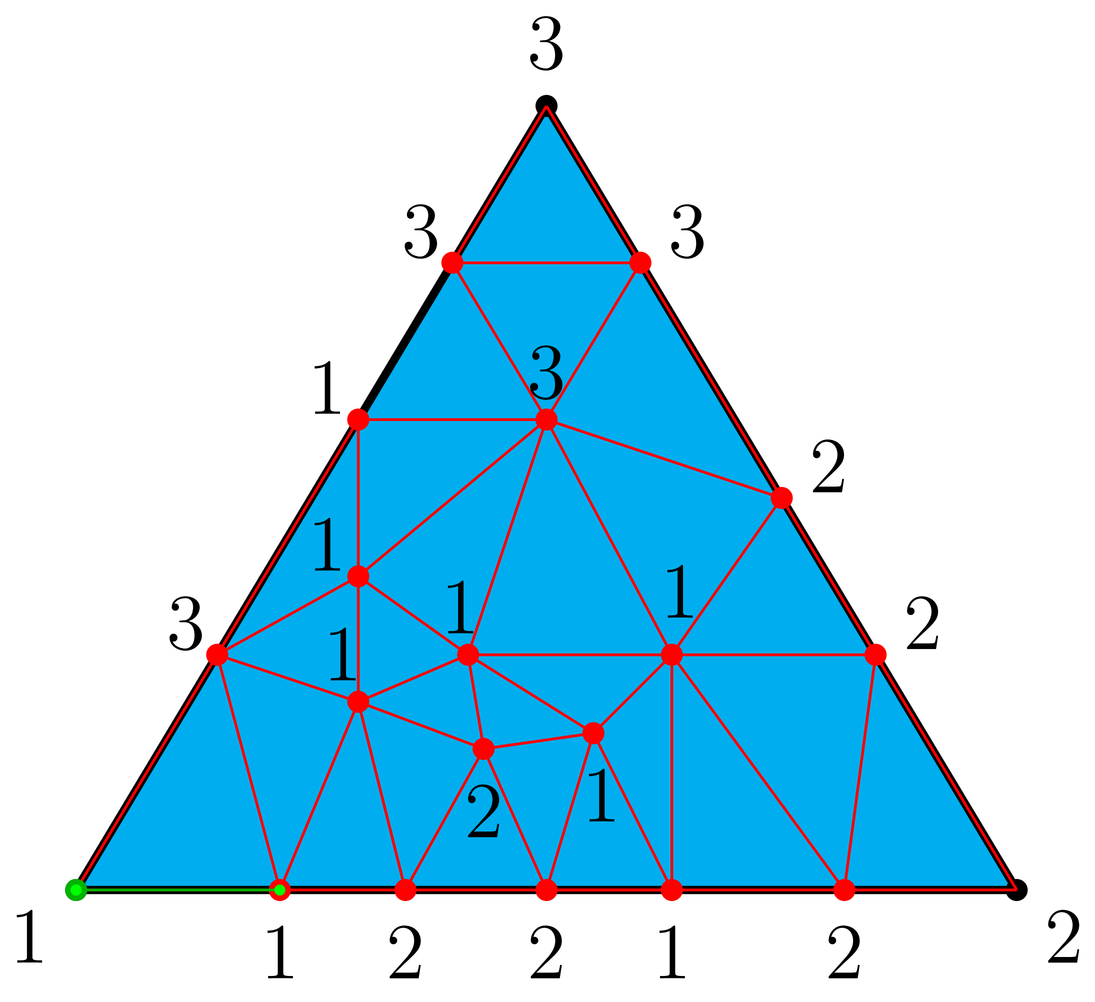

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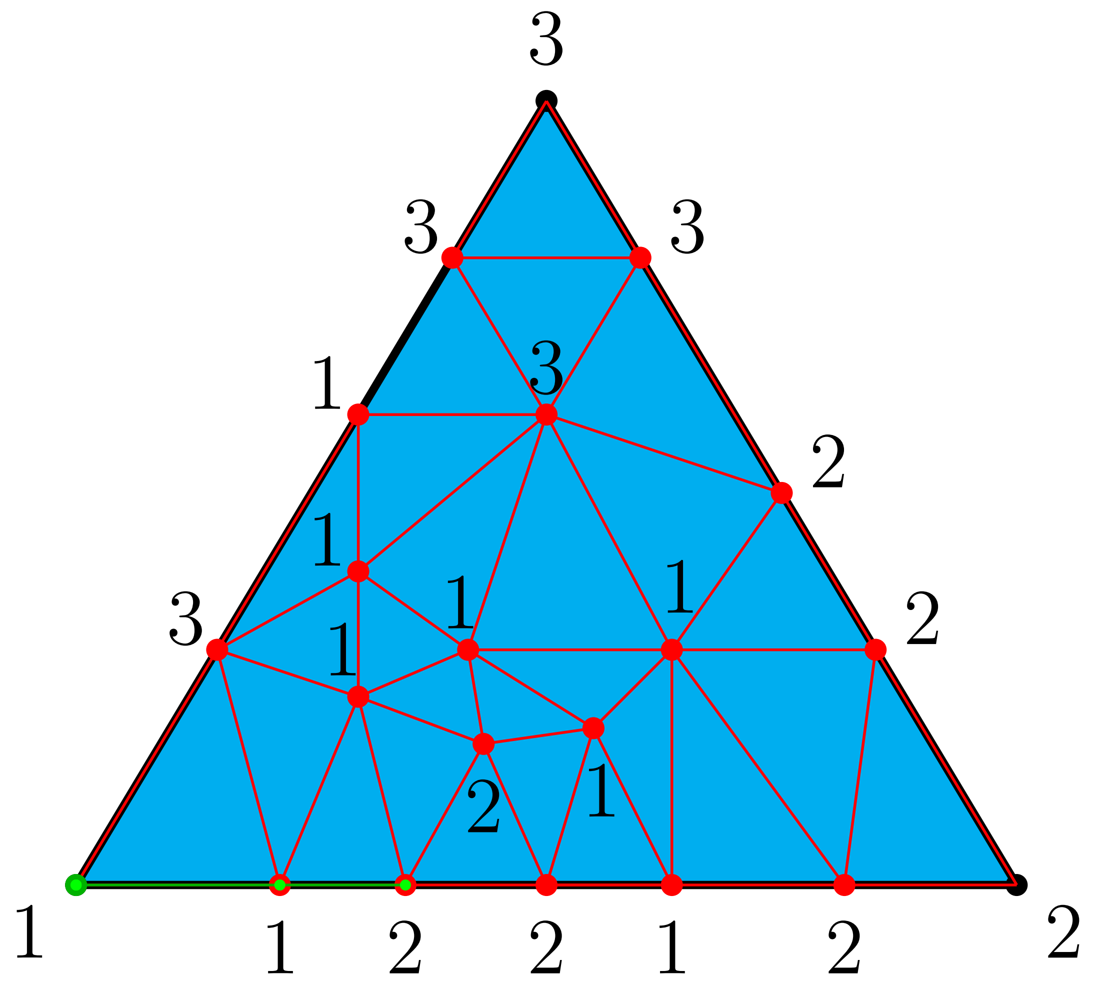

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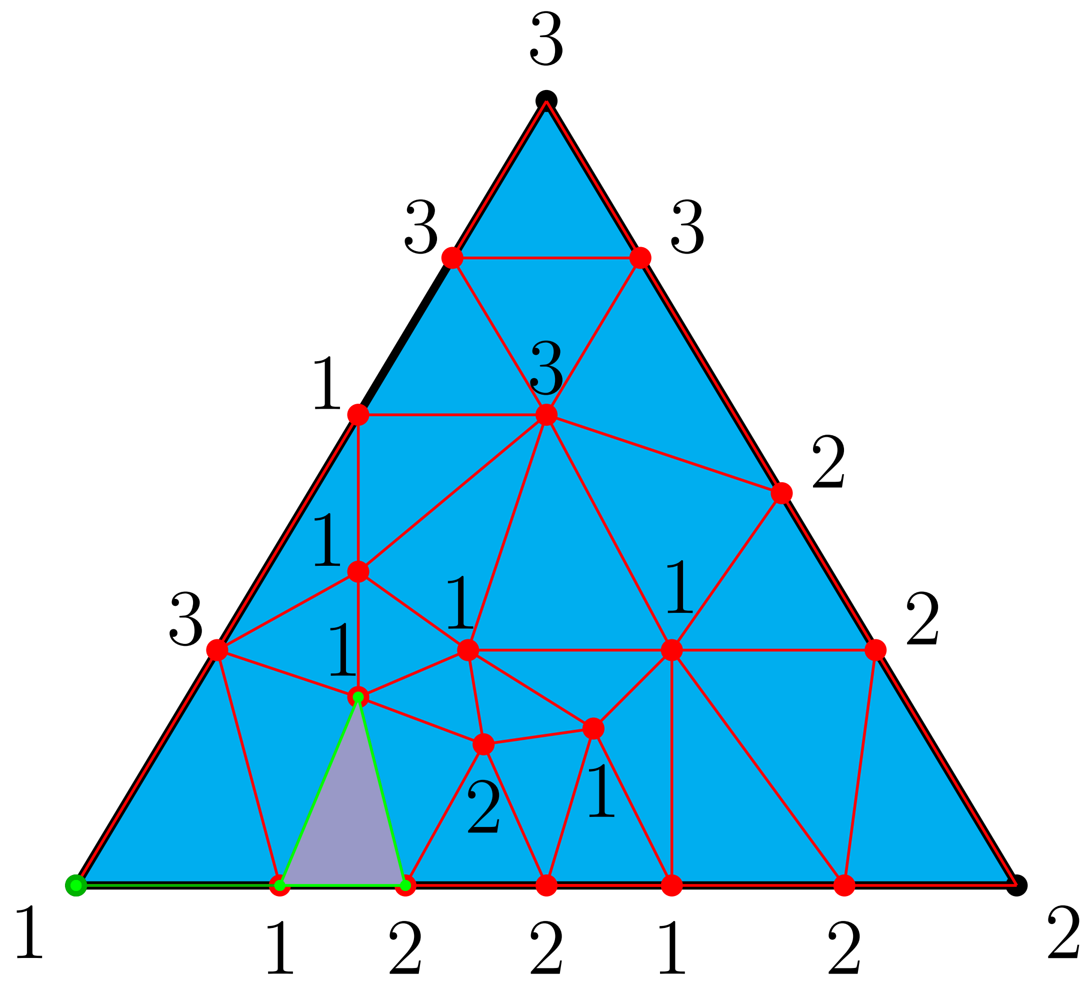

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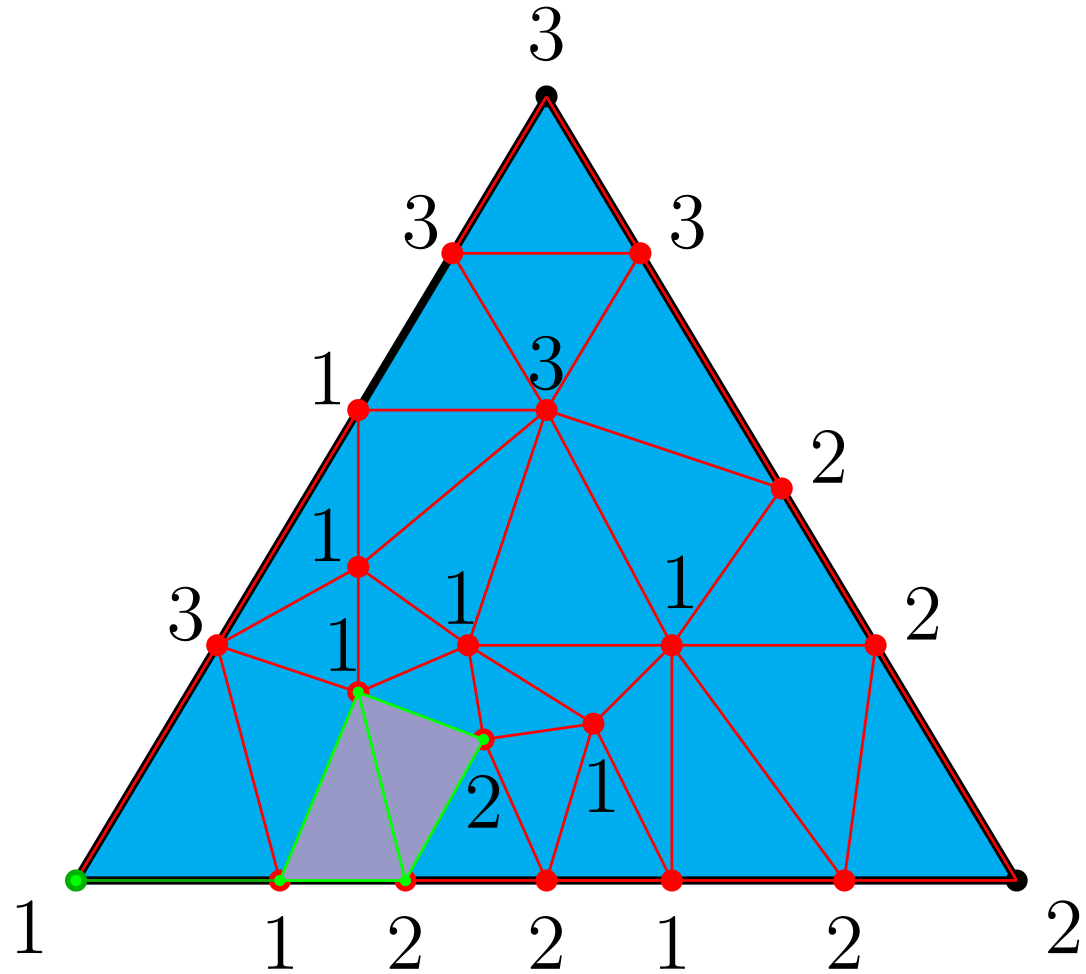

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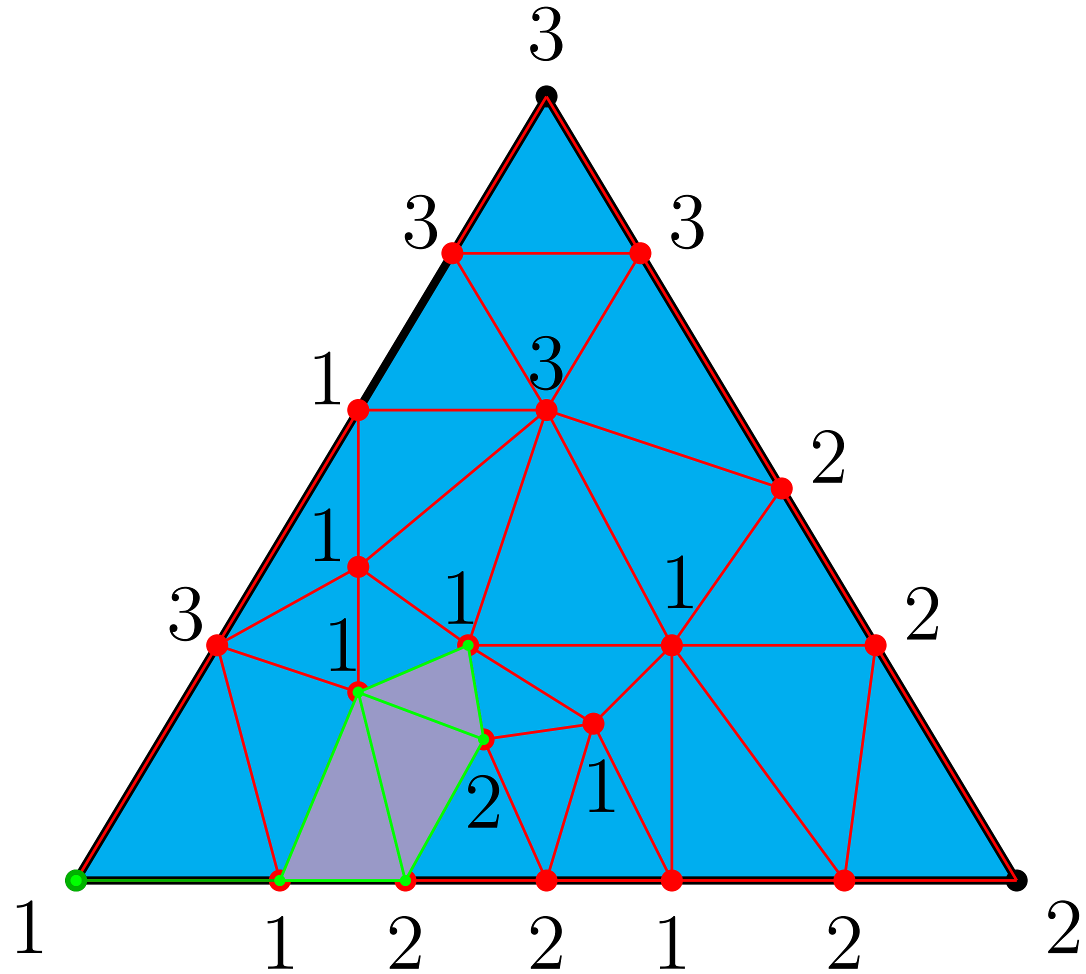

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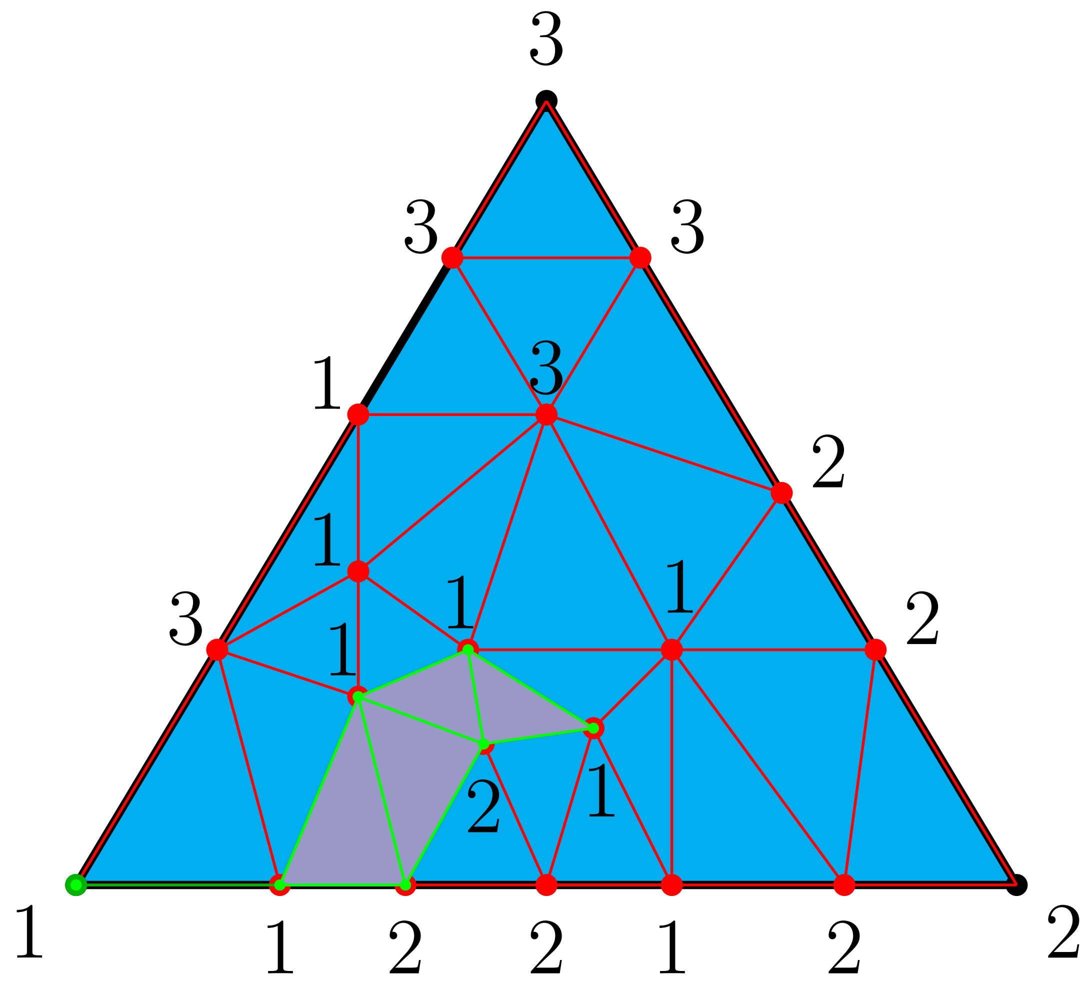

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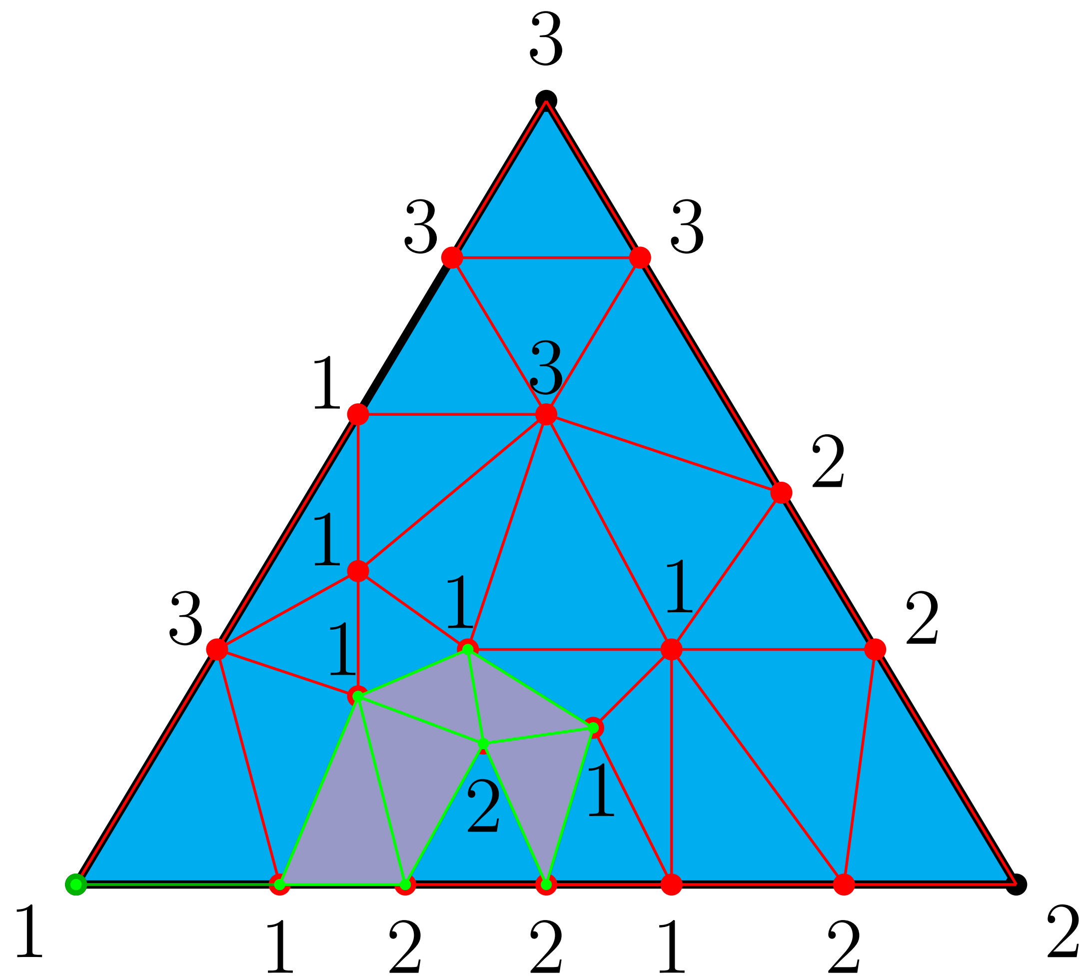

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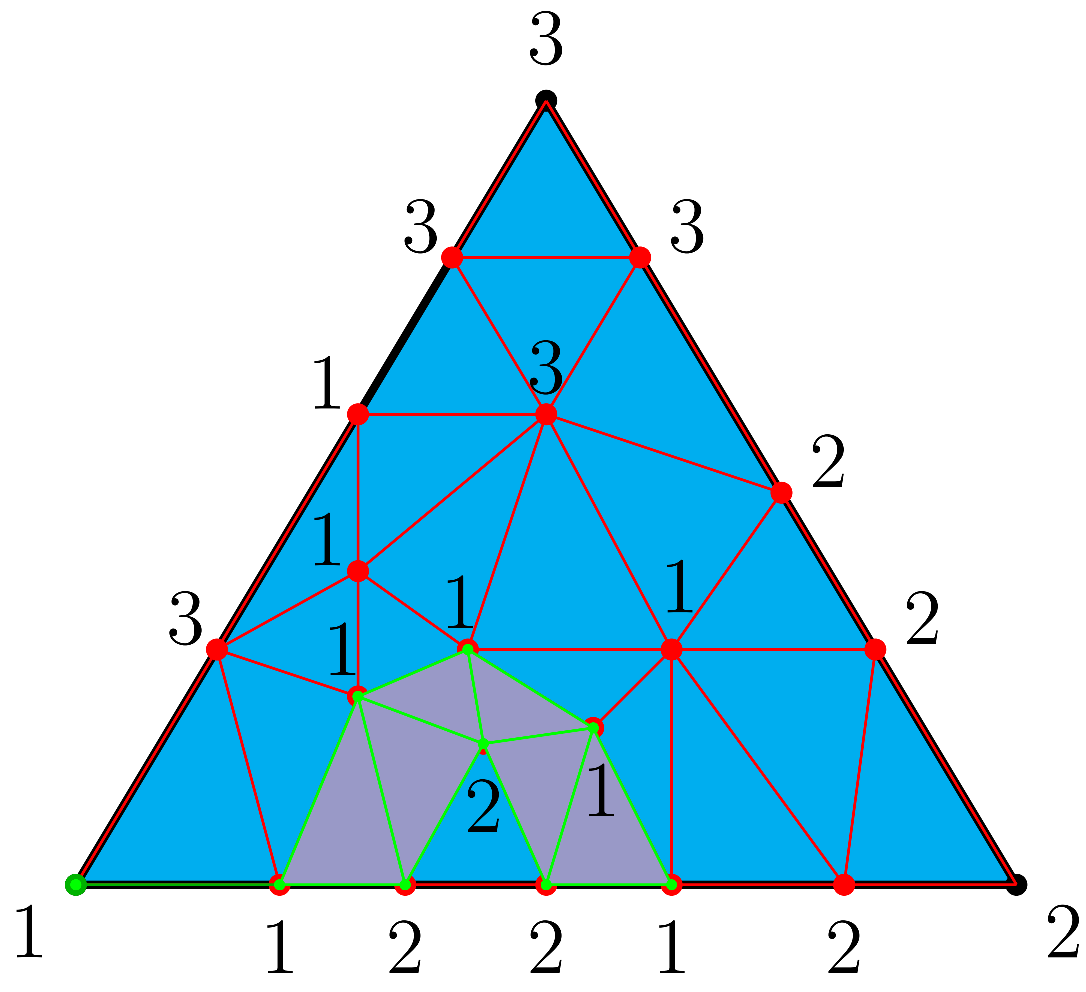

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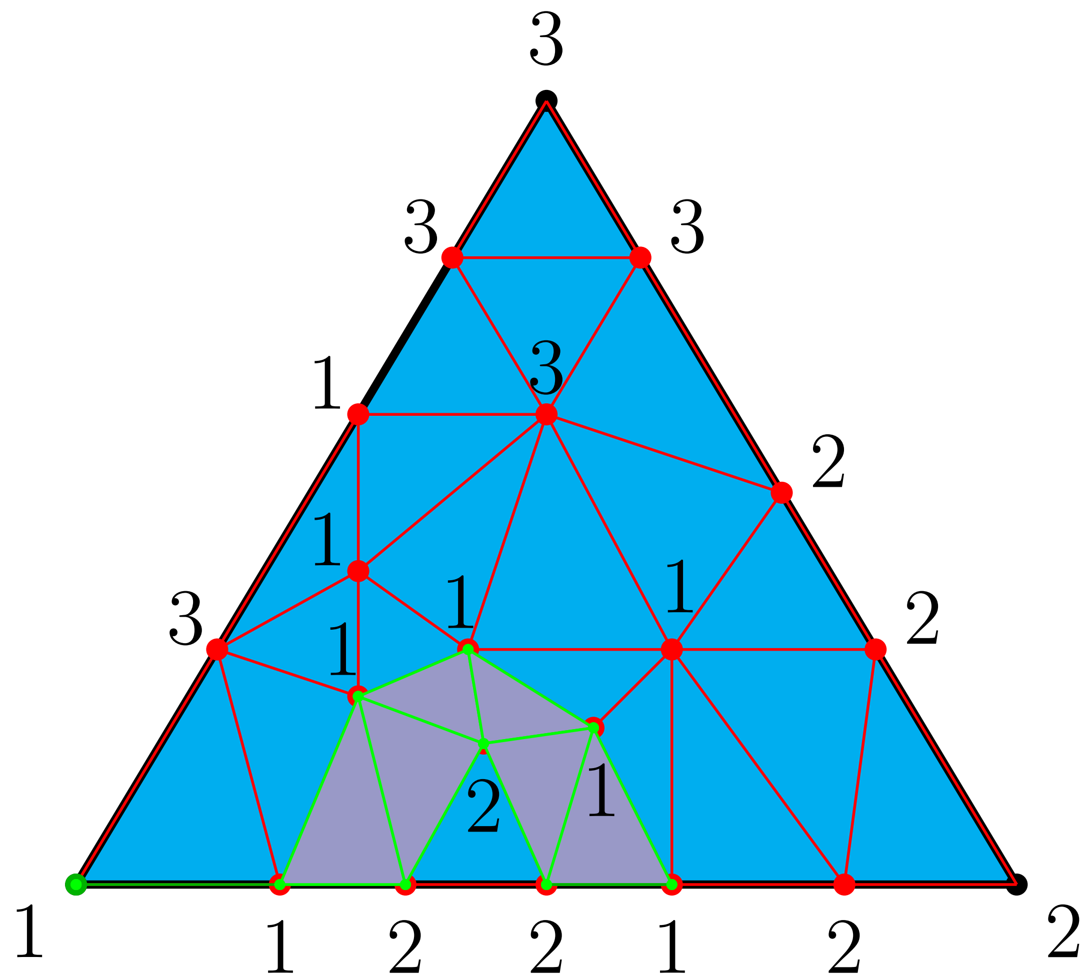

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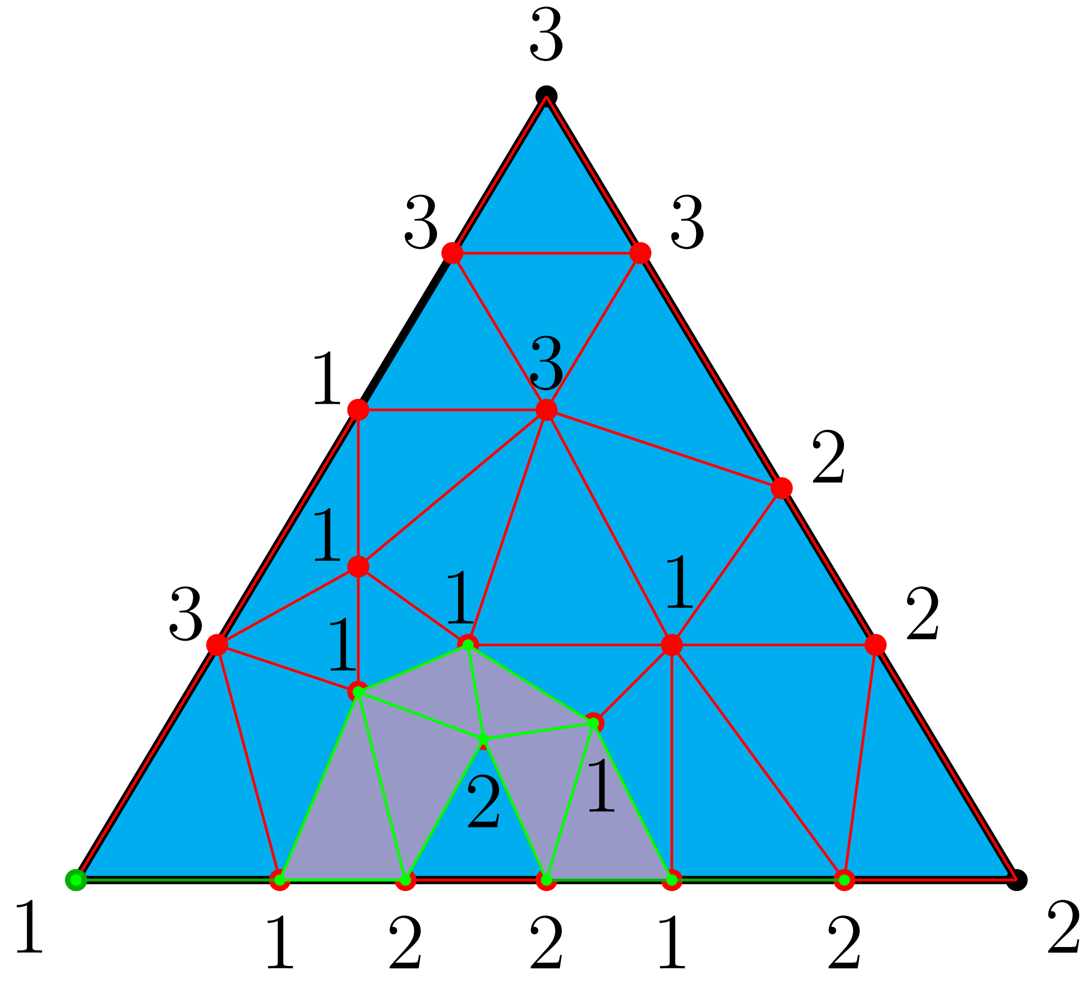

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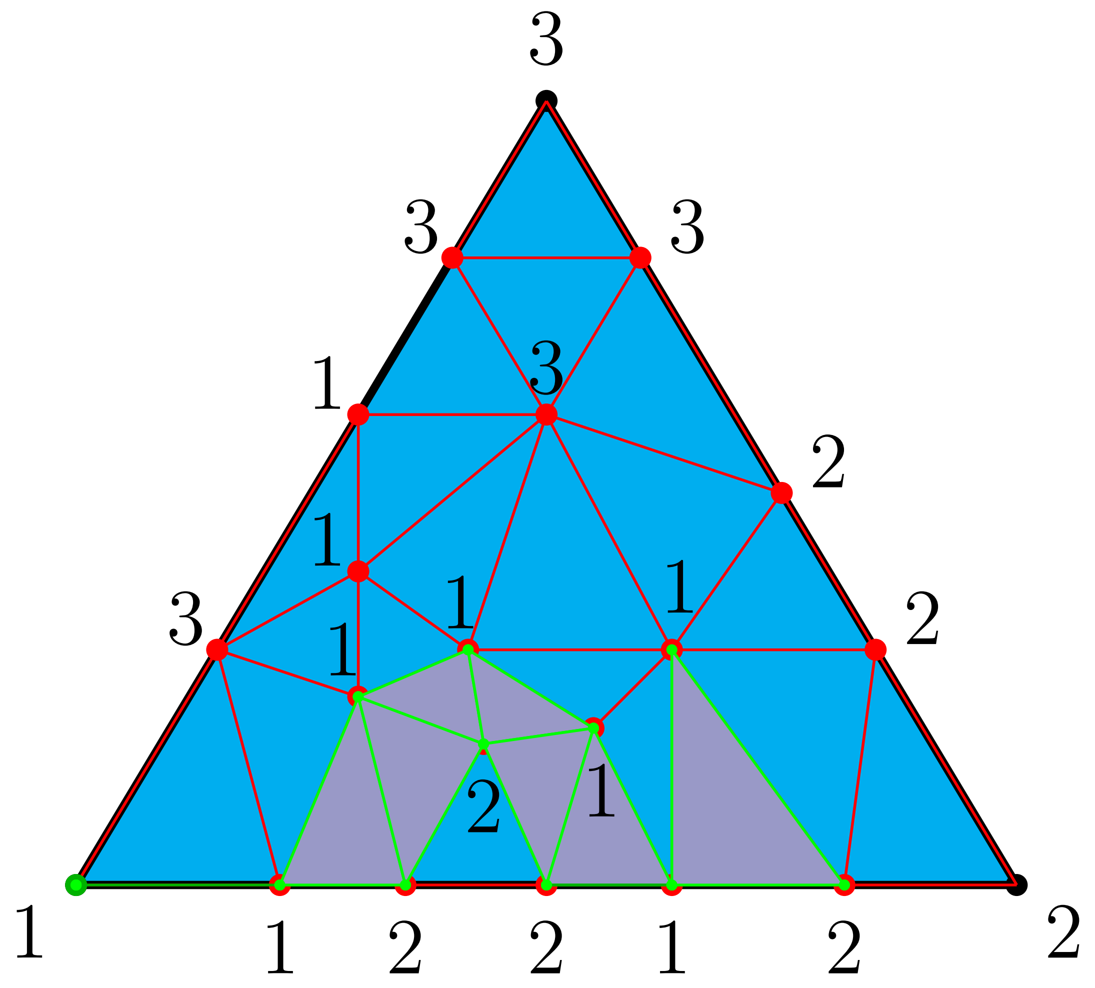

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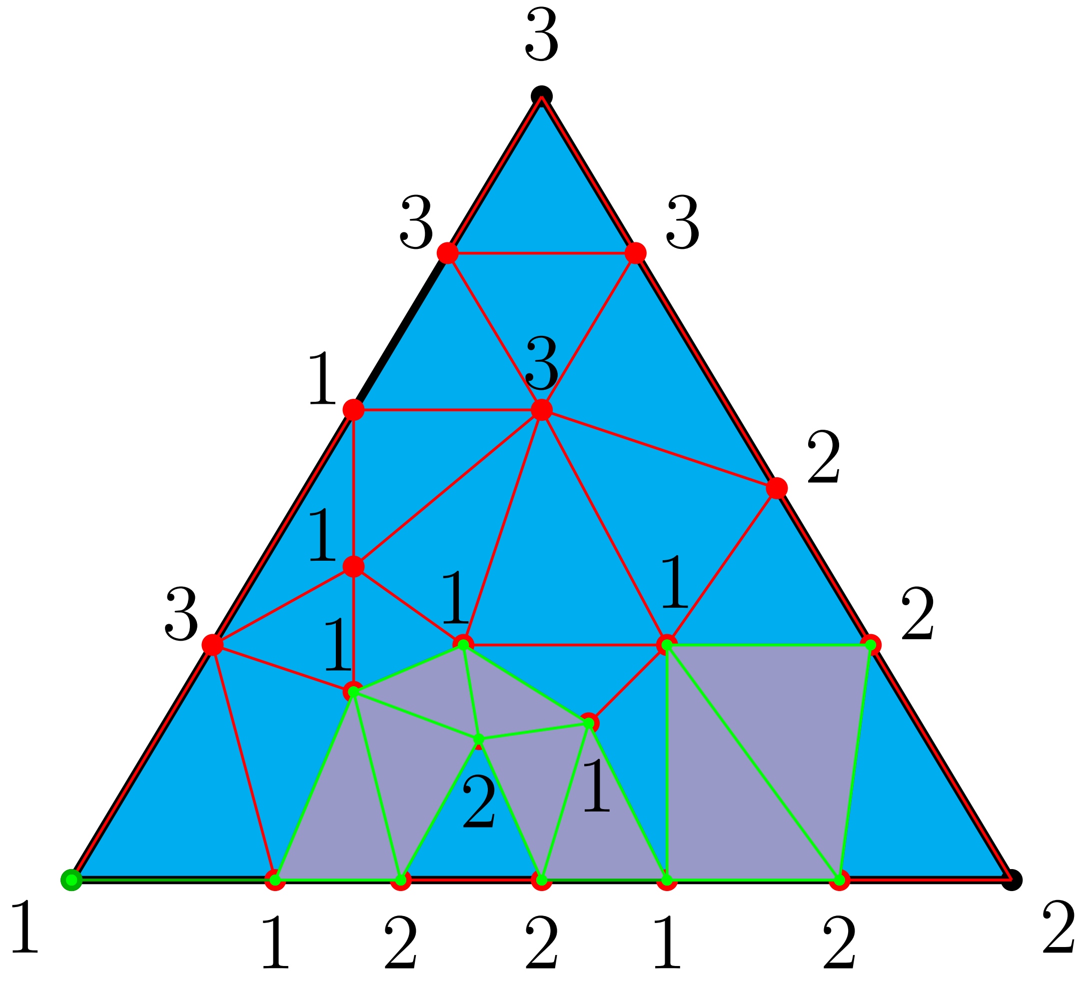

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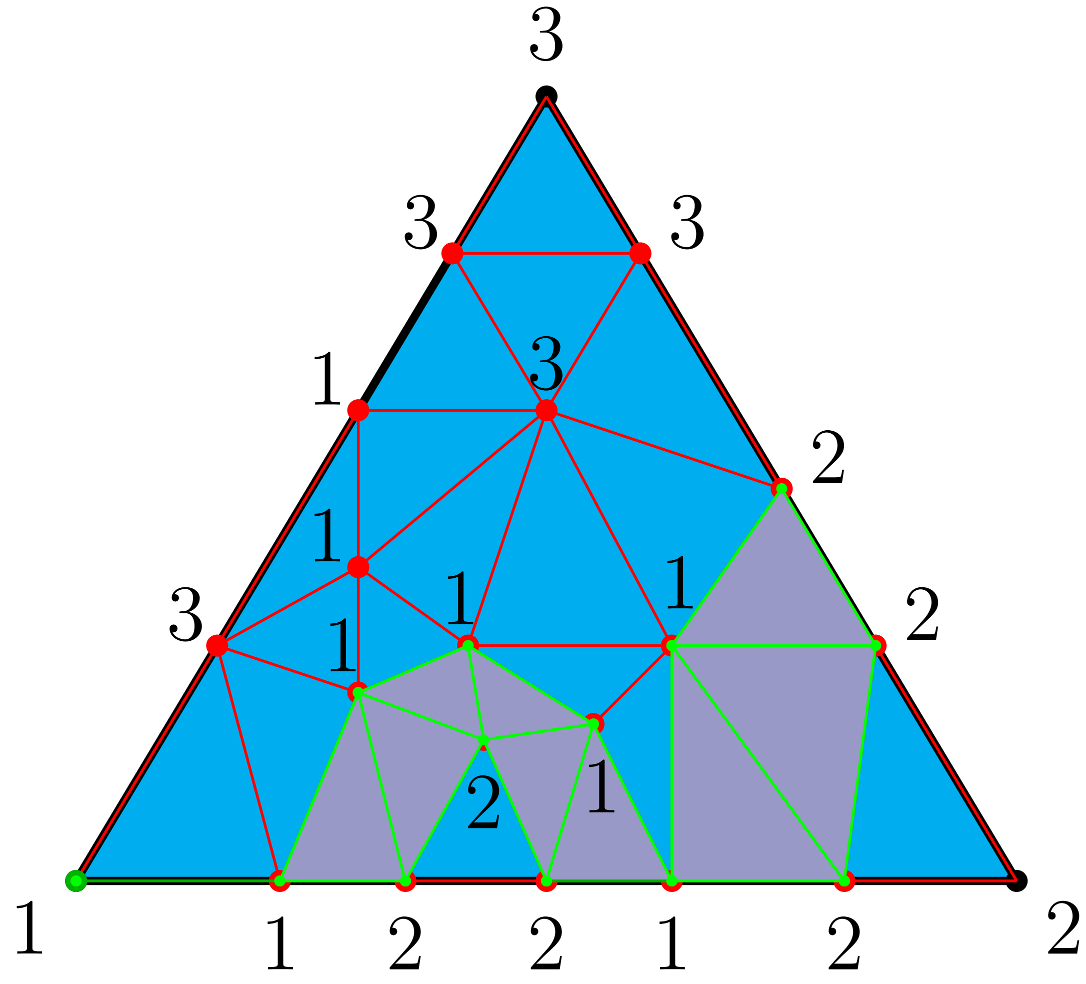

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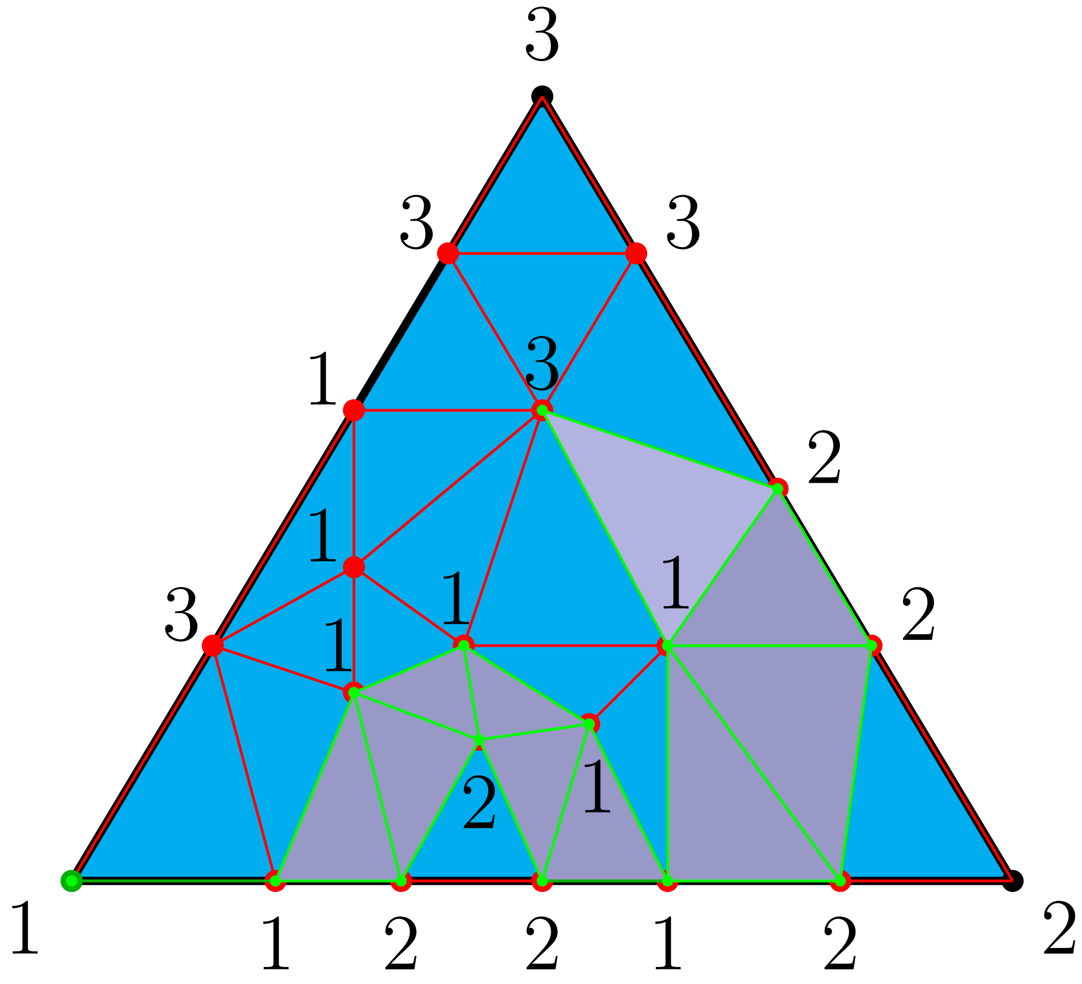

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Kakutani's Fixed Point Theorem: F has a fixed point: $x^* \in F(x^*)$ for some $x* \in C$.

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We study a similar game in which the common strategy set is a finite subset of C.

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The game (A, B) is an *imitation game* if $m = n$ and B is the $m \times m$ identity matrix, which we denote by I^m.

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We will show that any accumulation point x^* of $\{x_m\}$ is a fixed point of F.

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Proof of the Claim

Player 1's expected payoff if she plays i is

$$u(i, \tau^m) = \sum_{j=1}^{m} \tau_j^m a_{ij} = -\sum_{j=1}^{m} \tau_j^m \langle x_i - y_j, x_i - y_j \rangle.$$

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then use $\sum_j \tau_j^m (x_{m+1} - y_j) = 0$ to see that

$$u(i, \tau^m) = -\|x_i - x_{m+1}\|^2 - \sum_{j=1}^{m} \tau_j^m \|x_{m+1} - y_j\|^2.$$

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- Since this proves the existence of an equilibrium, in conjunction with the procedure above is gives a proof of the KFPT.
- Combined with the procedure above, it gives an algorithm for finding an *approximate fixed point* of a continuous function $f : C \to C$, since a condition for halting such as

$$\|x_{m+1} - f(x_{m+1})\| < \varepsilon$$

will eventually be satisfied.

Complexity of 2-Nash

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- In many senses 2-Nash "looks like" linear programming, and the Lemke-Howson algorithm closely resembles the simplex method.
- Like the simplex method, Lemke-Howson works well in practice, solving games with dozens or hundreds of strategies.

However, Chen and Deng (2005) showed that 2-Nash is complete for **PPAD**.

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- Hirsch, Papadimitriou, and Vavasis (1989) show that algorithms for computing an approximate fixed point that obtain all their information about the function from an “oracle” that evaluates it have exponential worst case running times.
- In view of this, it is exceedingly unlikely that a polynomial time algorithm for 2-Nash exists.

Advanced Fixed Point Theory

2. The Smooth Degree and Index

Degree and Index: Introduction

We now study “the same” mathematical structure from three different points of view.

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We study these first in the best behaved setting.

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- An *orientation* of X is a connected component of $F^m(V)$.

Proposition: There are exactly two orientations of X.

Proof

For any $(\mathbf{e}_1, \ldots, \mathbf{e}_m) \in F^m(X)$ the map

$$A \mapsto (\sum_j a_{1j}\mathbf{e}_j, \ldots, \sum_j a_{mj}\mathbf{e}_j)$$

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is a linear bijection between the set of $m \times m$ matrices $A = (a_{ij})$ and X^m.

- There are at least two orientations because there cannot be a continuous path in $F^m(X)$ between elements of $F^m(X)$ whose corresponding matrices have determinants of opposite signs.

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- Continuing from this point, such paths can be combined to eliminate all off diagonal coefficients, arriving at an ordered basis of the form $(c_1\mathbf{e}_1, \ldots, c_m\mathbf{e}_m)$.

- From here we can continuously rescale the coefficients, arriving at $(d_1\mathbf{e}_1, \ldots, d_m\mathbf{e}_m)$ with $d_i = \pm 1$ for all i.

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$$\theta \mapsto (\ldots, \cos\theta v_i + \sin\theta v_{i+1}, \cos\theta v_{i+1} - \sin\theta v_i, \ldots)$$

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  that goes to $(v_1, \ldots, -v_i, -v_{i+1}, \ldots, v_m)$.
- Thus there is a path in $F^m(X)$ from any m-frame to either $(\mathbf{e}_1, \ldots, \mathbf{e}_m)$ or $(-\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m)$.

Regular Points and Values

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- $y \in Y$ is a *critical value* of f if $f^{-1}(y)$ contains a critical value. Otherwise y is a *regular value* of f.

Oriented Vector Spaces

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Now let X and Y be oriented m-dimensional vector spaces.

- If $S \subset X$, $f : S \to Y$ is C^1, and x is a regular point of f, then f is *orientation preserving* (*orientation reversing*) at x if $Df(x)$ maps positively oriented m-frames of X to positively (negatively) oriented m-frames of Y.

Smooth Degree

For $C \subset X$ let $\partial C = C \cap \overline{X \setminus C}$. A C^∞ $f : C \to Y$ is *smoothly degree admissible* over $y \in Y$ if:

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If f is smoothly degree admissible over y, then the *degree* $\deg_y(f)$ of f over y is the number of $x \in f^{-1}(y)$ at which f is orientation preserving minus the number of $x \in f^{-1}(y)$ at which f is orientation reversing.

Properties of Smooth Degree

We will work axiomatically, specifying properties of the smooth degree that characterize it.

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- (Normalization) If f is smoothly degree admissible over y and $f^{-1}(y)$ is a single point at which f is orientation preserving, then $\deg_y(f) = 1$.
- (Additivity) If $f : C \to Y$ is smoothly degree admissible over y, $C_1, \ldots, C_r$ are pairwise disjoint compact subsets of C, $f^{-1}(y) \subset \bigcup_i C_i$, and $f^{-1}(y) \cap \partial C_i = \emptyset$ for all i, then

$$\deg_y(f) = \sum_i \deg_y(f|_{C_i}).$$

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Theorem: *The smooth degree is the unique integer valued function on the set of smoothly degree admissible functions over y that satisfies Normalization, Additivity, and Homotopy.*

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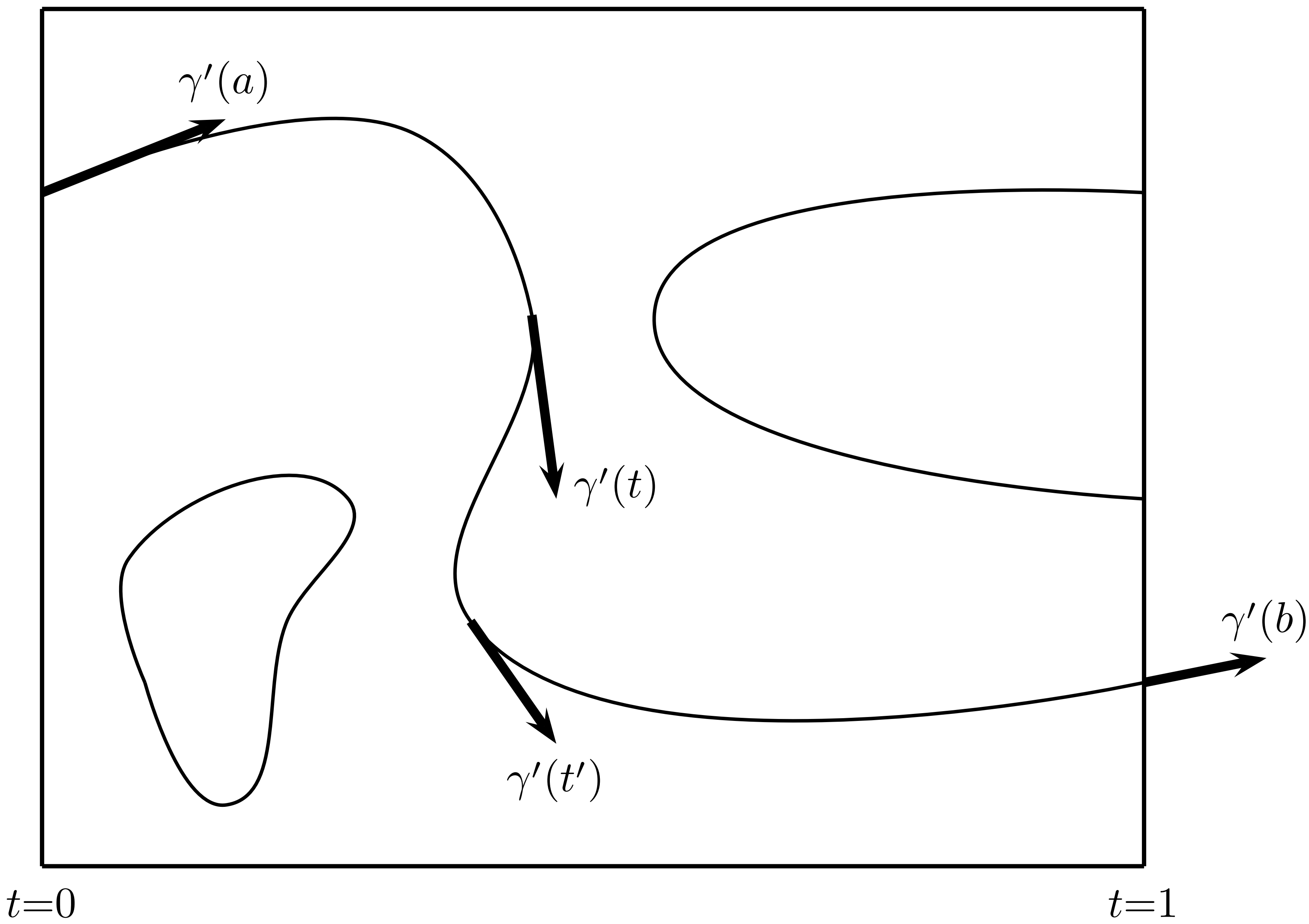
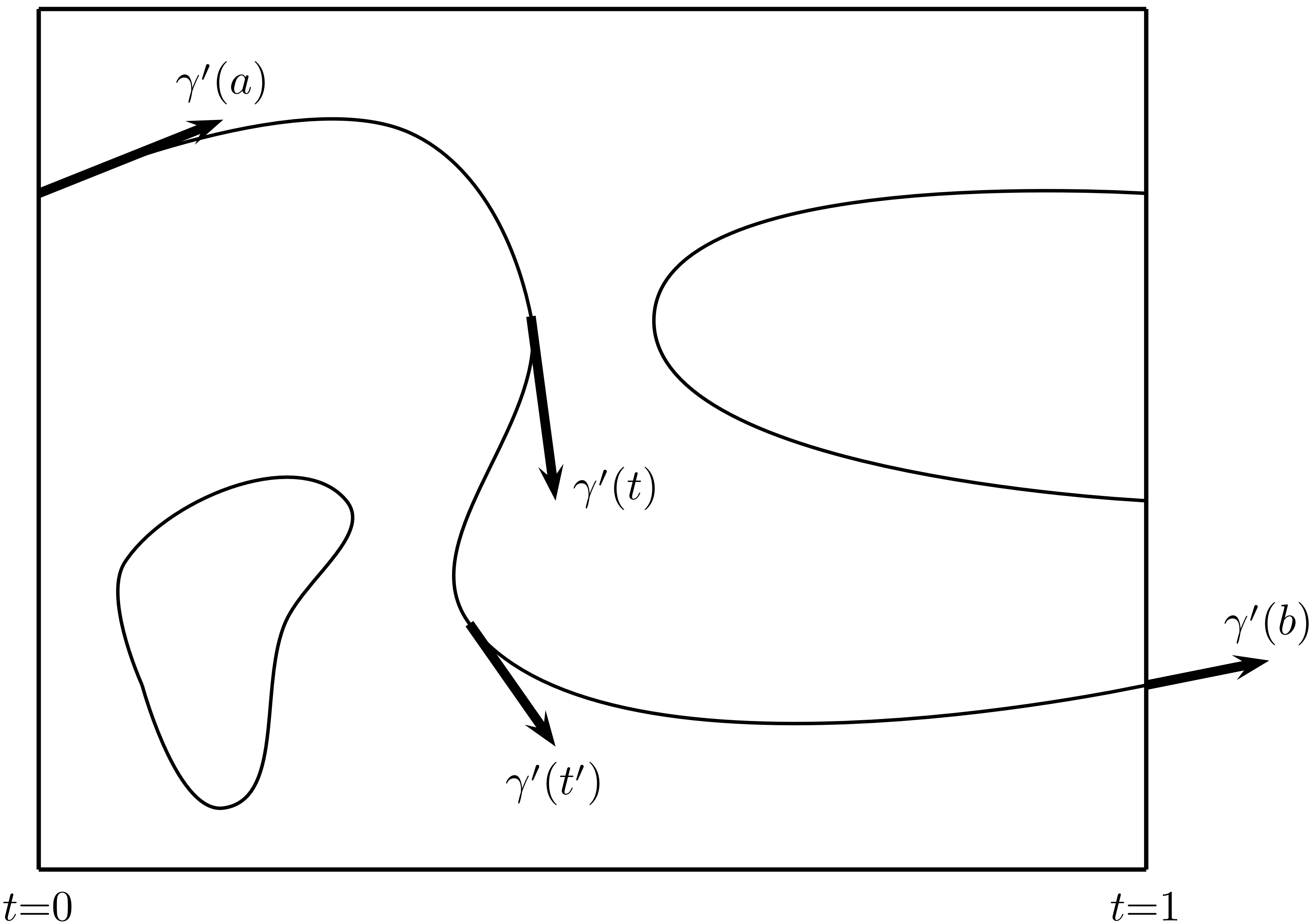
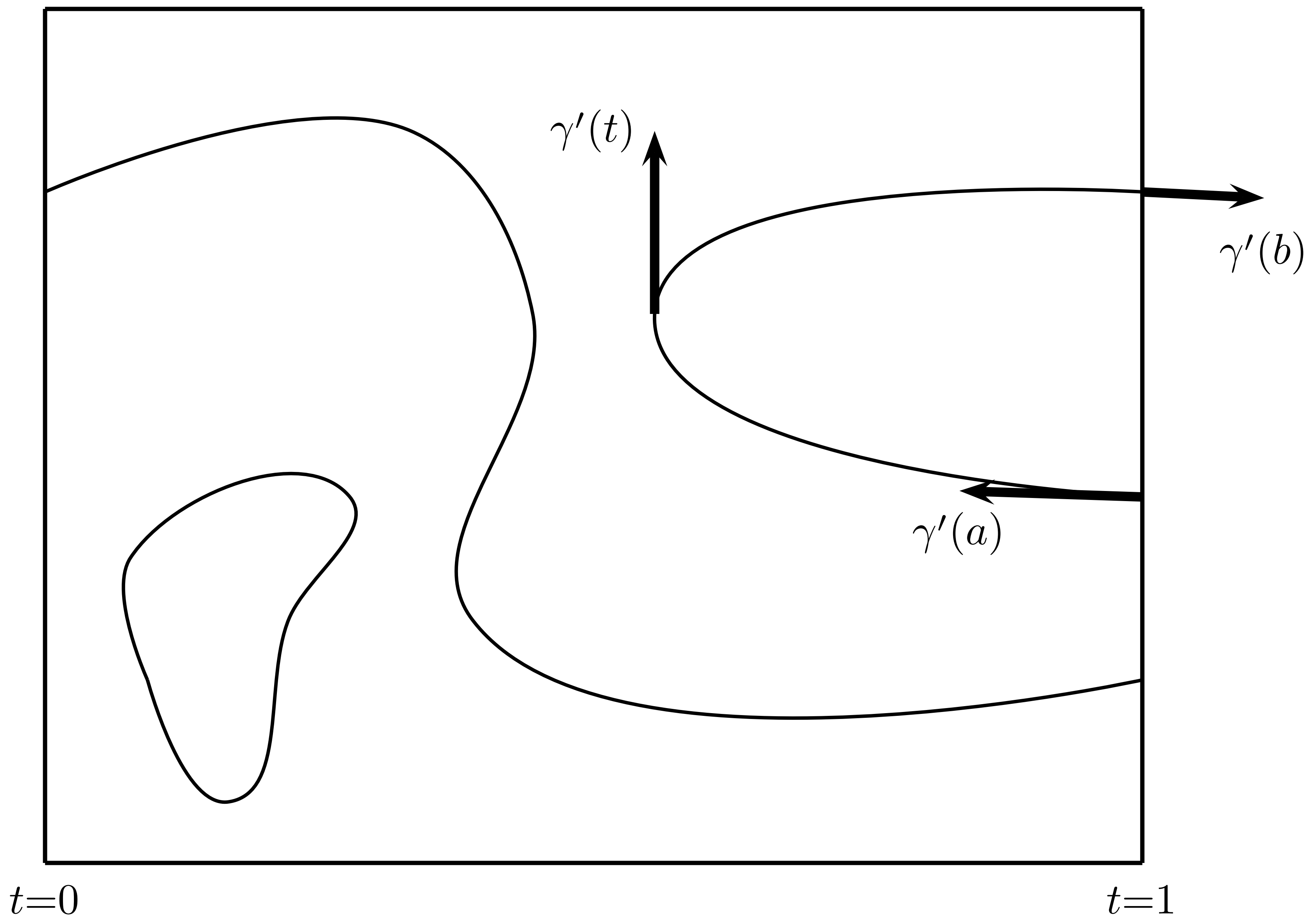
- The implicit function theorem (and a bit of sweating the details) implies that $h^{-1}(y)$ is a finite union of smooth loops and line segments.
- Suppose that $\gamma : [a,b] \to C \times [0,1]$ is a smooth parameterization of the one of the line segments, with $\gamma'(t) \neq 0$ for all t.



$\gamma'(a)$
$\gamma'(t)$
$\gamma'(b)$
$\gamma'(t')$
$t=0$
$t=1$



- $\gamma'_{m+1}(a) \neq 0 \neq \gamma'_{m+1}(b)$ because y is a regular value of h_0 and h_1.



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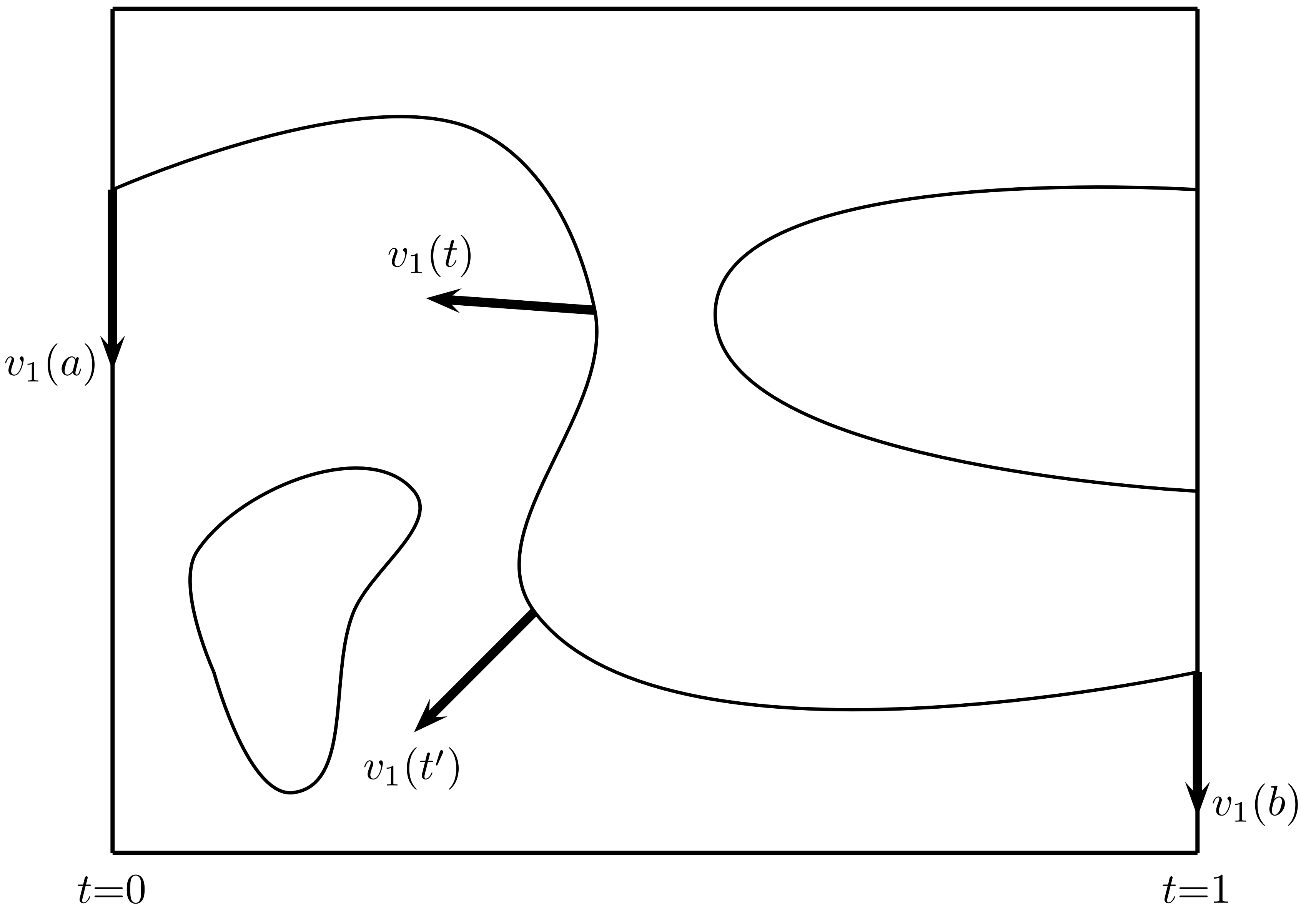
- Proving that such $v_1, \ldots, v_m$ exist is a rather tedious construction.



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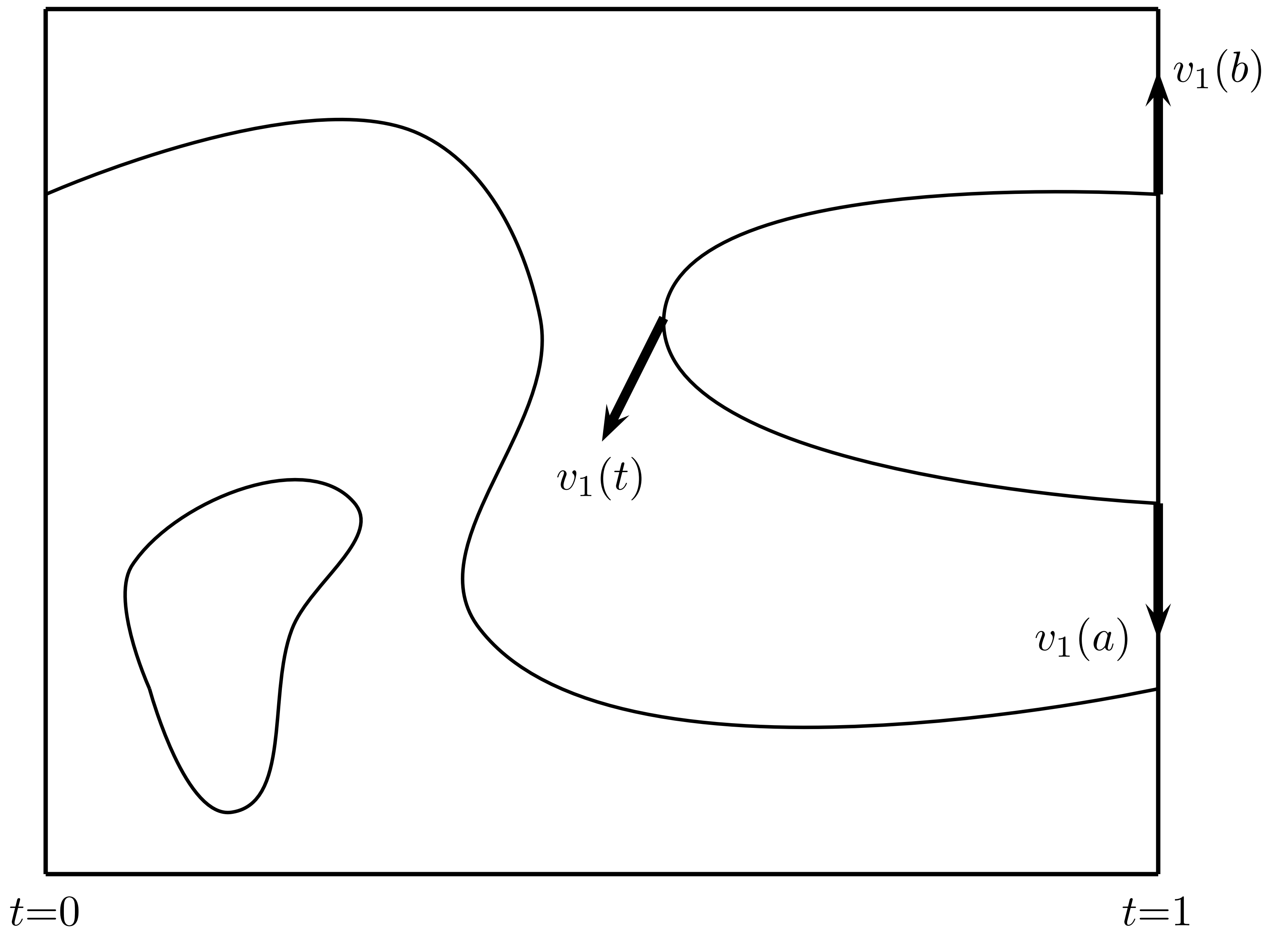
$v_1(b)$
$v_1(t)$
$v_1(a)$
$t=0$
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- If $\gamma_{m+1}(a) = 0$ and $\gamma_{m+1}(b) = 1$, then $(v_1(a), \ldots, v_m(a))$ and $(v_1(b), \ldots, v_m(b))$ have the same orientation.

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$$(Dh(\gamma(t))v_1(t), \ldots, Dh(\gamma(t))v_m(t)) \in F^m(Y).$$

  - By continuity the orientation of this m-frame is the same for all t.
- If $\gamma_{m+1}(a) = 0$ and $\gamma_{m+1}(b) = 1$, then the contribution of $\gamma(a)$ to $\deg_y(h_0)$ is the same as the contribution of $\gamma(b)$ to $\deg_y(h_1)$.
- If $\gamma_{m+1}(a) = \gamma_{m+1}(b) = 0$, then the total contribution of $\gamma(a)$ and $\gamma(b)$ to $\deg_y(h_0)$ is zero.
- Summing over all the components of $h^{-1}(y)$ that are paths gives $\deg_y(h_0) = \deg_y(h_1)$.

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- By Additivity it suffices to show that if $f^{-1}(y) = \{x\}$, then $\deg_y(f)$ is $+1$ or -1 according to whether $Df(x)$ is orientation preserving or reversing.
- If f is orientation preserving at x, this is Normalization.
- If f is orientation reversing at x, one can construct a homotopy between a function that has no preimages of y (and thus has degree zero by Additivity) and a function that agrees with f in a neighborhood of x and has another preimage of y at which the function is orientation preserving.

Other Properties of Degree

The determinant of a block diagonal matrix is the product of the determinants of the blocks, so:

Proposition: If X and Y are m-dimensional vector spaces, X' and Y' are m'-dimensional vector spaces, $C \subset X$ and $C' \subset X'$ are compact, and $f : C \to Y$ and $f' : C' \to Y'$ are smoothly degree admissable over y and y' respectively, then

$$f \times f' : C \times C' \to Y \times Y', \quad (x, x') \mapsto (f(x), f'(x')),$$

is smoothly degree admissible over (y, y'), and

$$\deg_{(y,y')}(f \times f') = \deg_y(f) \cdot \deg_{y'}(f').$$

The chain rule gives:

Proposition: If X, Y, and Z are m-dimensional vector spaces, $C \subset X$ and $D \subset Y$ are compact, $f : C \to D$ and $g : D \to Z$ are C^∞, g is smoothly degree admissable over z, $g^{-1}(z) = \{y_1, \ldots, y_r\}$, and f is smoothly degree admissible over each y_i, then $g \circ f$ is smoothly admissible over z. If, in addition, $D_1, \ldots, D_r$ are pairwise disjoint compact subsets of D such that for each i, y_i is contained in the interior of D_i, then

$$\deg_z(g \circ f) = \sum_i \deg_{y_i}(f) \cdot \deg_z(g|_{D_i}).$$

Smooth Fixed Point Index

For $C \subset X$ and $f : C \to X$ let

$$\mathcal{F}(f) = \{\, x \in C : f(x) = x \,\}$$

be the set of fixed points of f. A C^∞ function $f : C \to X$ is *smoothly index admissible* if:

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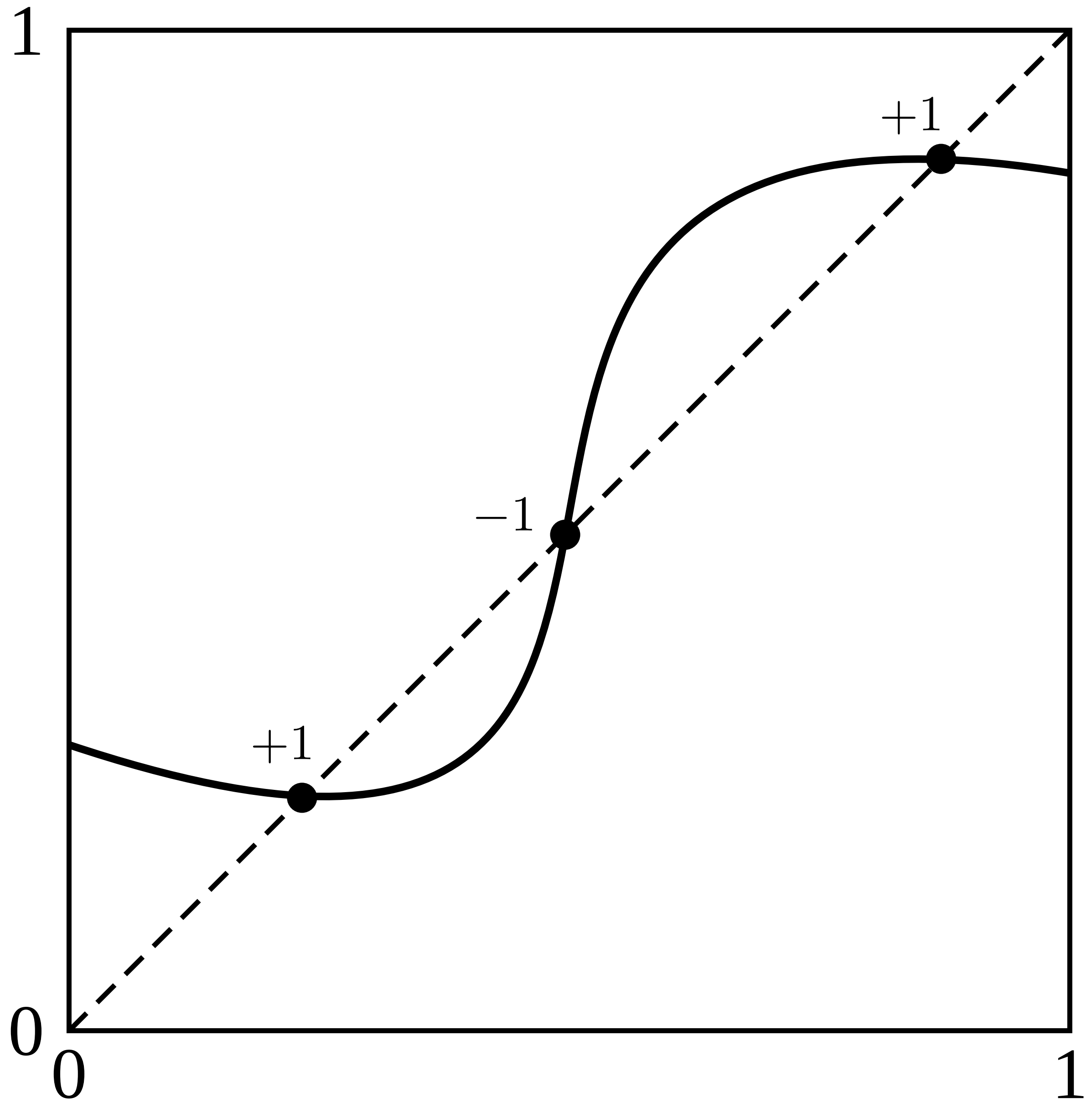
If f is smoothly index admissible, then the *fixed point index* of f is

$$\Lambda_X(f) = \deg_0(\mathrm{Id}_C - f).$$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The axiomatic characterization of the smooth fixed point index is a translation of the characterization of the smooth degree.

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The axiomatic characterization of the smooth fixed point index is a translation of the characterization of the smooth degree.

- (Normalization) If $C \subset X$ is compact, $x \in C \setminus \partial C$, and $c : C \to X$ is the constant function with value x, then $\Lambda_X(c) = 1$.
- (Additivity) If $f : C \to X$ is smoothly index admissible, $C_1, \ldots, C_r$ are pairwise disjoint compact subsets of C, $\mathcal{F}(f) \subset \bigcup_i C_i$, and $\mathcal{F}(f) \cap \partial C_i = \emptyset$ for all i, then

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(Homotopy) If $h : C \times [0,1] \to X$ is a smoothly admissible homotopy, then

$$\Lambda_X(h_0) = \Lambda_X(h_1).$$

Theorem: *The smooth index is the unique integer valued function on the set of smoothly index admissible functions that satisfies Normalization, Additivity, and Homotopy.*

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- This is an immediate consequence of the axiomatic characterization of the smooth degree.

Other Properties of the Index

The analogous property of the degree implies that:

Proposition: If X and X' are finite dimensional vector spaces, $C \subset X$ and $C' \subset X'$ are compact, and $f : C \to Y$ and $f' : C' \to Y'$ are smoothly index admissable, then

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is smoothly index admissible, and

$$\Lambda_X(f \times f') = \Lambda_X(f) \cdot \Lambda_{X'}(f').$$

Commutativity

Proposition: If X and Y are finite dimensional vector spaces, $C \subset X$ and $D \subset Y$ are compact, $f : C \to D$ and $g : D \to Z$ are C^∞, and $g \circ f$ and $f \circ g$ are smoothly index admissible, then

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- This is *not* a consequence of the earlier result concerning the degree of a composition.
- In fact the underlying linear algebra result was not known prior to the 1948 Ph.D. thesis of Felix Browder.
- This property of the index is the key to extending the index to very general spaces.

Linear Algebra

Proposition: (Jacobson (1953) pp. 103–106) *S*uppose $K : V \to W$ and $L : W \to V$ are linear transformations, where V and W are vector spaces of dimensions m and n respectively over an arbitrary field. Suppose $m \leq n$. Then the characteristic polynomials κ_{KL} and κ_{LK} of KL and LK are related by the equation $\kappa_{KL}(\lambda) = \lambda^{n-m}\kappa_{LK}(\lambda)$. In particular,

$$\kappa_{LK}(1) = |\mathrm{Id}_V - LK| = |\mathrm{Id}_W - KL| = \kappa_{KL}(1).$$

*P*roof. We can decompose V and W as direct sums $V = V_1 \oplus V_2 \oplus V_3 \oplus V_4$, $W = W_1 \oplus W_2 \oplus W_3 \oplus W_4$ where

$$V_1 = \ker K \cap \mathrm{im} L, \quad V_1 \oplus V_2 = \mathrm{im} L, \quad V_1 \oplus V_3 = \ker K,$$

and similarly for W. With suitably chosen bases the matrices of K and L have the forms

$$\begin{bmatrix} 0 & K_{12} & 0 & K_{14} \\ 0 & K_{22} & 0 & K_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & L_{12} & 0 & L_{14} \\ 0 & L_{22} & 0 & L_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Computing the product of these matrices, we find that

$$\kappa_{KL}(\lambda) = \begin{vmatrix} \lambda I & -K_{12}L_{22} & 0 & -K_{12}L_{24} \\ 0 & \lambda I - K_{22}L_{22} & 0 & -K_{22}L_{24} \\ 0 & 0 & \lambda I & 0 \\ 0 & 0 & 0 & \lambda I \end{vmatrix}$$

Using elementary facts about determinants, this reduces to $\kappa_{KL}(\lambda) = \lambda^{n-k}|\lambda I - K_{22}L_{22}|$, where $k = \dim V_2 = \dim W_2$.

In effect this reduces the proof to the special case $V_2 = V$ and $W_2 = W$, i.e. K and L are isomorphisms. But this case follows from the computation

$$|\lambda \mathrm{Id}_V - LK| = |L^{-1}| \cdot |\lambda \mathrm{Id}_V - LK| \cdot |L|$$

$$= |L^{-1}(\lambda \mathrm{Id}_V - LK)L| = |\lambda \mathrm{Id}_W - KL|.$$

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A *vector field* on $C \subset X$ is a continuous function $z : C \to X$. An *equilibrium* of z is a point $x \in C$ such that $z(x) = 0$. Let $\mathcal{E}(z) = \{\, x \in C : z(x) = 0 \,\}$ be the set of equilibria of z.

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If z is smoothly vector field admissible, then the *vector field index* $\mathrm{vfind}(z)$ of z is $\deg_0(z)$. The interpretation (a dynamical system) is different, but mathematically this is just a special case of the degree, so it has the same properties.

Advanced Fixed Point Theory

3. Using Sard's Theorem to Extend the Degree and Index

Continuous Degree and Index

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- Study the approximation of continuous functions by C^∞ functions.
- State (but not prove) Sard's theorem.

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  - The continuity of addition implies that the topology is characterized by the neighborhoods of the origin.
- A TVS is *locally convex* if every neighborhood of the origin contains a convex neighborhood of the origin.

Normed Spaces

A *norm* on a vector space V is a function $\| \cdot \| : V \to [0, \infty)$ such that for all $x, y \in V$ and $\alpha \in \mathbb{R}$:

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A *normed space* is vector space endowed with a norm. It is automatically endowed with the topology derived from the associated metric $(x, y) \mapsto \|x - y\|$, and is thus a locally convex TVS.

Banach Spaces

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- *Exercise:* Prove that any metric space (X, d) can be isometrically embedded in a complete metric space $(\overline{X}, \overline{d})$ such that $\overline{X}$ is the closure of X, and $(\overline{X}, \overline{d})$ is unique up to isometry.

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- *Exercise:* Prove that if W is a Banach space, then $C(K, W)$ is a Banach space.

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- $\alpha(t) \cdot \alpha(1-t)$ is a “bump” function with compact support. Multidimensional bump functions can be added together, etc.

Sard's Theorem

For $\alpha > 0$, a set $S \subset \mathbb{R}^n$ *has* α*-dimensional Hausdorff measure zero* if, for any $\varepsilon > 0$, there is a sequence $\{(x_j, \delta_j)\}_{j=1}^{\infty}$ such that

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Theorem: Let $U \subset \mathbb{R}^m$ be open, and let $f : U \to \mathbb{R}^n$ be a C^r function. For $0 \leq p < m$ let R_p be the set of points $x \in M$ such that the rank of $Df(x)$ is less than or equal to p. Then $f(R_p)$ has α-dimensional measure zero for all $\alpha \geq p + \frac{m-p}{r}$.

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- See Milnor's *Topology from the Differentiable Viewpoint*. The special case $m = n$ is proved on p. 85 of Spivak's *Calculus on Manifolds*.

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In the argument below the *degree* of f over y will be defined to be $\deg_y(f')$ for $f' : C \to Y$ sufficiently close to f that are smoothly degree admissible over y.

Properties of Degree

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- (Additivity) If $f : C \to Y$ is degree admissible over y, $C_1, \ldots, C_r$ are pairwise disjoint compact subsets of C, $f^{-1}(y) \subset \bigcup_i C_i$, and $f^{-1}(y) \cap \partial C_i = \emptyset$ for all i, then

$$\deg_y(f) = \sum_i \deg_y(f|_{C_i}).$$

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  - *Exercise:* If $h : C \times [0, 1] \to Y$ is a homotopy, then $t \mapsto h_t$ is a continuous path in $C(C, Y)$, and conversely.

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  - *Exercise:* If $h : C \times [0, 1] \to Y$ is a homotopy, then $t \mapsto h_t$ is a continuous path in $C(C, Y)$, and conversely.
  - Thus Continuity implies Homotopy.

Multiplication

(Multiplication) If X and Y are m-dimensional vector spaces, X' and Y' are m'-dimensional vector spaces, $C \subset X$ and $C' \subset X'$ are compact, and $f : C \to Y$ and $f' : C' \to Y'$ are degree admissable over y and y' respectively, then $f \times f' : C \times C' \to Y \times Y'$ is degree admissible over (y, y'), and

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Theorem: *T*here is a unique integer valued function on the set of degree admissible functions over y, called the *degree*, that satisfies Normalization, Additivity, and Continuity. It also satisfies Multiplication.

Proof

Since Continuity implies Homotopy, the restriction of any function satisfying our conditions to the functions that are smoothly degree admissible over y must agree with the smooth degree.

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- Sard's theorem implies that the set of regular values of such an f' are dense, and if y' is a regular value of f', then y is a regular value of $x \mapsto f'(x) + y - y'$.
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- In view of the uniqueness of the smooth degree, this establishes the uniqueness claim.
- This definition of the degree of f over y is unambiguous: if f' and f'' are smoothly degree admissible over y, then

$$(x,t) \mapsto (1-t)f'(x) + tf''(x)$$

is a C^∞ homotopy, which is smoothly degree admissible over y if f' and f'' are sufficiently close to f.

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- Normalization is satisfied automatically.
- The construction implies that Continuity holds.
- Additivity and Multiplication follow easily from the fact that these conditions are satisfied by the smooth degree.

The Fixed Point Index

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If f is index admissible, then the *fixed point index* of f is

$$\Lambda_X(f) = \deg_0(\mathrm{Id}_C - f).$$

Properties of the Index

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- (Commutativity) If X and Y are finite dimensional vector spaces, $C \subset X$ and $D \subset Y$ are compact, $f : C \to D$ and $g : D \to C$ are C^∞, and $g \circ f$ and $f \circ g$ are index admissible, then

$$\Lambda_X(g \circ f) = \Lambda_Y(f \circ g).$$

- (Multiplication) If X and X' are finite dimensional vector spaces, $C \subset X$ and $C' \subset X'$ are compact, and $f : C \to Y$ and $f' : C' \to Y'$ are index admissable, then

$$f \times f' : C \times C' \to X \times X', \quad (x, x') \mapsto (f(x), f'(x')),$$

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Theorem: *The index is the unique integer valued function on the set of index admissible functions that satisfies Normalization, Additivity, and Continuity. It also satisfies Commutativity and Multiplication.*

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A vector field $z : C \to X$ is *vector field admissible* if:

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- As before, the main properties of the vector field index are inherited from the properties of the degree.

Advanced Fixed Point Theory

4. Absolute Neighborhood Retracts

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- Kinoshita's example shows that the answer is no!

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- Fix a number $\varepsilon \in (0, 2\pi)$.

We work with polar coordinates, identifying $(r, \theta) \in \mathbb{R}_+ \times \mathbb{R}$ with $(r \cos \theta, r \sin \theta)$.

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  - $S \times [0, 1]$ = "roll of toilet paper."

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- $f_1(1, \theta, z) \neq (1, \theta, z)$ ($\kappa(z) = z$ implies that $z = 0, 1$, and ε is not a multiple of 2π).

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- It stretches $\{\, (\rho(t), \theta) : \varepsilon \leq t, \theta \in \mathbb{R} \,\}$ onto all of D while rotating it by ε.
- f_2 does not have any fixed points (the first components (that is, the radius) of $f_2(\rho(t), \theta, 0)$ is strictly less than $\rho(t)$ except when $t = 0$).

Define $f_3 : S \times [0,1] \to X$ by setting

$$f_3(s(\tau), z) := (s((\tau + \varepsilon)z), 1 - (1 - \kappa(z))\tau/\varepsilon)$$

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- The case $\tau = \varepsilon$: $f_3(s(\varepsilon), z) = (s(2\varepsilon z), \kappa(z))$.
- The equation $z = 1 - (1 - \kappa(z))\tau/\varepsilon$ is equivalent to $(1 - \kappa(z))\tau = (1 - z)\varepsilon$, which is impossible when $0 < z < 1$ because $\kappa(z) > z$.

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- There cannot be any fixed point with $0 < z < 1$ because in this case $z < \kappa(z)$.

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- *Exercise:* Prove that if $\{(s(\tau_n), z_n)\}$ is a sequence in $S \times [0, 1]$ converging to $(1, \theta, z) \in C \times \{0\}$, then $f_3(s(\tau_n), z_n) \to f_1(1, \theta, z)$.

Retracts

We need some additional restriction beyond contractibility to get fixed point theory to work. The dominant response in the literature is to require that the space also be an absolute neighborhood retract (ANR). We develop the parts of the theory that clarify what sorts of spaces these are, and how fixed point theory is extended to them.

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- Let X be a metric space, and let A be a subset.
  - A *retraction* of X onto A is a continuous function $r : X \to A$ with $r(a) = a$ for all $a \in A$.
  - We say that A is a *retract* of X if such a retraction exists.

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- To give sufficient conditions for a space to be an ANR that imply that lots of spaces are ANR's.
- To explain the main result used to extend the theory of the fixed point index to ANR's.

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A brief proof is given by Rudin, ME (1969) Proc. Am. Math. Soc. 20:603.

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Theorem: *I*f X is normal and $\{U_\alpha\}_{\alpha \in I}$ is a locally finite open cover, then there is a partition of unity subordinate to $\{U_\alpha\}$.

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Urysohn's Lemma: *If* X is normal and $C \subset U \subset X$ with C closed and U open, then there is a continuous $\varphi : X \to [0, 1]$ with $\varphi(x) = 0$ for all $x \in X \setminus U$ and $\varphi(x) = 1$ for all $x \in C$.

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- For a metric X set $\varphi(x) = \frac{d(x,X\setminus U)}{d(x,C)+d(x,X\setminus U)}$.

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Dugundji's Theorem

The following generalizes Tietze' extension theorem for metric spaces.

Theorem: If A is a closed subset of a metric space (X, d), Y is a locally convex topological vector space, and $f : A \to Y$ is continuous, then there is a continuous extension $\overline{f} : X \to Y$ whose image is contained in the convex hull of $f(A)$.

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To prove this:

- Since $X \setminus A$ is metric, hence paracompact, the open cover $\{\mathrm{B}_{d(x,A)/2}(x)\}_{x \in X \setminus I}$ has a locally finite refinement $\{U_\alpha\}_{\alpha \in I}$.

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Exercise: Prove that $\overline{f}$ is continuous.

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- Then $V := j^{-1}(U)$ is a neighborhood of $h(A)$, and $h \circ r \circ j|_V : V \to h(A)$ is a retraction.

Examples

- *Exercise:* Prove that any finite simplicial complex $\mathcal{T}$ is a neighborhood retract of an open subset of a Euclidean space. (Hint: $\mathcal{T}$ is a subcomplex of the set Δ of probability measures on its set of vertices.)

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- The tubular neighborhood theorem implies that a C^2 submanifold of a Euclidean space is an ANR.

A Necessary Condition

Proposition: (Kuratowski, Wojdyslawski) A metric space X has an embedding $\iota : X \to C(X)$ such that $\iota(X)$ is a relatively closed subset of a convex set, and is a closed subset of $C(X)$ if X is complete.

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- *Exercise:* For each $x \in X$ let $f_x \in C(X)$ be the function $f_x(y) := \min\{1, d(x, y)\}$. Prove that $\iota : x \mapsto f_x$ is satisfactory.

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- *Proof.* Let $h : A \to Z$ (where Z is a Banach space) such that h maps A homeomorphically onto $h(A)$ and $h(A)$ is closed in the relative topology of its convex hull C. Since A is an ANR, there is a relatively open $U \subset C$ and a retraction $r : U \to h(A)$.

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- A topological space D ε-*dominates* $C \subset X$ if there are continuous functions $\varphi : C \to D$ and $\psi : D \to X$ such that $\psi \circ \varphi : C \to X$ is ε-homotopic to Id_C.

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- One can easily show that if this is true for the metric d, then it is true for any equivalent metric. Therefore we may assume that:

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  - X is a closed subset of convex subset C of a Banach space.
  - There is a retraction $r : U \to X$ where $U \subset C$ is relatively open.

- For each $x \in X$ choose $\rho_x > 0$ such that

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- Notation: for each $V = \mathbf{B}_{\rho_{x_i}}(x_i) \in \mathcal{U}$ let $x_V = x_i$, and let $\mathbf{e}_V$ be the corresponding standard unit basis vector of $\mathbb{R}^{\mathcal{U}}$.

- The *nerve* of $\mathcal{U}$ is the simplicial complex

$$\mathcal{N}_{\mathcal{U}} = (\mathcal{U}, \Sigma_{\mathcal{U}})$$

where the elements of $\Sigma_{\mathcal{U}}$ are $\emptyset$ and the simplices

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  - Let $\{\alpha_V : X \to [0,1]\}_{V \in \mathcal{U}}$ be a partition of unity of C subordinate to $\mathcal{U}$.

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$$\eta(x,t) = r\Big((1-t)\sum_{V} \alpha_V(x) x_V + tx\Big).$$

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  - Let $g_A(X, Y)$ be the set of such germs.
  - For f as above let $\gamma_A(f)$ be its equivalence class.
  - We say that f is a *representative* of $\gamma_A(f)$.

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  - Actually, we only need f and g to be defined on neighborhoods of $\mathcal{F}(g \circ f)$ and $\mathcal{F}(f \circ g)$.

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- For $\gamma \in g_A(X,Y)$ and $\gamma' \in g_{A'}(X',Y')$ we define $\gamma \times \gamma' \in g_{A \times A''}(X \times X', Y \times Y')$ to be the germ of $f \times f'$, $(x,x') \mapsto (f(x), f'(x'))$ where f and f' are representatives of γ and γ'.

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- Let $\pi_X : \mathcal{I}_X \to \mathcal{G}_X$ be the function $f \mapsto \gamma_{\mathcal{F}(f)}(f)$.

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$$\Lambda_X(\gamma_{\{x\}}(c_x)) = 1.$$
- (Additivity) If $X \in \mathcal{ANR}$, $\gamma \in \mathcal{G}_X$ has domain A, and $A_1, \ldots, A_r$ are pairwise disjoint compact subsets of A with $\mathcal{F}(\gamma) \subset A_1 \cup \ldots \cup A_r$, then
$$\Lambda_X(\gamma) = \sum_i \Lambda_X(\gamma|_{A_i}) \; .$$

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In addition these functions satisfy:

- (Multiplication) If $X, X' \in \mathcal{ANR}$, $\gamma \in \mathcal{G}_X$, and $\gamma' \in \mathcal{G}_{X'}$, then

$$\Lambda_{X\times X'}(\gamma \times \gamma') = \Lambda_X(\gamma) \cdot \Lambda_{X'}(\gamma').$$

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- For "sufficiently small" $\varepsilon > 0$ let $\mathcal{T}$ be a finite simplicial complex that ε-dominates C by virtue of the maps $\varphi : C \to |\mathcal{T}|$ and $\psi : |\mathcal{T}| \to X$ and the ε-homotopy $\eta : C \times [0, 1] \to X$ with $\eta_0 = \mathrm{Id}_C$ and $\eta_1 = \psi \circ \varphi$.

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- Embed $|\mathcal{T}|$ in some $\mathbb{R}^m$. Since $|\mathcal{T}|$ is an ANR there is a retraction $r : \overline{U} \to |\mathcal{T}|$ where U is a neighborhood of $|\mathcal{T}|$ whose closure is compact.

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- We are motivated by the calculation

$$\Lambda_{\mathbb{R}^m}(\varphi \circ f \circ \psi \circ r|_{r^{-1}(\psi^{-1}(B))}) = \Lambda_X(\psi \circ r \circ \varphi \circ f|_B)$$

$$= \Lambda_X(\psi \circ \varphi \circ f|_B) = \Lambda_X(\eta_1 \circ f|_B)$$

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    - We omit these numerous and lengthy verifications.

Advanced Fixed Point Theory

5. Approximation of Correspondences by Functions

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Theorem: Suppose that X and Y are ANR's, X is separable, and C and D are compact subsets of X with $C \subset \operatorname{int} D$. Let $F : D \to Y$ be an upper hemicontinuous contractible valued correspondence. Then for any neighborhood W of $\operatorname{Gr}(F|_C)$ there are:

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(a) a continuous $f : C \to Y$ with $\operatorname{Gr}(f) \subset W$;

(b) a neighborhood W' of $\operatorname{Gr}(F)$ such that, for any two continuous functions $f_0, f_1 : D \to Y$ with $\operatorname{Gr}(f_0), \operatorname{Gr}(f_1) \subset W'$, there is a homotopy $h : C \times [0,1] \to Y$ with $h_0 = f_0|_C$, $h_1 = f_1|_C$, and $\operatorname{Gr}(h_t) \subset W$ for all $0 \leq t \leq 1$.

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Notation: If C and Y are metric spaces with C compoact, then $\mathcal{U}(C, Y)$ is the space of upper hemicontinuous correspondences $F : C \to Y$, with the topology generated by the base of open sets $U_W := \{ F : \mathrm{Gr}(F) \subset W \}$ where $W \subset C \times Y$ is open.

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- Upper hemicontinuous correspondences $F : U \to Y$, $F' : U' \to Y$ *have the same germ at* A if $F|_V = F'|_V$ for some neighborhood $V \subset U \cap U'$ of A.
  - The equivalence classes of this equivalence relation are *germs* of upper hemicontinuous correspondences from X to Y at A.
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  - We say that F is a *representative* of $\Gamma_A(F)$.

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- For $\Gamma \in G_A(X, Y)$ and $\Gamma' \in G_{A'}(X', Y')$ we define $\Gamma \times \Gamma' \in G_{A \times A''}(X \times X', Y \times Y')$ to be the germ of $F \times F'$, $(x, x') \mapsto F(x) \times F'(x')$ where F and F' are representatives of Γ and Γ'.

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  - Let $\mathcal{C}_X$ be the set of such germs.
- For $F \in \mathcal{I}_X$ let $\pi_X(F) := \Gamma_{\mathcal{F}(F)}(F)$.

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- (Additivity) If $X \in \mathcal{ANR}$, $\Gamma \in \mathcal{C}_X$ has domain A, and $A_1, \ldots, A_r$ are pairwise disjoint compact subsets of A with $\mathcal{F}(\Gamma) \subset A_1 \cup \ldots \cup A_r$, then
$$\Lambda_X(\Gamma) = \sum_i \Lambda_X(\Gamma|_{A_i}) \ .$$

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In addition these functions satisfy:

- (Multiplication) If $X, X' \in \mathcal{ANR}$, $\Gamma \in \mathcal{C}_X$, and $\Gamma' \in \mathcal{C}_{X'}$, then

$$\Lambda_{X\times X'}(\Gamma \times \Gamma') = \Lambda_X(\Gamma) \cdot \Lambda_{X'}(\Gamma').$$

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- The Theorem gives a neighborhood $V \subset W$ of $\mathrm{Gr}(G)$ such that for any maps $f_0, f_1 : C \to X$ with $\mathrm{Gr}(f_0), \mathrm{Gr}(f_1) \subset V$ there is a homotopy $h : C \to [0,1] \to X$ with $h_0 = f_0$, $h_1 = f_1$, and $\mathrm{Gr}(h_t) \subset W$ for all t.

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- The Theorem also gives a continuous $f : C \to X$ with $\mathrm{Gr}(f') \subset V$, and we set $\Lambda_X(\Gamma) := \Lambda_X(f)$.

Advanced Fixed Point Theory

6. Sequential Equilibria as Fixed Points

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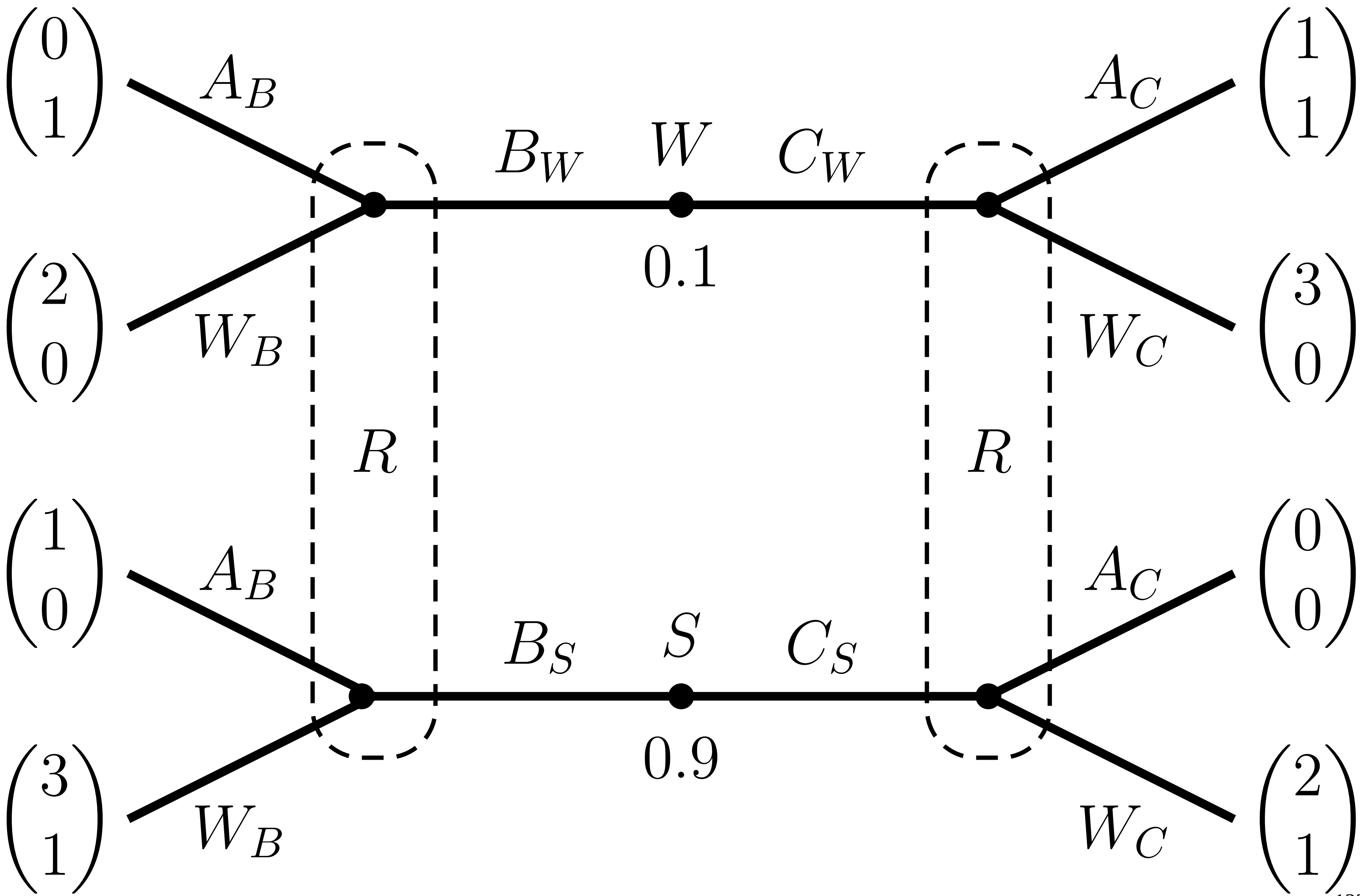
- Sequential equilibria is defined.
- A well behaved (uhccv) correspondence is introduced that (roughly speaking) has the set of sequential equilibria as its fixed points.
    - By perturbing this correspondence, or defining subcorrespondences, we may define refinements of the sequential equilibrium concept.

Signalling Games

The Beer-Quiche game from Cho and Kreps (1987):



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Notation: Let:

$$W = \{\, w \in T : P(w) = \emptyset \,\}, \quad X = T \backslash Z, \quad Y = T \backslash W,$$

$$\text{and } \; Z = \{\, z \in T : P^{-1}(z) = \emptyset \,\}.$$

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  - We require that for each h and $x \in H$, $c(x, \cdot) : A_h \to Y$ maps A_h bijectively onto $p^{-1}(x)$.
    - Define $\alpha : Y \to \bigcup_h A_h$ implicitly by requiring that $c(p(y), \alpha(y)) = y$.

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  - In words, anything that happened to $\iota(h)$ on the way to x must also have happened on the way to x'.

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- There is a *payoff function*

$$u = (u_1, \ldots, u_n) : Z \to \mathbb{R}^I.$$

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- For $t \in T$ let $C(t) \subset W \times A$ be the set of initial node-pure strategy pairs that induce t: $C(t) = \{w\} \times C(t|w)$ where w is the initial predecessor of t.

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$$\mathbf{P}^\pi(t) = (\rho \times \prod_h \pi_h)(C(t)).$$

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- Given an assessment (π, μ), agent i's expected payoff conditional on information set h is

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$$b_h(\pi, \mu) := \text{argmax}_{a \in A_h} \mathbf{E}^{\pi|a,\mu}(u_{\iota(h)}|h).$$

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  - An inductive calculation shows that a sequential equilibrium is rational in the conceptually correct sense: there is no information set such that the agent who chooses there can increase her expected payoff, conditional on that information set, by changing her behavior at that information set and/or other information sets she controls.

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- It is easy to see that the fixed points of Φ are precisely the sequential equilibria.

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  - *These questions are open.*

Conditional Systems

A *conditional system* p on a finite set A is an assignment of a probability measure $p(\cdot|E) \in \Delta(E)$ to each nonempty $E \subset A$ such that

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  - *Exercise:* Prove that $\Delta^*(A)$ is the closure of $\delta^{-1}(\Delta^\circ(A))$.

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  - An induced map $\theta_C : \Delta^*(B) \to \Delta^*(A)$ is given by

$$\theta_C(q)(D|E) := q\Big(\bigcup_{a \in D} C(a) \Big| \bigcup_{a \in E} C(a)\Big) \ .$$

- If $\rho \in \Delta^\circ(B)$, and $p \in \Delta^*(A)$, we define $\rho \otimes p \in \Delta^*(B \times A)$ by setting

$$(\rho \otimes p)(D|E) := \frac{\sum_{(b,a)\in D} \rho(b) \cdot p(\{a\}|E')}{\sum_{(b,a)\in E} \rho(b) \cdot p(\{a\}|E')}$$

where $E' = \{\, a : (b,a) \in E \,\}$.

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- Since γ is continuous, it follows that $\gamma(\Xi) = \Psi$.

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- Theorem: Γ is contractible valued.
  - Since Ξ is a possible value of Γ, it follows that Ξ is contractible.

The Key Idea

If $p \in \Delta^*(A)$ and $a, b \in A$, let

$$\lambda_p(a,b) = \ln p(a|\{a,b\}) - \ln p(b|\{a,b\}) \in [-\infty, \infty].$$

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  for all a, b, c such that the sum is defined.
- Conversely, for a system of (extended) numbers $\lambda(a,b)$ satisfying these conditions define $p_\lambda \in \Delta^*(A)$ by setting

$$p_\lambda(D|E) = \frac{\sum_{a \in D} \exp(\lambda(a,b))}{\sum_{a \in E} \exp(\lambda(a,b))}$$

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- The logarithmic framework has a major advantage: linear algebra can be applied.
  - In particular, the image of $\prod_h \Delta^\circ(A_h)$ is a linear subspace.
- The general idea is to show that (the homeomorphic image of) of Ξ is a simplicial complex, and a Euclidean ball. The contractibility of best response sets is established by repeatedly retracting simplices.

Advanced Fixed Point Theory

7. Dynamic Stability and the Fixed Point Index

Dynamics and the Index

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- Review basics of ordinary differential equations.
- Introduce dynamic stability concepts and results.
- Sketch the proof of the general necessary condition.

ODE Existence-Uniqueness

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- f is *Lipschitz* if it is L-Lipschitz for some L.
- f is *locally Lipschitz* if each $x \in X$ has a neighborhood U such that $f|_U$ is Lipschitz.

Theorem: (Picard-Lindelöf) If z is a locally Lipschitz, then for any compact $C \subset U$ there is an $\varepsilon > 0$ such that there is a unique function $F : C \times [-\varepsilon, \varepsilon] \to U$ such that for each $x \in C$, $F(x, 0) = x$ and $F(x, \cdot)$ is a trajectory of z. In addition F is continuous, and if z is C^s $(1 \leq s \leq \infty)$ then so is F.

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Idea: If z is L-Lipschitz and $\|z(x)\| < M$ for all x, $\mathbf{B}_\delta(C) \subset U$, and $M\varepsilon < \delta$, then the operator

$$\Gamma : C(C \times [-\varepsilon, \varepsilon], \mathbf{B}_\delta(C)) \to C(C \times [-\varepsilon, \varepsilon], \mathbf{B}_\delta(C))$$

defined by $\Gamma(F)(x, t) = x + \int_0^t z(F(x, s))\, ds$ is a contraction.

The Flow

The *flow domain* of z is the set of $(p, t) \in U \times \mathbb{R}$ such that if $t \leq 0$ ($t \geq 0$) there is a trajectory $\gamma : [t, 0] \to U$ ($\gamma : [0, t] \to U$) of z with $\gamma(0) = p$.

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Exercise: Prove this.

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- A is *uniformly attractive* if there is a neighborhood V of A such that $V \times [0, \infty) \subset W$ and for any neighborhood Z of A, there is a $T \geq 0$ such that $\Phi(V \times [T, \infty)) \subset Z$.

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The invariant set is *uniformly asymptotically stable* if it is compact, stable, and uniformly attractive.

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- If A is asymptotically stable, then there is a Lyapunov function. (F. Wesley Wilson)

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This is probably false! Check carefully.

Claim: If A is an asymptotically stable set for z, and an ANR, then the index of A with respect to z is the *Euler characteristic* $\chi_A = \Lambda(\mathrm{Id}_A)$ of A.

Sketch of the Proof

- It suffices to show that χ_A is the fixed point index of A with respect to the function

$$\beta_\delta : p \mapsto p + \delta z(p)$$

for some $\delta > 0$.

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- If $i : A \to N$ is the inclusion, Commutativity gives

$$\Lambda(r) = \Lambda(i \circ r) = \Lambda(r \circ i) = \Lambda(\mathrm{Id}_A) = \chi_A.$$