Equilibrium Finding and Equilibrium Index in Two-Player Games

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Overview

- Nash Equilibrium of a bimatrix game: elementary existence proof with the Lemke-Howson method
- Geometric view of equilibrium [Shapley 1974]
- Definition of the Index of an Equilibrium via an orientation of the Lemke-Howson path

Nash equilibria of bimatrix games

$$A = \begin{pmatrix} 0 & 6 \\ 2 & 5 \\ 3 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 3 \end{pmatrix}$$

Nash equilibrium =

pair of strategies x , y with

x best response to y and
y best response to x.

Mixed equilibria

$$A = \begin{bmatrix} 0 & 6 \\ 2 & 5 \\ 3 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 3 \end{bmatrix}$$

$$x = \begin{bmatrix} 2/3 \\ 1/3 \\ 0 \end{bmatrix} \qquad x^T B = \begin{bmatrix} 5/3 & 5/3 \end{bmatrix}$$

$$Ay = \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix} \qquad y^T = \begin{bmatrix} 1/3 & 2/3 \end{bmatrix}$$

only **pure best responses** can have probability > 0

Best response condition

Let $\mathbf{x}$ and $\mathbf{y}$ be mixed strategies of player I and II, respectively. Then $\mathbf{x}$ is a best response to $\mathbf{y}$

$\iff$ for all pure strategies i of player I:

$$x_i > 0 \implies (\mathbf{A}\mathbf{y})_i = u = \max\{(\mathbf{A}\mathbf{y})_k \mid 1 \le k \le m\}.$$

Here, $(\mathbf{A}\mathbf{y})_i$ is the ith component of $\mathbf{A}\mathbf{y}$, which is the expected payoff to player I when playing row i.

Proof.

$$\mathbf{x}\mathbf{A}\mathbf{y} = \sum_{i=1}^{m} \mathbf{x}_i (\mathbf{A}\mathbf{y})_i = \sum_{i=1}^{m} \mathbf{x}_i (u - (u - (\mathbf{A}\mathbf{y})_i)$$

$$= \sum_{i=1}^{m} \mathbf{x}_i u - \sum_{i=1}^{m} \mathbf{x}_i (u - (\mathbf{A}\mathbf{y})_i) = u - \sum_{i=1}^{m} \mathbf{x}_i (u - (\mathbf{A}\mathbf{y})_i) \le u,$$

because $\mathbf{x}_i \ge 0$ and $u - (\mathbf{A}\mathbf{y})_i \ge 0$ for all i. Furthermore, $\mathbf{x}\mathbf{A}\mathbf{y} = u \iff \mathbf{x}_i > 0$ implies $(\mathbf{A}\mathbf{y})_i = u$, as claimed.

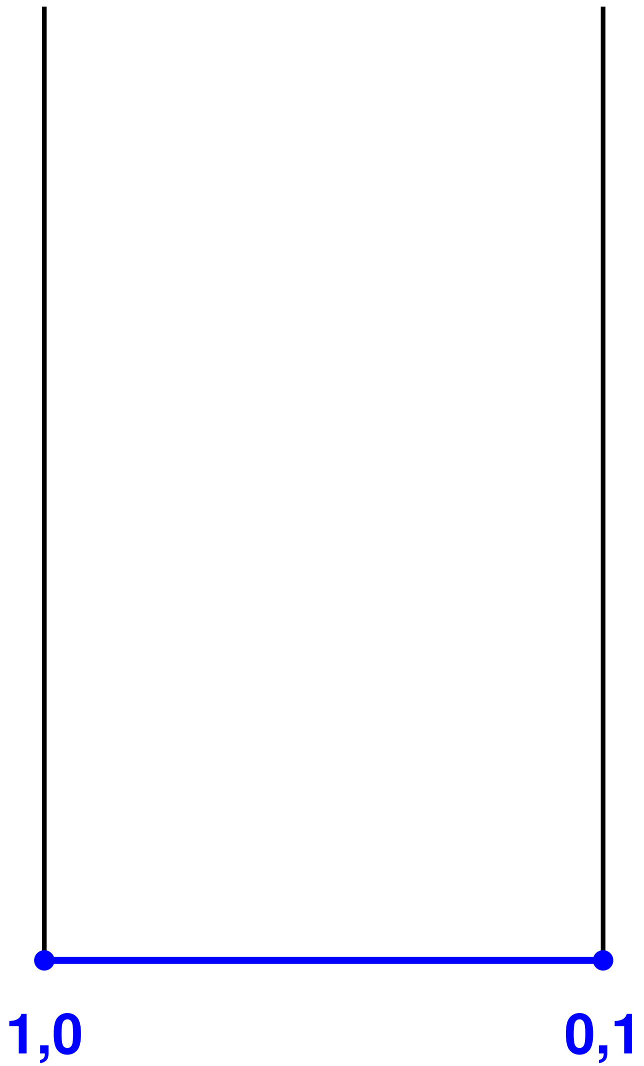
Best responses to mixed strategy of player 2

	4	5
1	0	6
2	2	5
3	3	3

= A

payoffs to
player I

Best responses to mixed strategy of player 2

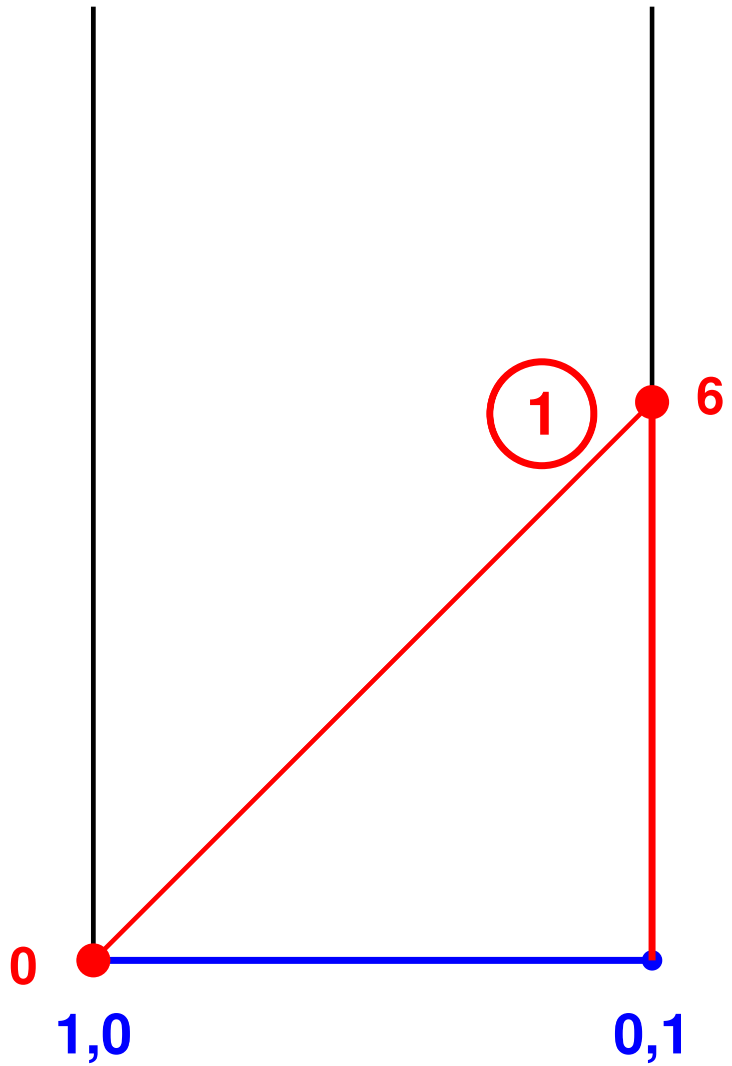


	4	5
1	0	6
2	2	5
3	3	3

= A

payoffs to player I

Best responses to mixed strategy of player 2

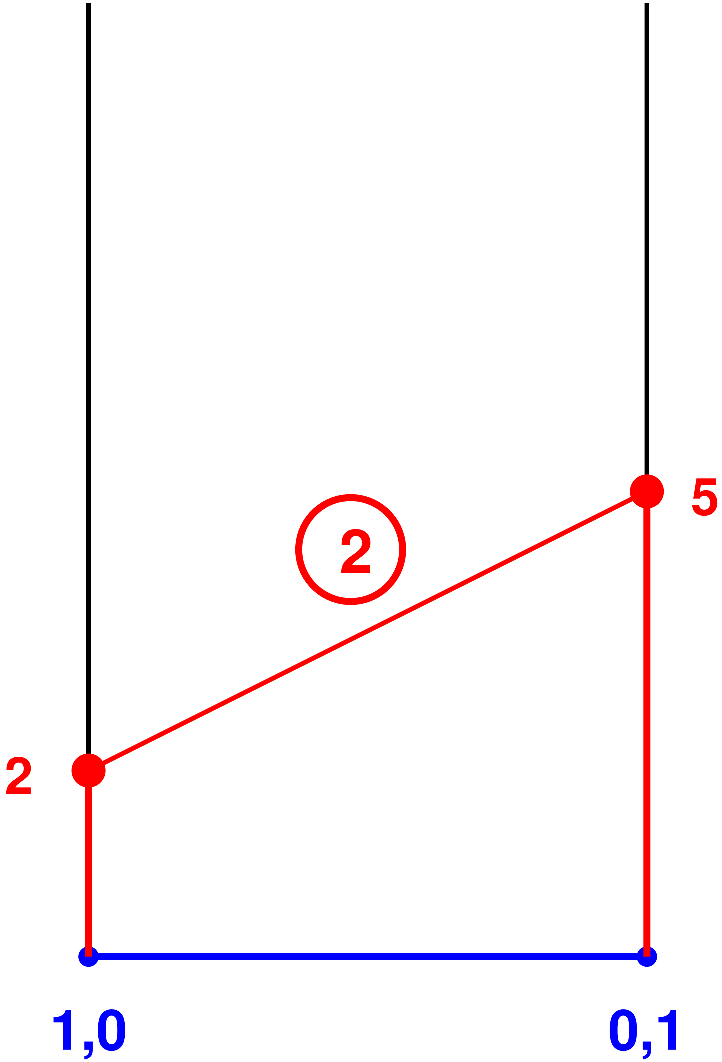


	4	5
1	0	6
2	2	5
3	3	3

= A

payoffs to player I

Best responses to mixed strategy of player 2

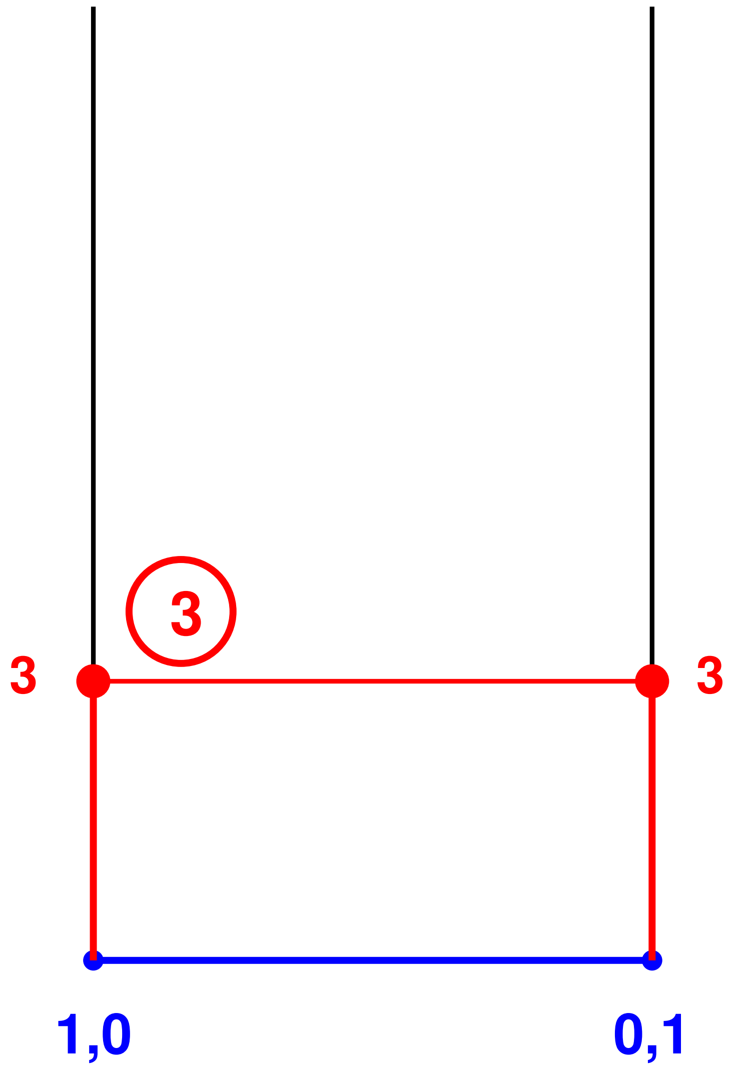


	4	5
1	0	6
2	2	5
3	3	3

= A

payoffs to
player I

Best responses to mixed strategy of player 2

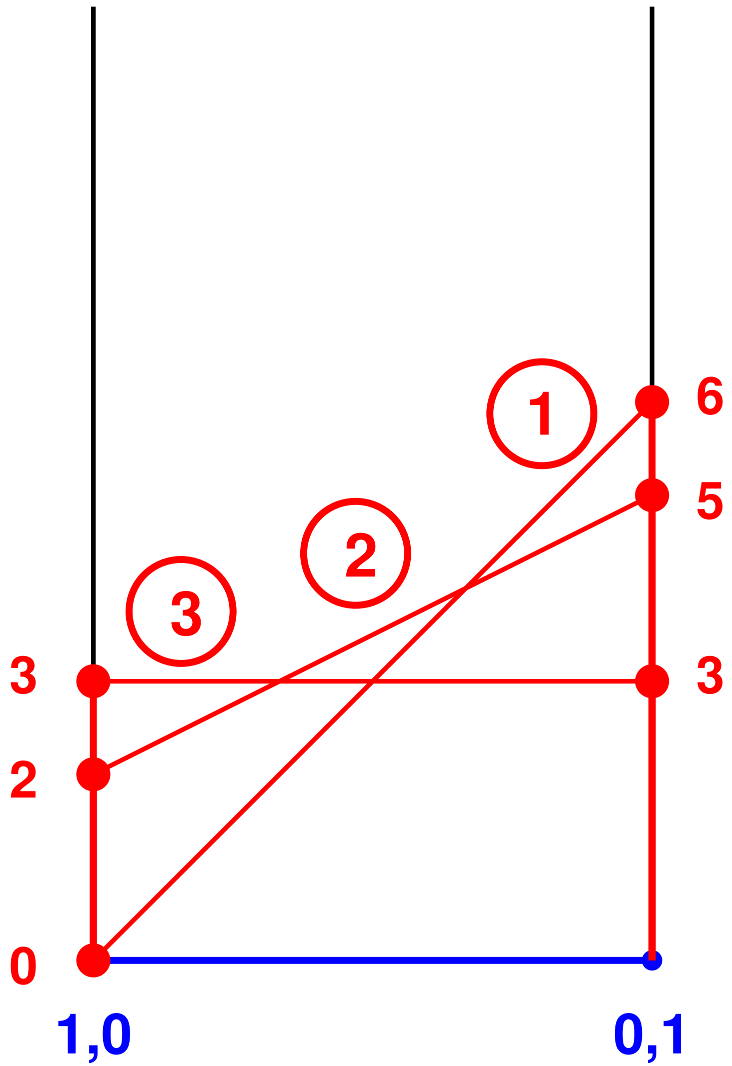


	4	5
1	0	6
2	2	5
3	3	3

= A

payoffs to
player I

Best responses to mixed strategy of player 2

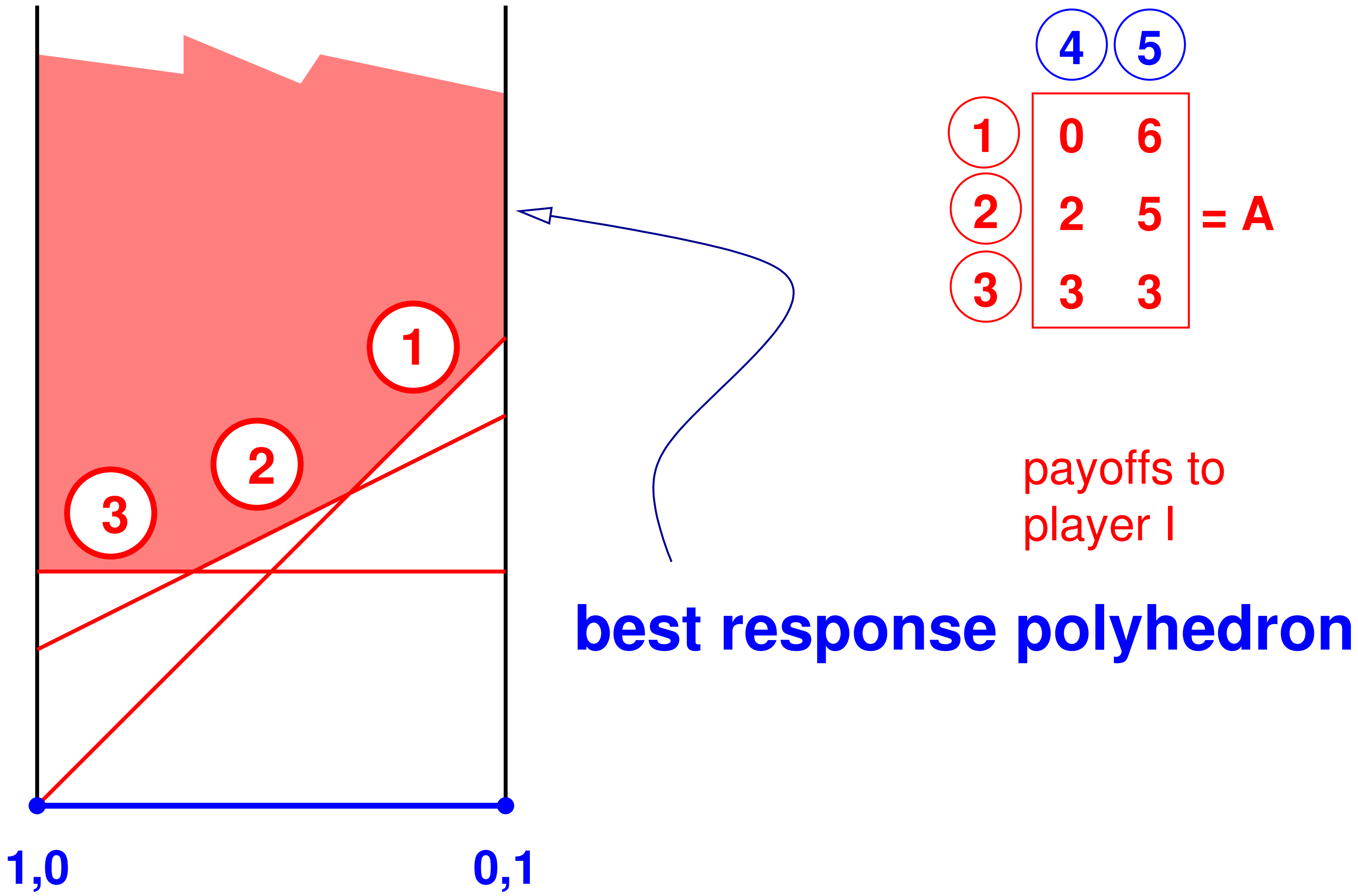


	4	5
1	0	6
2	2	5
3	3	3

= A

payoffs to
player I

Best responses to mixed strategy of player 2

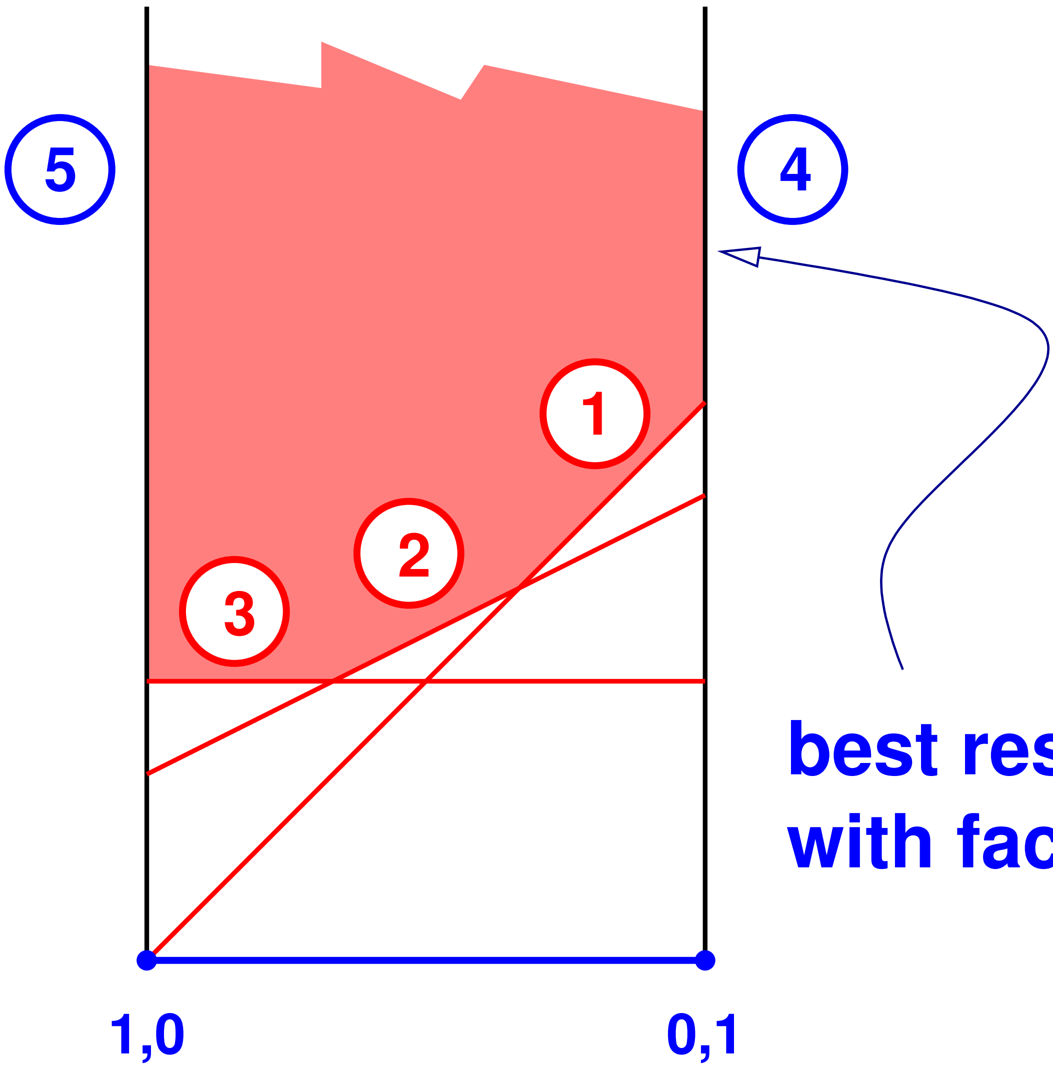


	4	5
1	0	6
2	2	5
3	3	3

= A

payoffs to player I

Best responses to mixed strategy of player 2



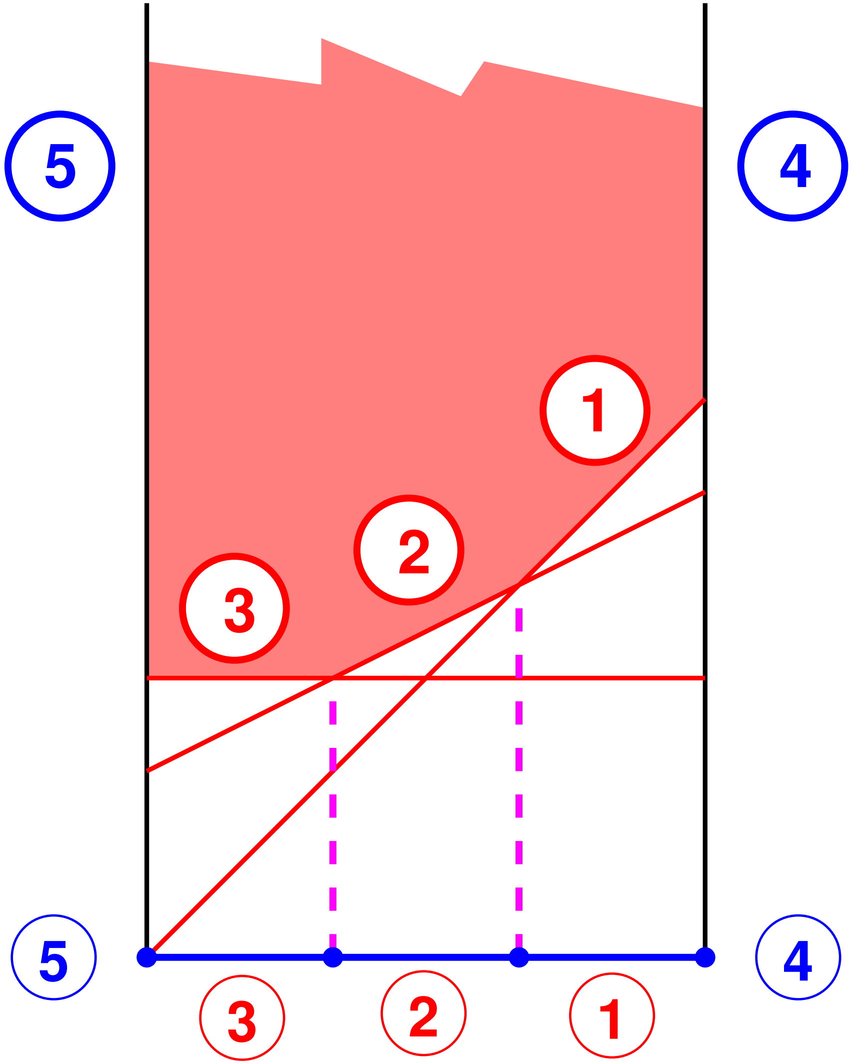
	4	5
1	0	6
2	2	5
3	3	3

= A

payoffs to player I

best response polyhedron with facet labels

Best responses to mixed strategy of player 2



	4	5
1	0	6
2	2	5
3	3	3

= A

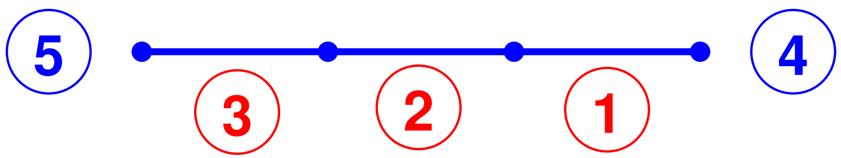
payoffs to
player I

Best responses to mixed strategy of player 2

	4	5
1	0	6
2	2	5
3	3	3

= A

payoffs to player I

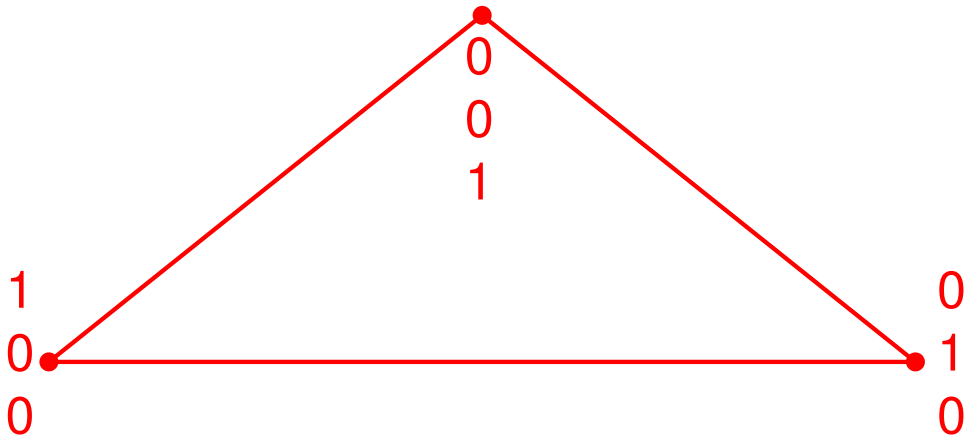


Best responses to mixed strategy of player 1

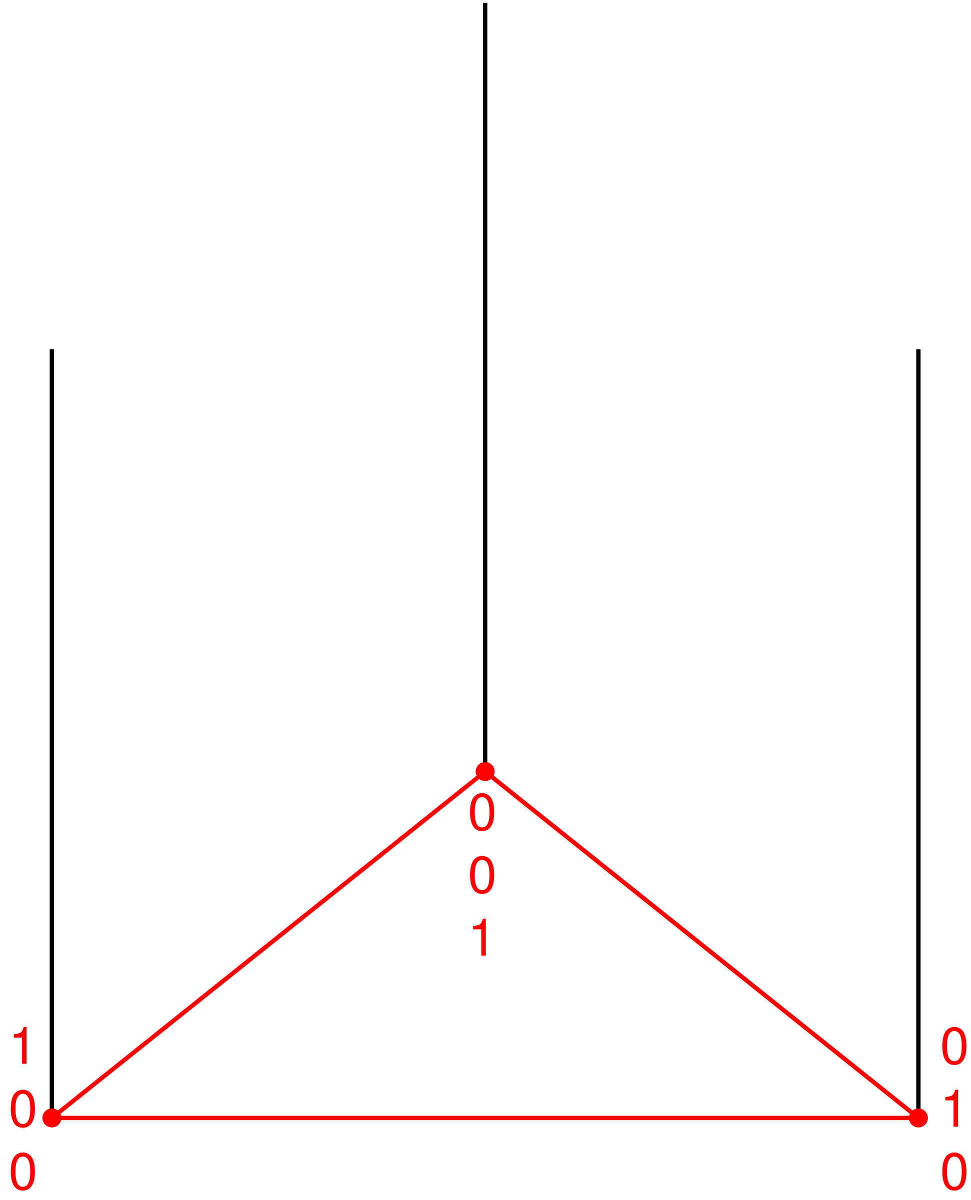
	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to
player II



Best responses to mixed strategy of player 1

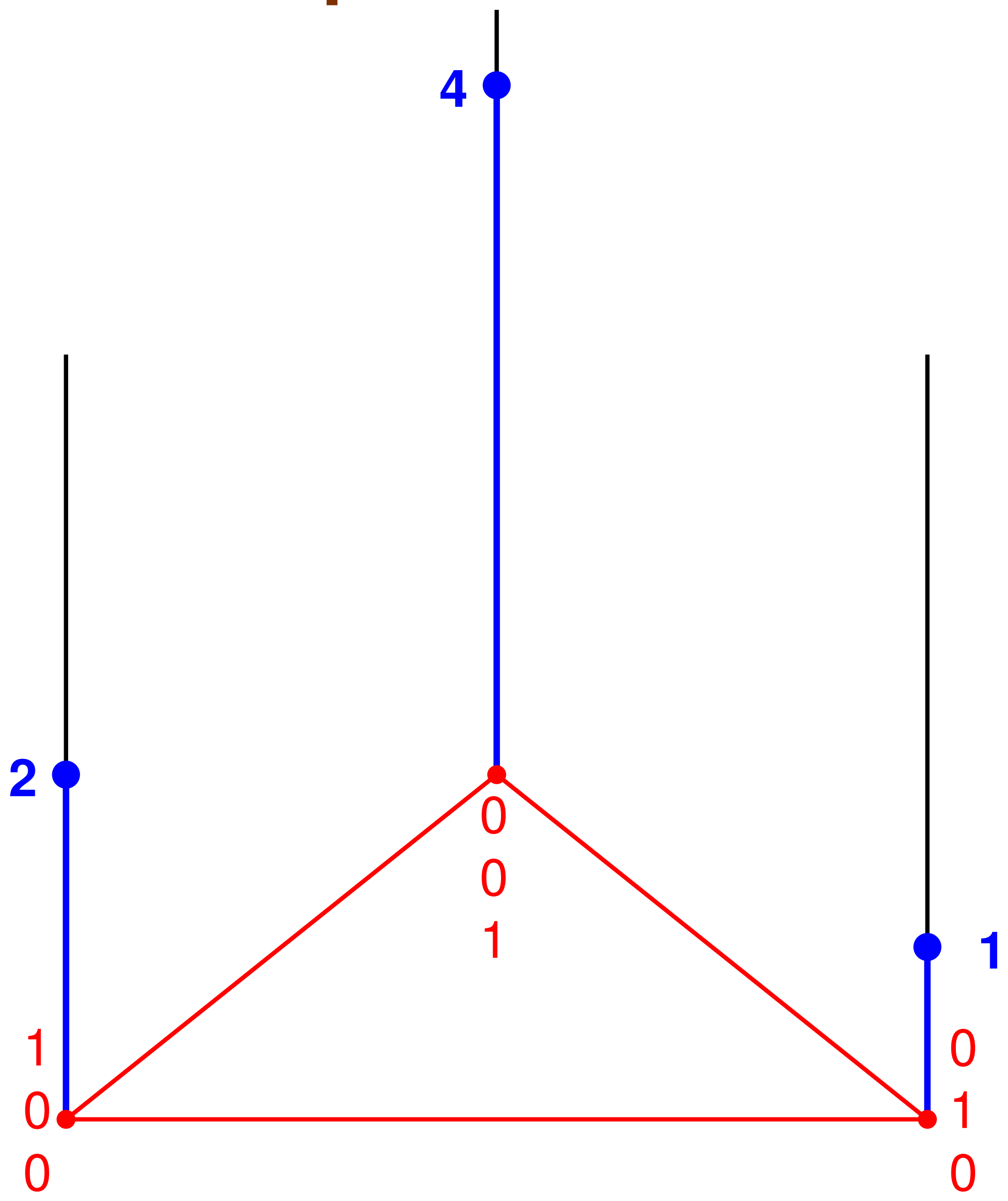


	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to player II

Best responses to mixed strategy of player 1

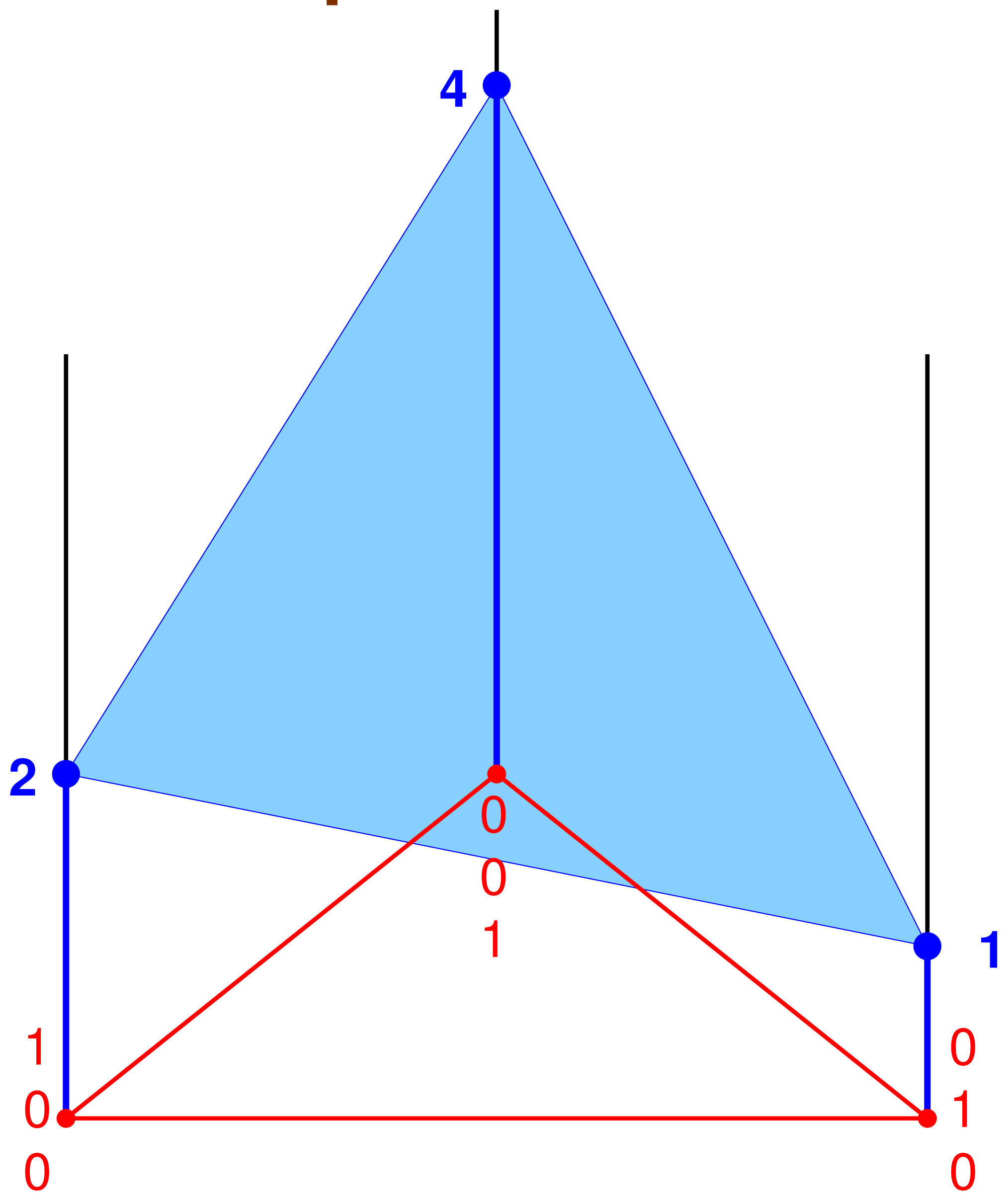


	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to
player II

Best responses to mixed strategy of player 1

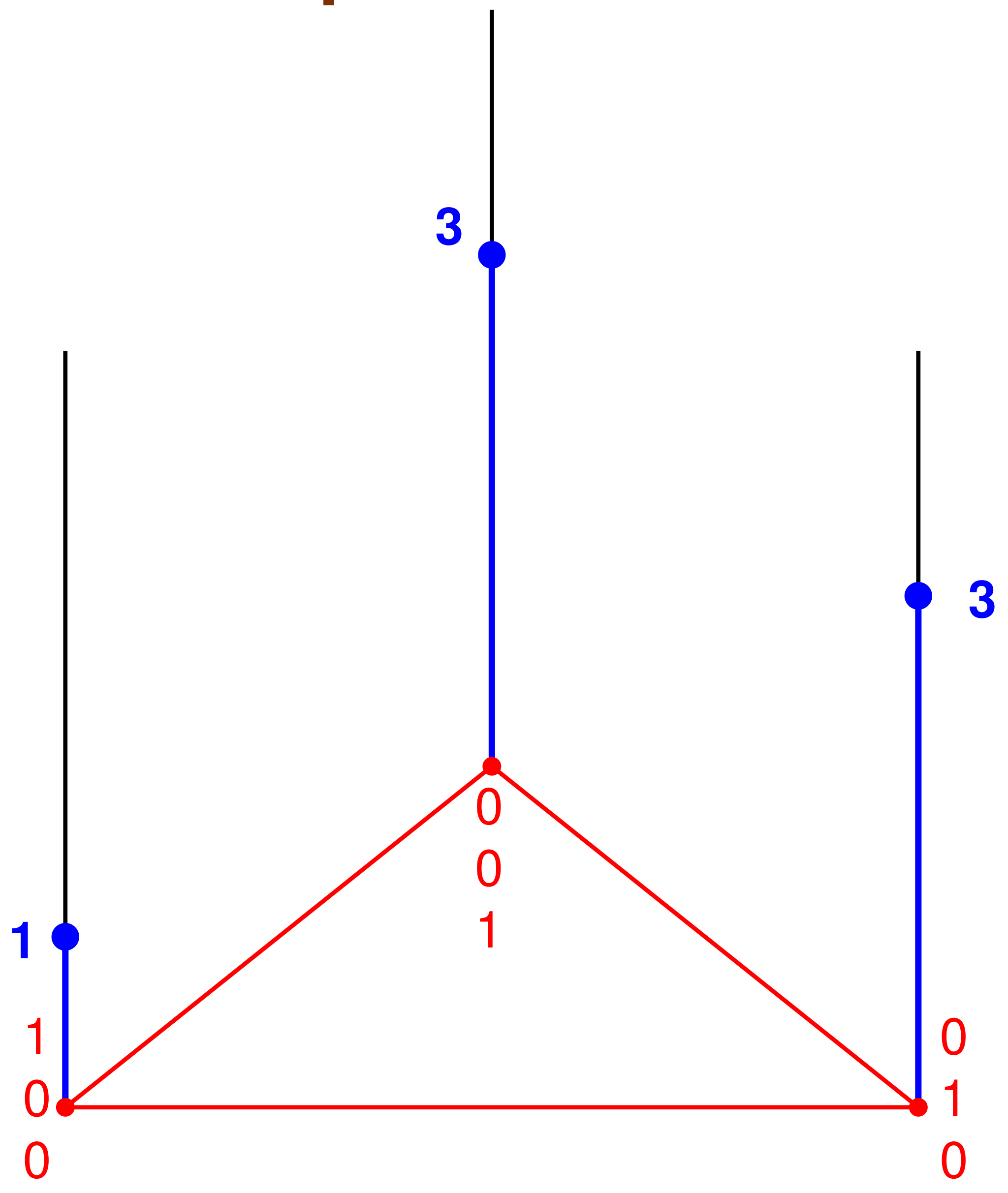


	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to
player II

Best responses to mixed strategy of player 1

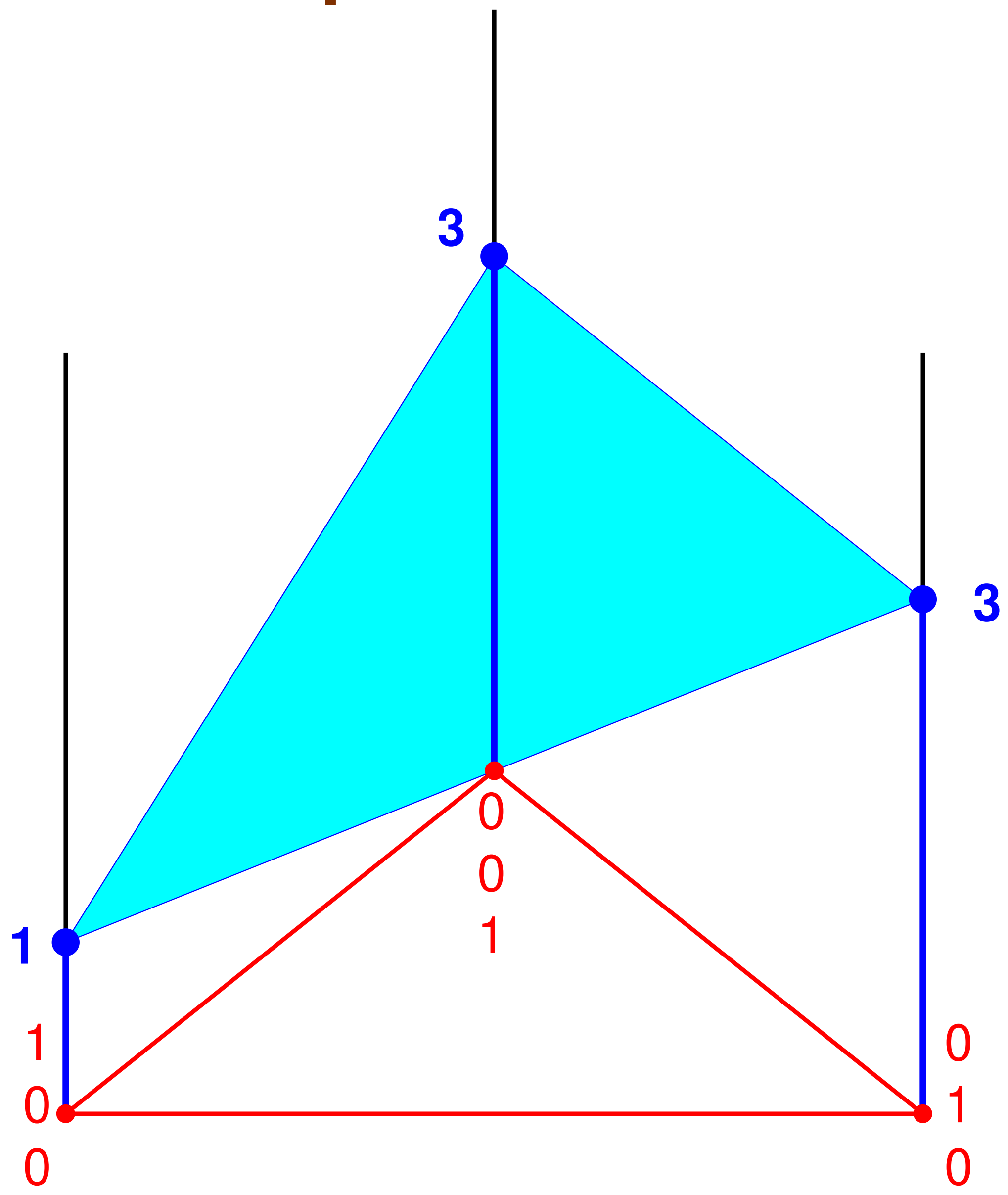


	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to
player II

Best responses to mixed strategy of player 1

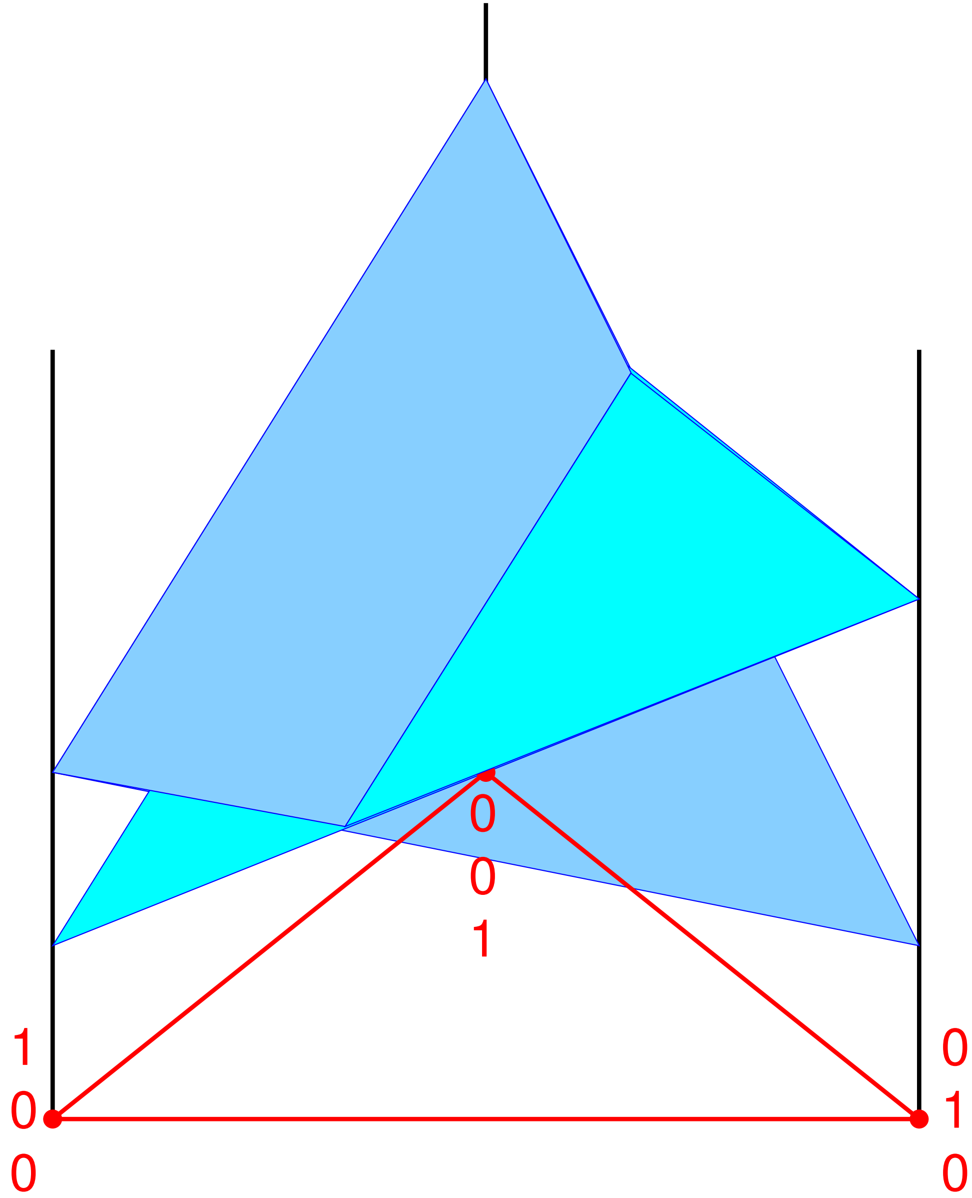


	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to player II

Best responses to mixed strategy of player 1

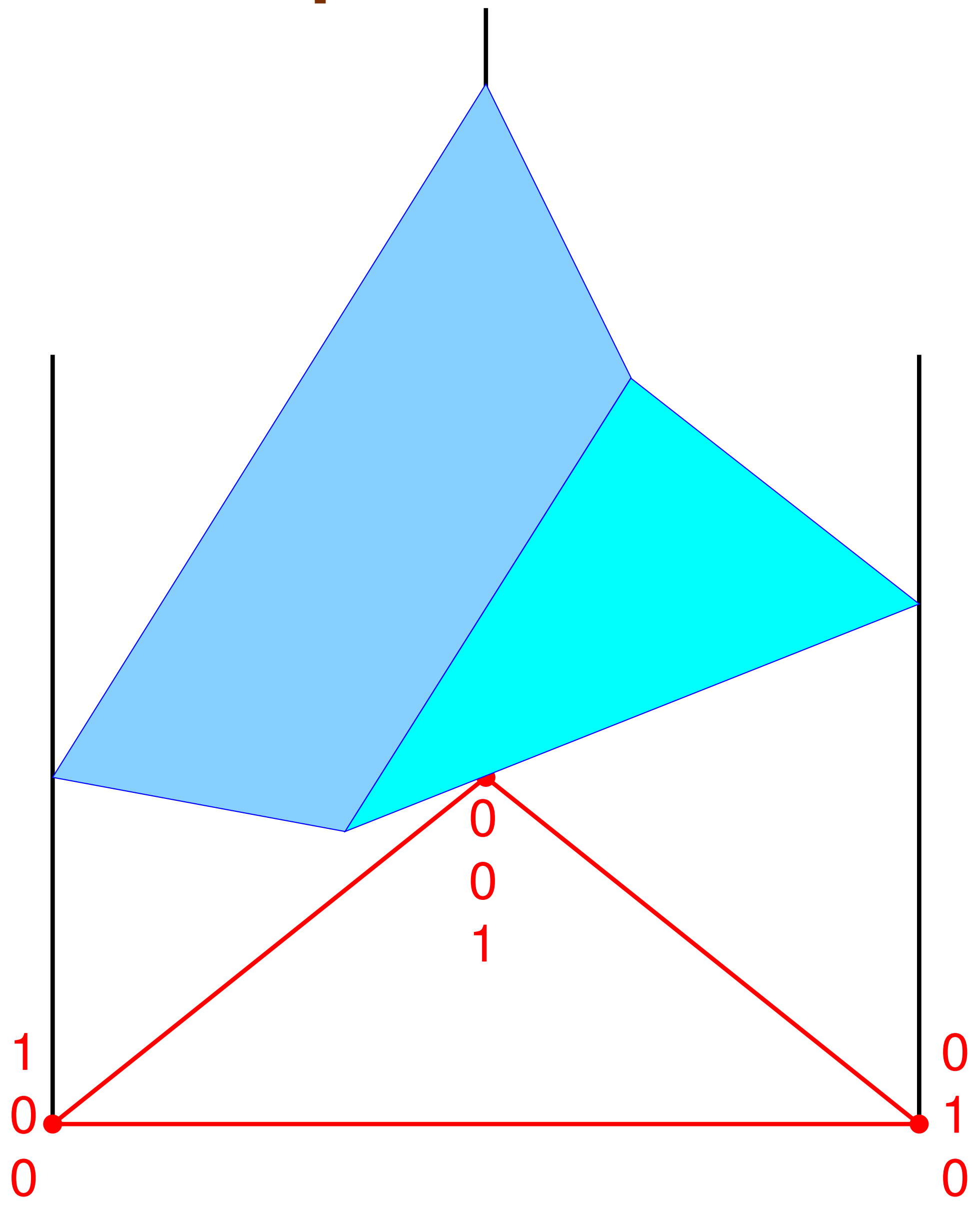


	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to
player II

Best responses to mixed strategy of player 1

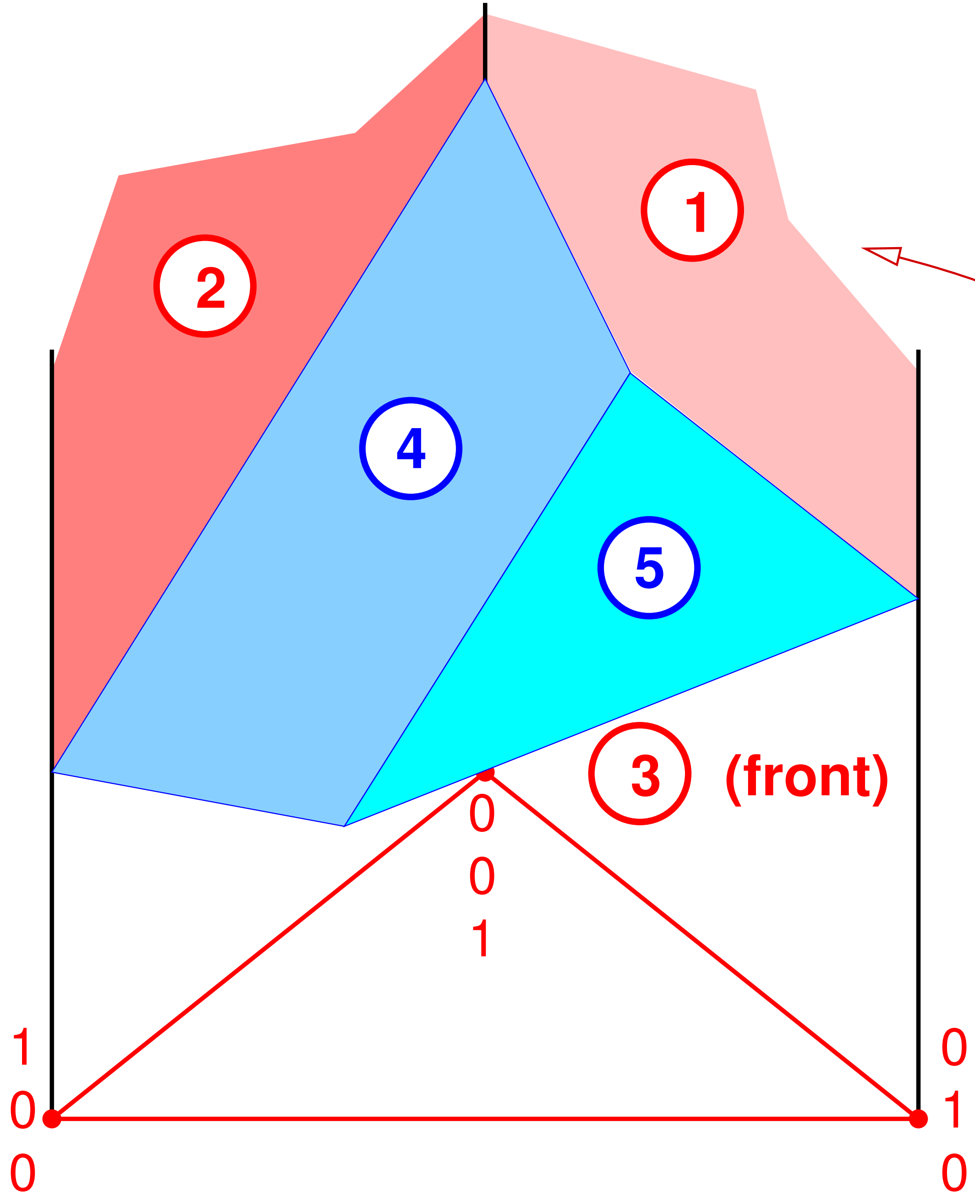


	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to
player II

Best responses to mixed strategy of player 1



	4	5
1	2	1
2	1	3
3	4	3

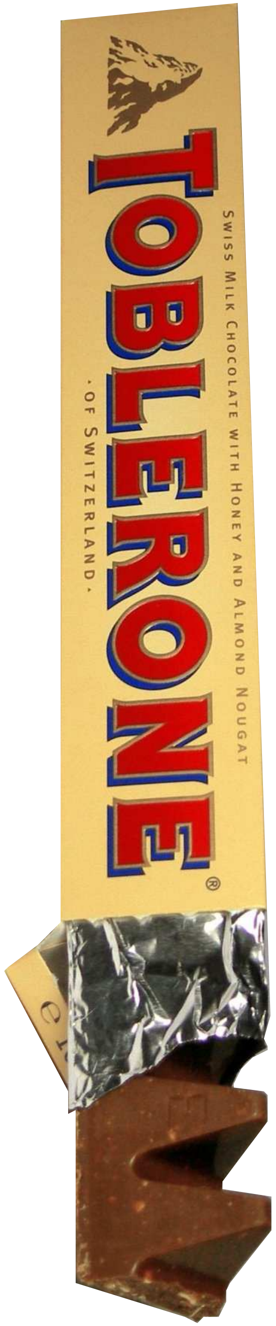
= B

payoffs to player II

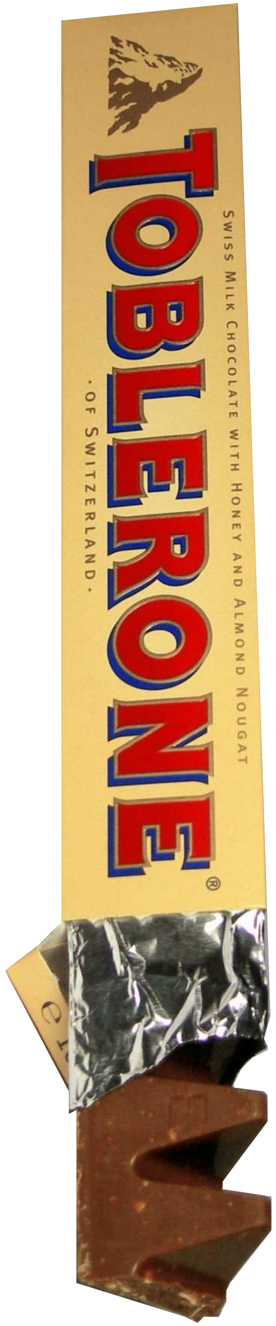
best response polyhedron with facet labels

Alternative view

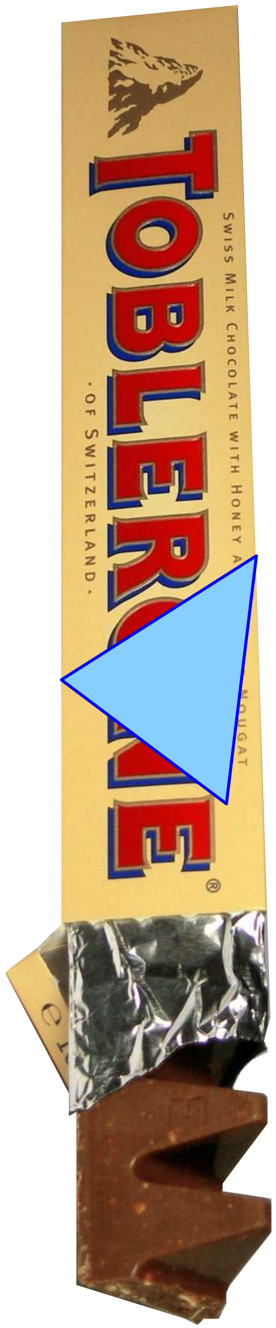
Chop off Toblerone prism



Chop off Toblerone prism



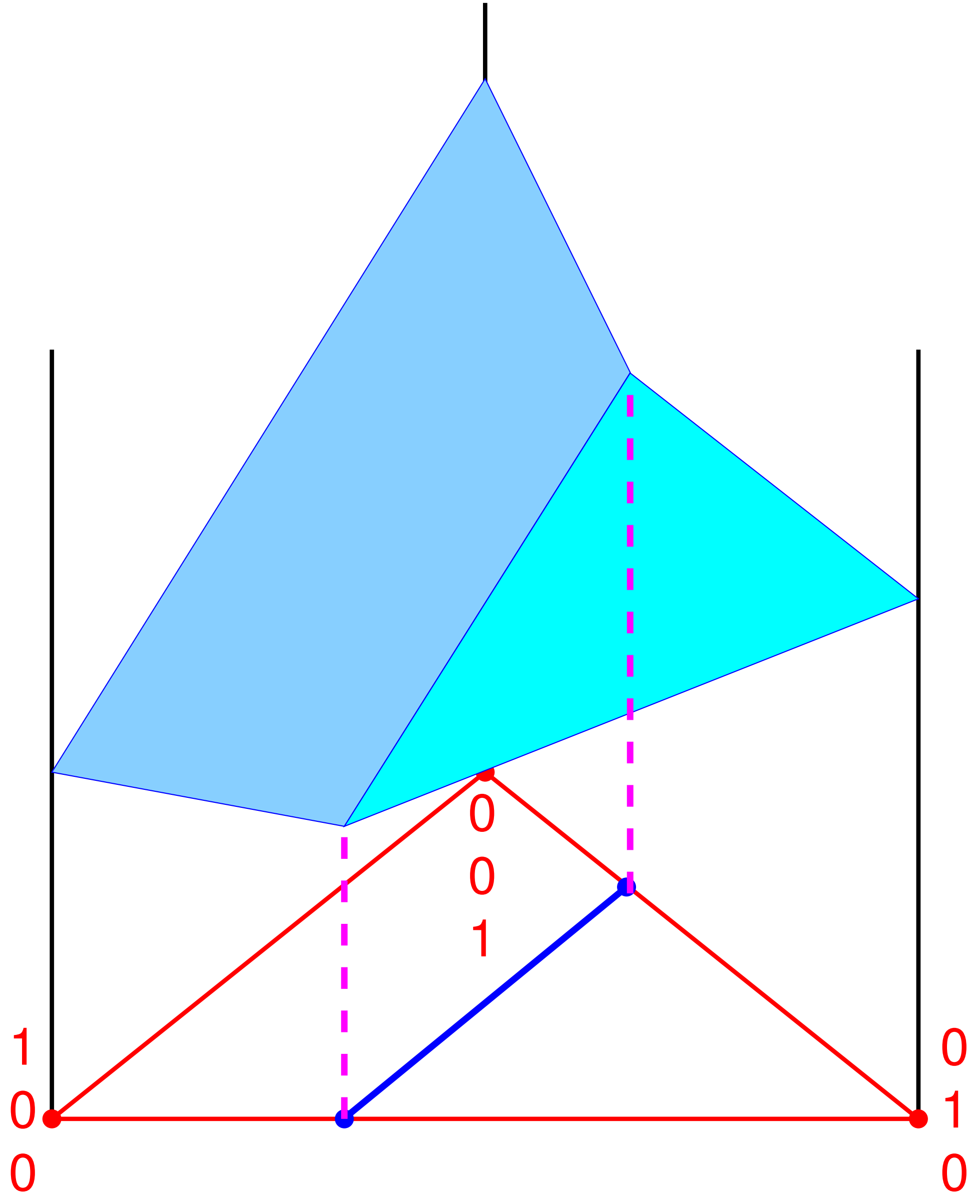
Chop off Toblerone prism



Chop off Toblerone prism

Chop off Toblerone prism

Best responses to mixed strategy of player 1



	4	5
1	2	1
2	1	3
3	4	3

= B

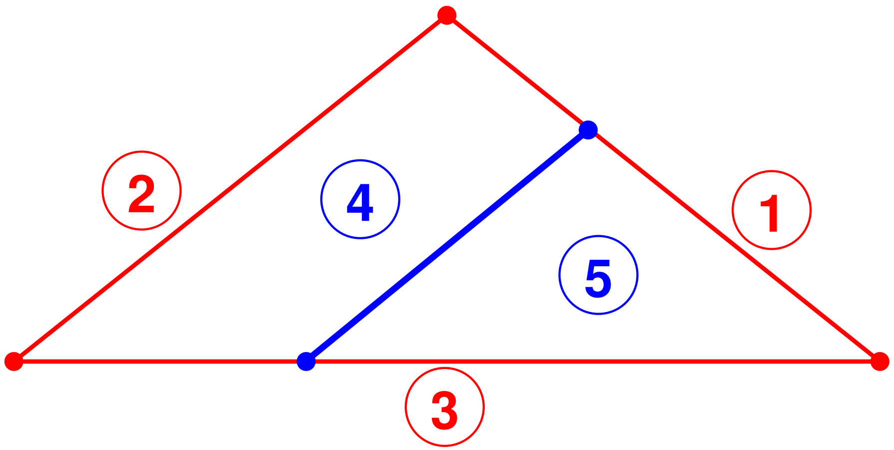
payoffs to
player II

Best responses to mixed strategy of player 1

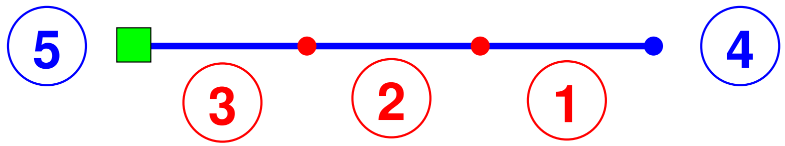
	4	5
1	2	1
2	1	3
3	4	3

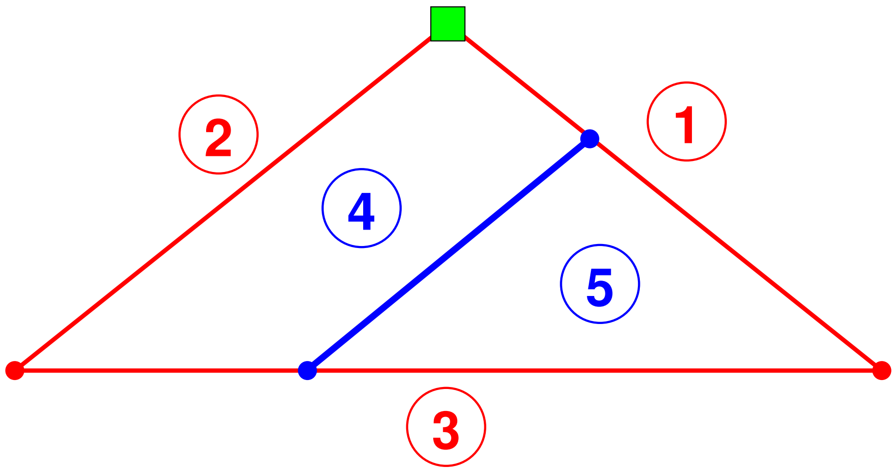
= B

payoffs to
player II

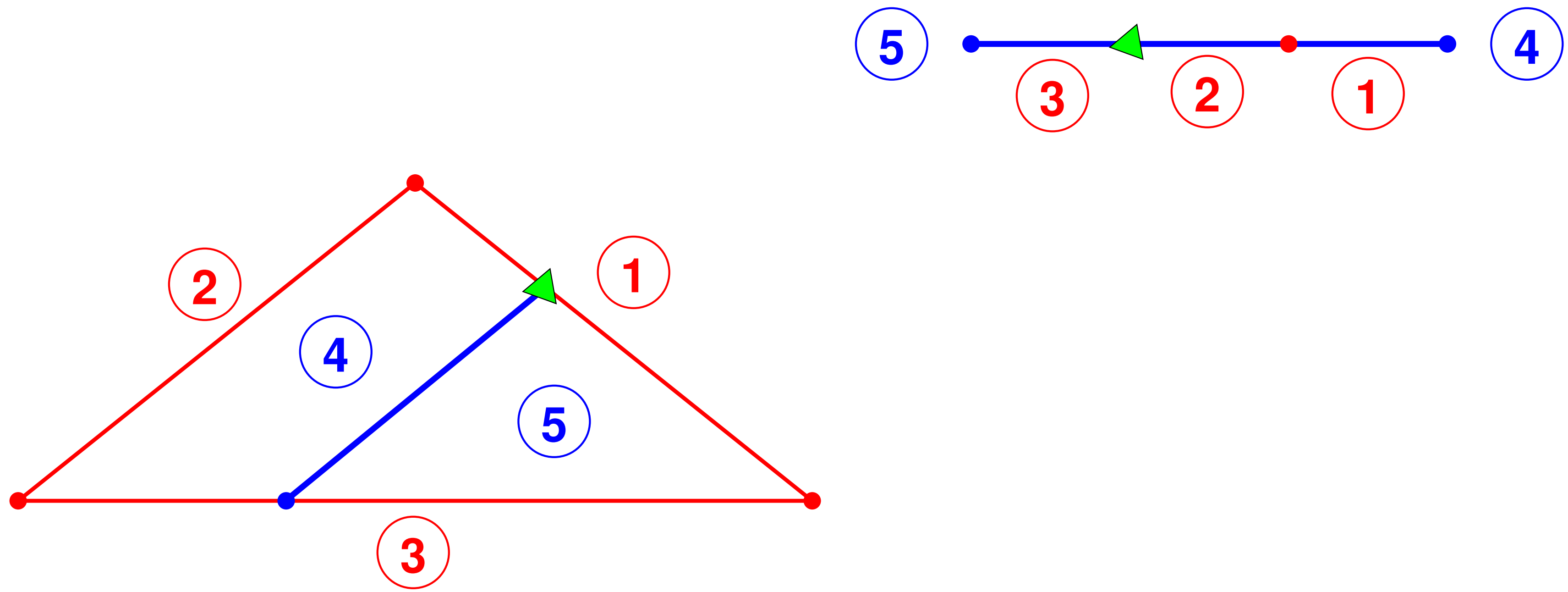


Equilibrium = completely labeled strategy pair

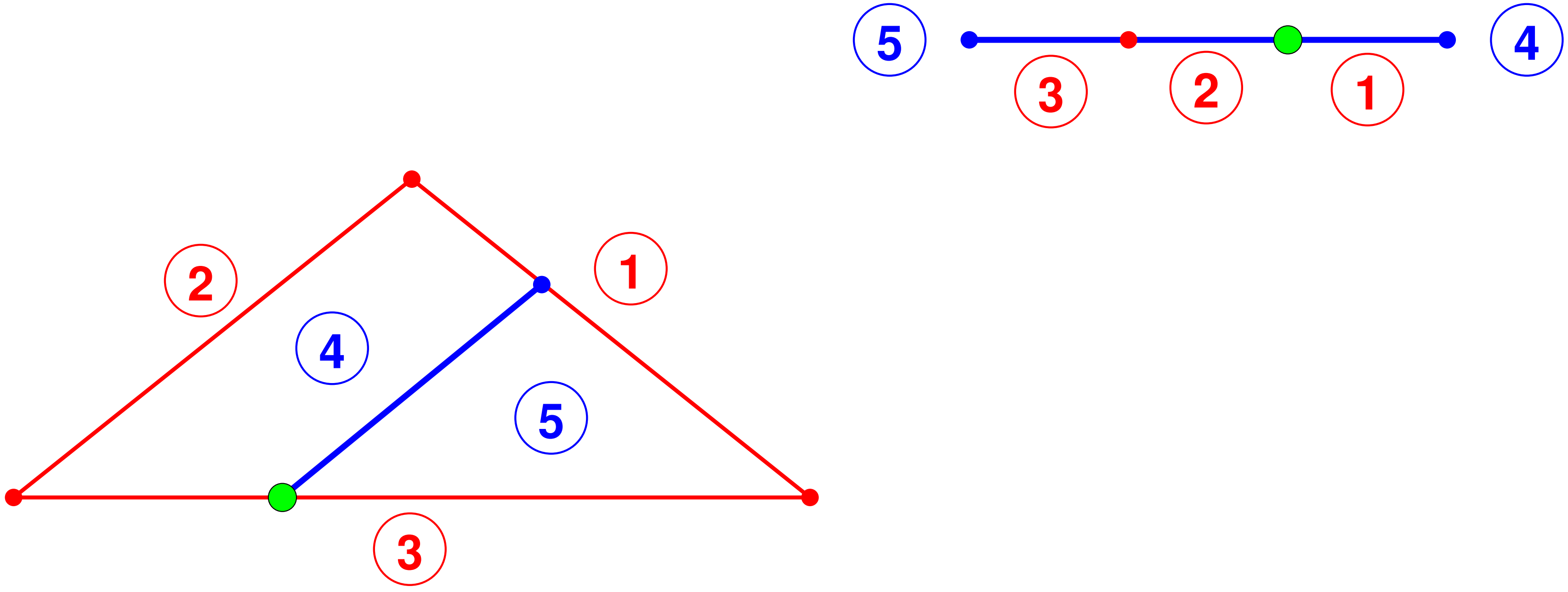




Equilibrium = completely labeled strategy pair



Equilibrium = completely labeled strategy pair



Constructing games using geometry

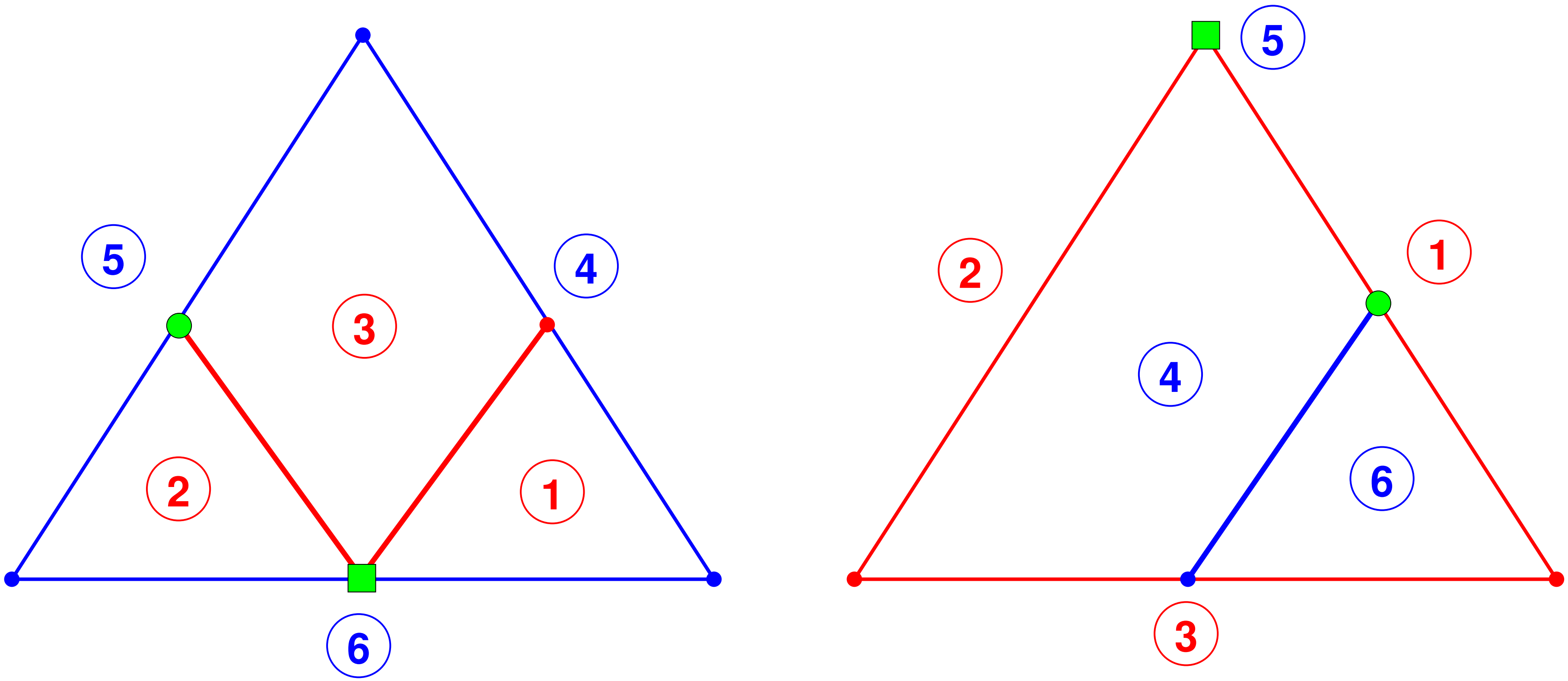
low dimension: 2, 3, (4) pure strategies:

subdivide mixed strategy simplex into
response regions, label suitably

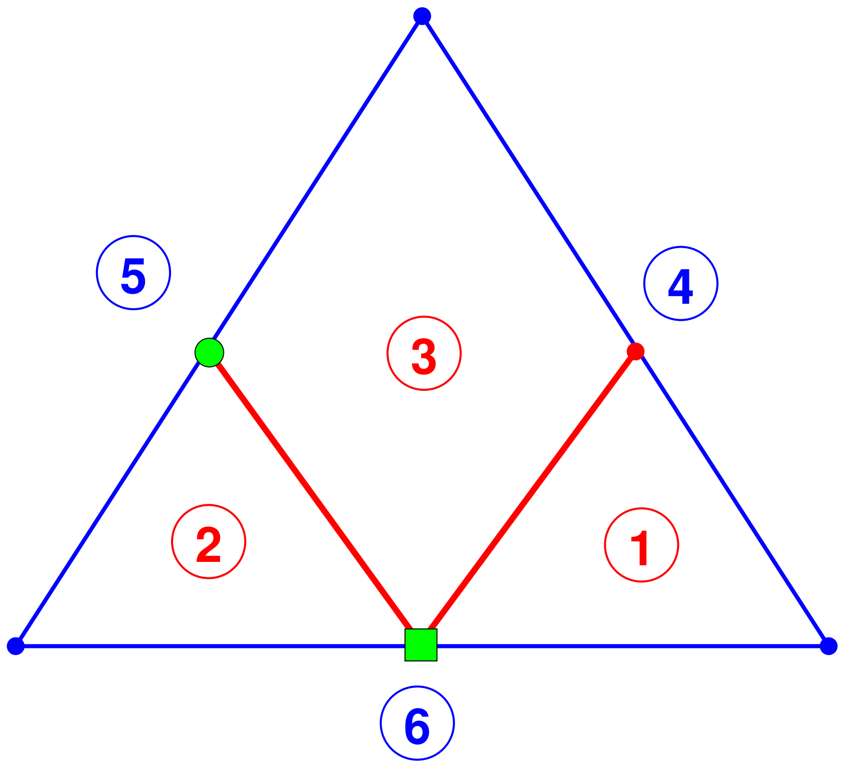
high dimension:

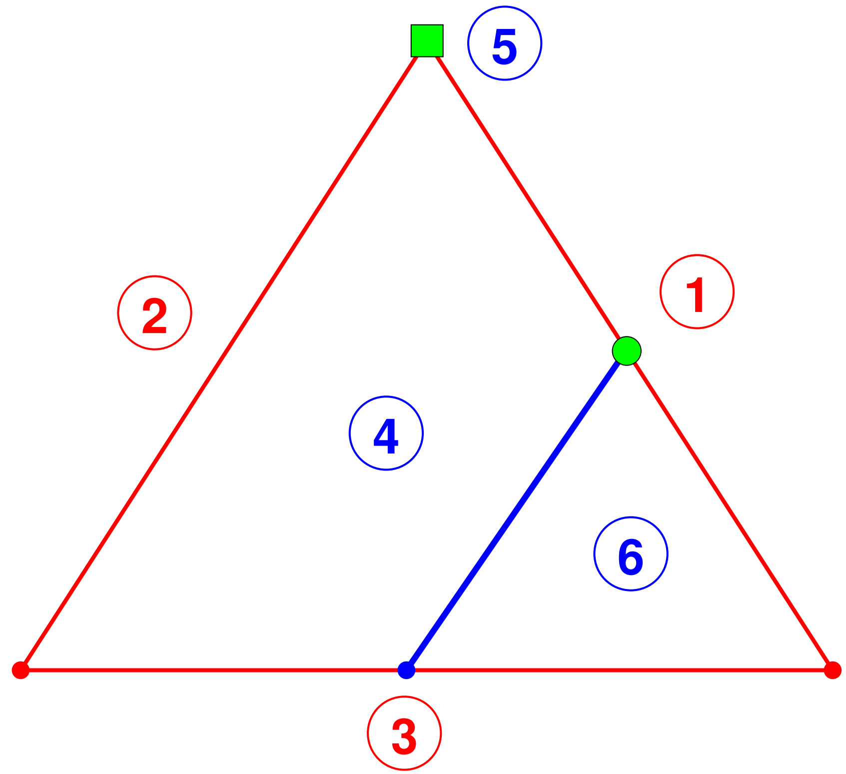
use polytopes with known combinatorial structure
e.g. for constructing games with many equilibria,
or long Lemke-Howson computations
[Savani & von Stengel, *FOCS 2004*,
Econometrica 2006]

Construct isolated non–quasi–strict equilibrium



Construct isolated non–quasi–strict equilibrium





A =

0	2	0	1
2	0	0	2
1	1	1	3

B =

4	5	6
1	0	0
1	0	2
1	1	0

Best response polyhedron H_2 for player 2

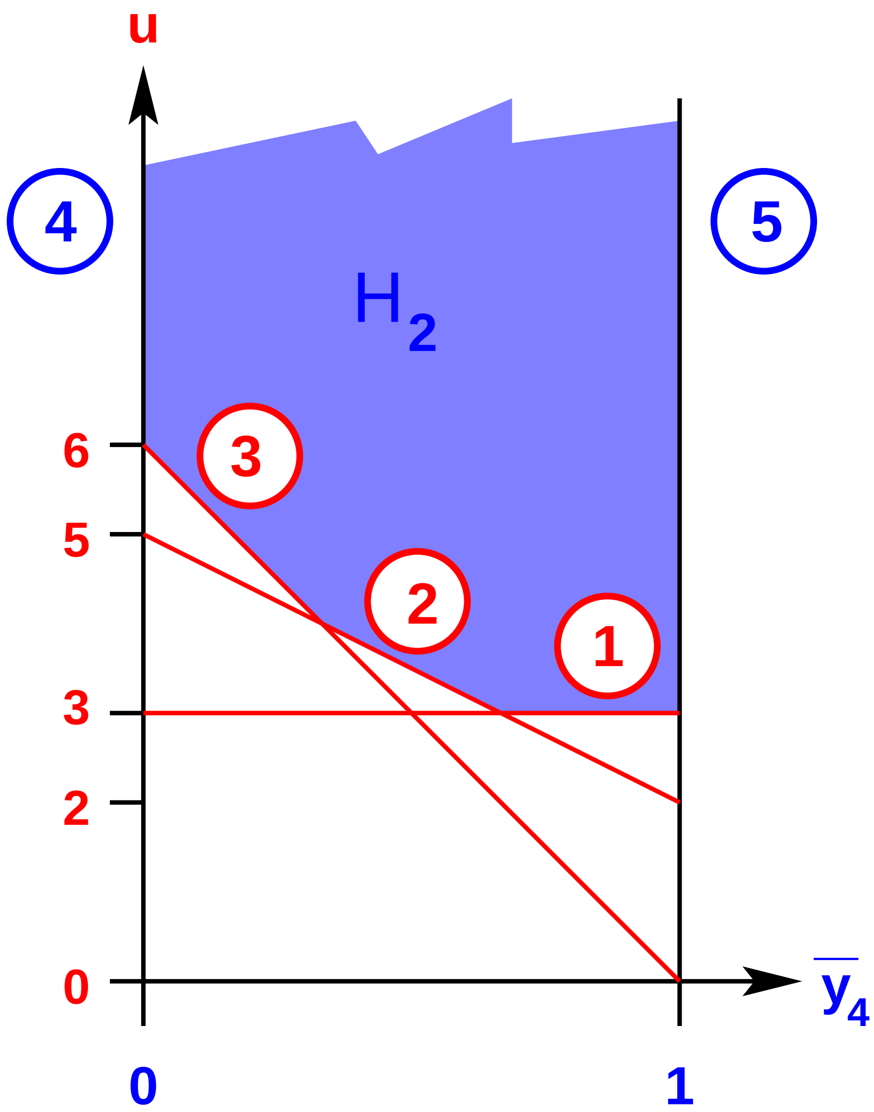
$$A = \begin{array}{c|cc|} & \overline{y}_4 & \overline{y}_5 \\ \hline ① & 3 & 3 \\ ② & 2 & 5 \\ ③ & 0 & 6 \\ \hline \end{array}$$

$$H_2 = \{ (\overline{y}_4, \overline{y}_5, u) \mid$$

$$\begin{array}{llcl} ①: & 3\overline{y}_4 + 3\overline{y}_5 & \leq & u \\ ②: & 2\overline{y}_4 + 5\overline{y}_5 & \leq & u \\ ③: & 6\overline{y}_5 & \leq & u \end{array}$$

$$\overline{y}_4 + \overline{y}_5 = 1$$

$$\begin{array}{llcl} ④: & \overline{y}_4 & \geq & 0 \\ ⑤: & \overline{y}_5 & \geq & 0 \ \} \end{array}$$



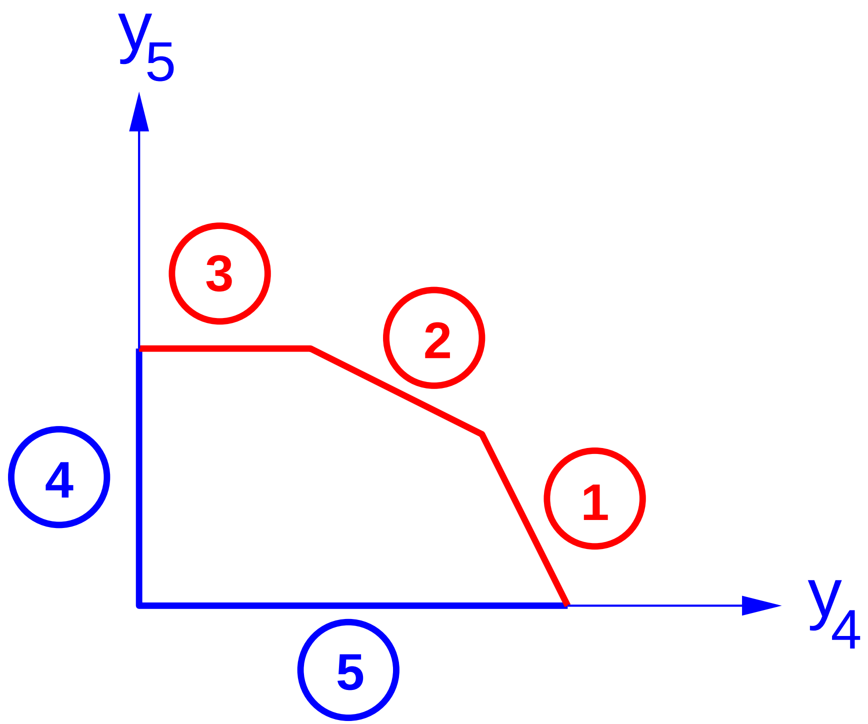
Best response polytope Q for player 2

$$A = \begin{array}{c|cc} & y_4 & y_5 \\ \hline (1) & 3 & 3 \\ (2) & 2 & 5 \\ (3) & 0 & 6 \end{array}$$

$$Q = \{ y \mid Ay \le 1,\ y \ge 0 \}$$

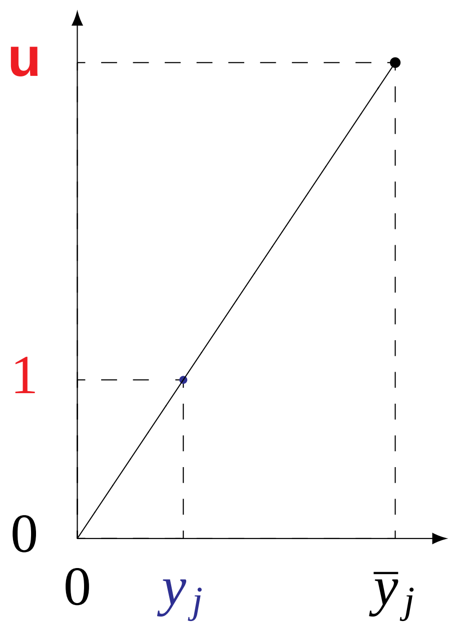
$$Q = \{ (y_4, y_5) \mid$$

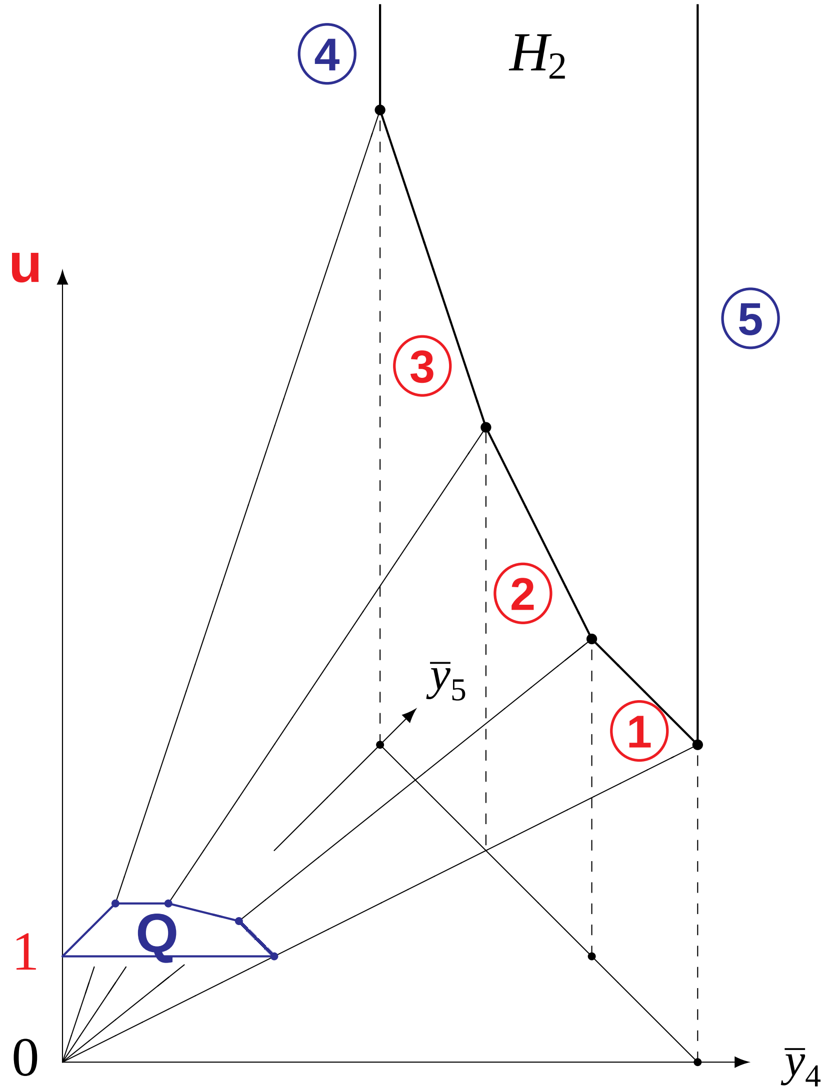
$$\begin{aligned} (1):&\quad 3y_4 + 3y_5 \le 1 \\ (2):&\quad 2y_4 + 5y_5 \le 1 \\ (3):&\quad 6y_5 \le 1 \\ (4):&\quad y_4 \ge 0 \\ (5):&\quad y_5 \ge 0 \ \} \end{aligned}$$



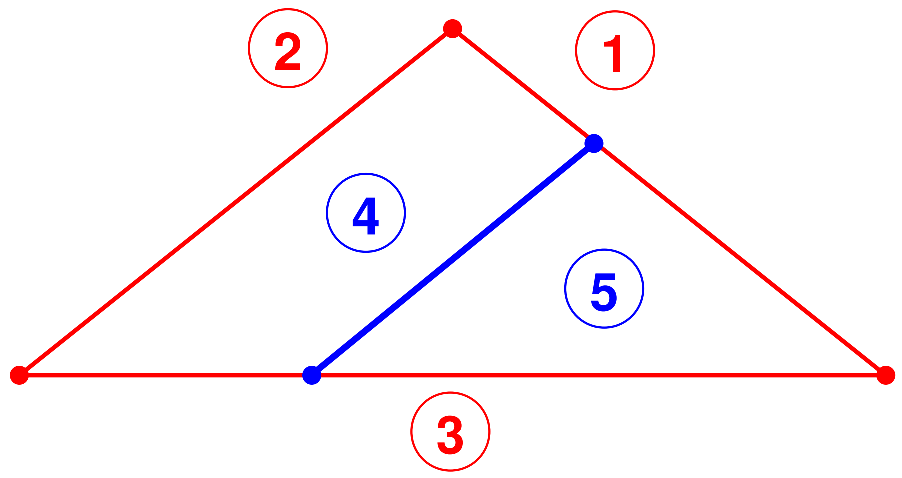
Projective transformation

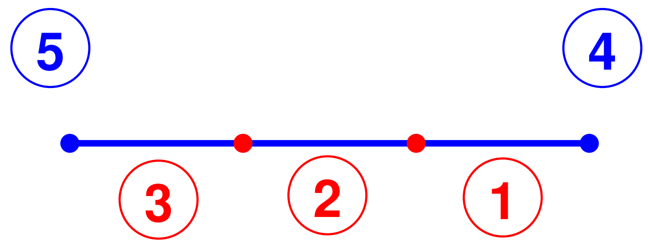
H_2, **Q** same face incidences





The Lemke–Howson algorithm





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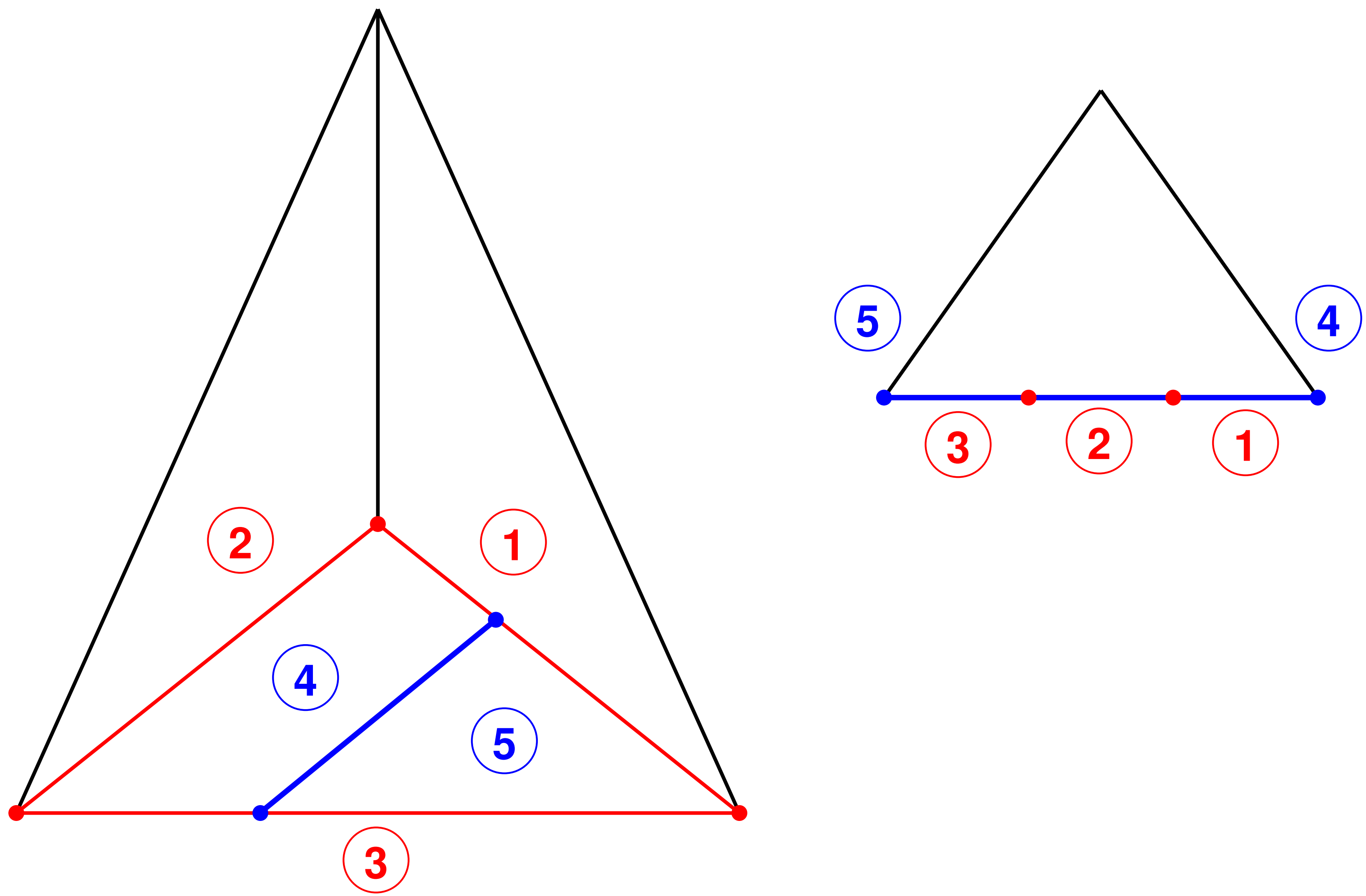

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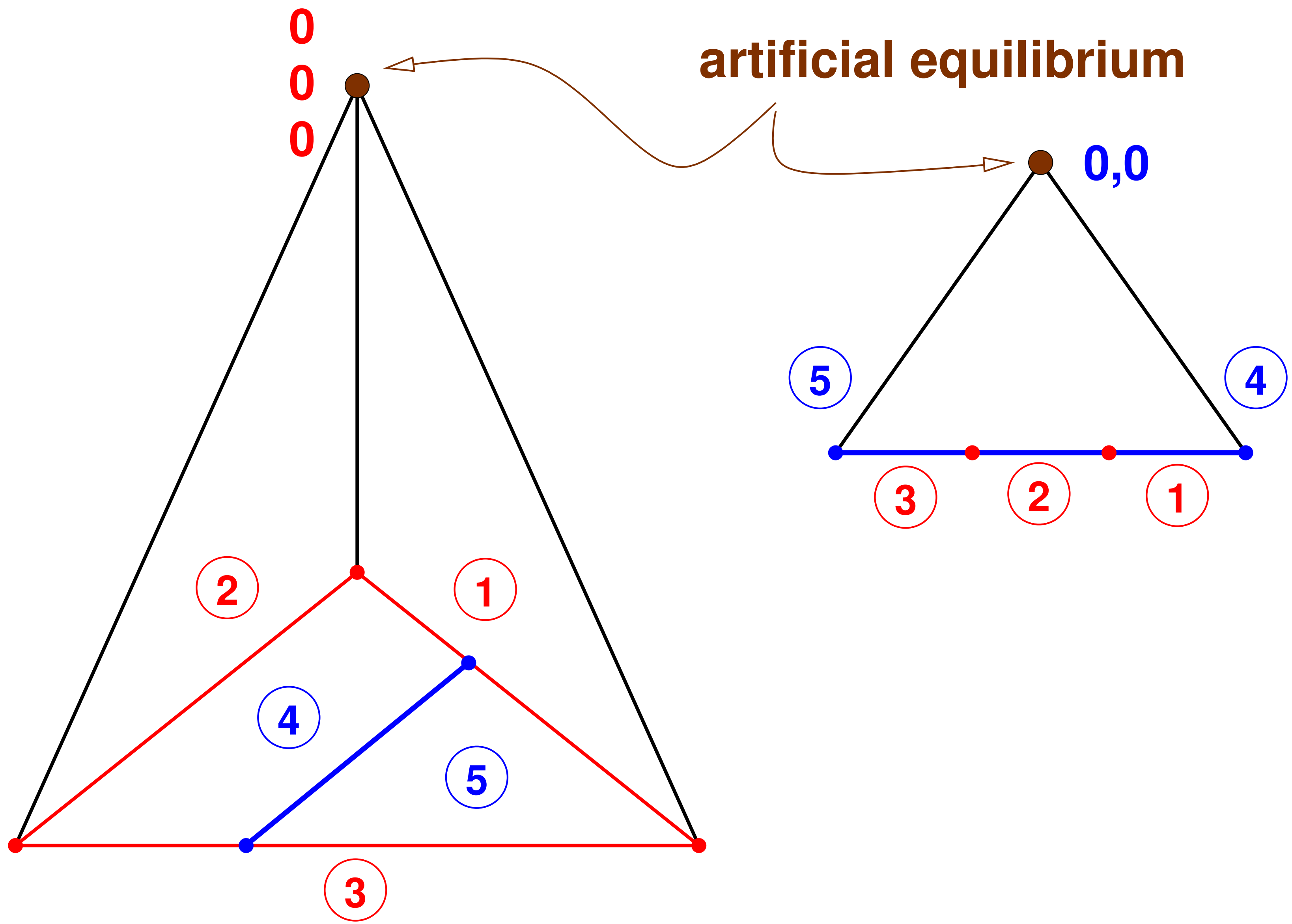

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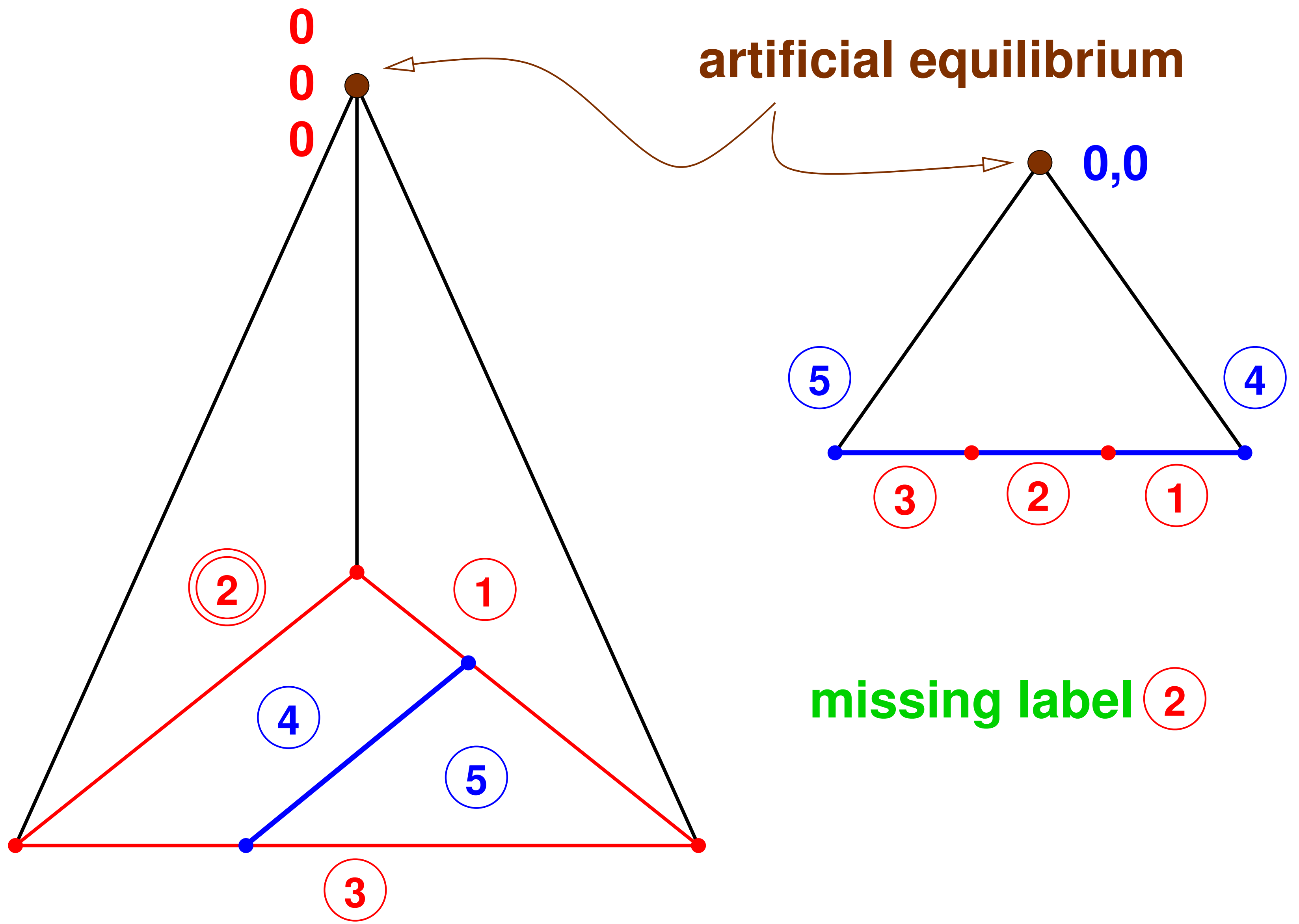

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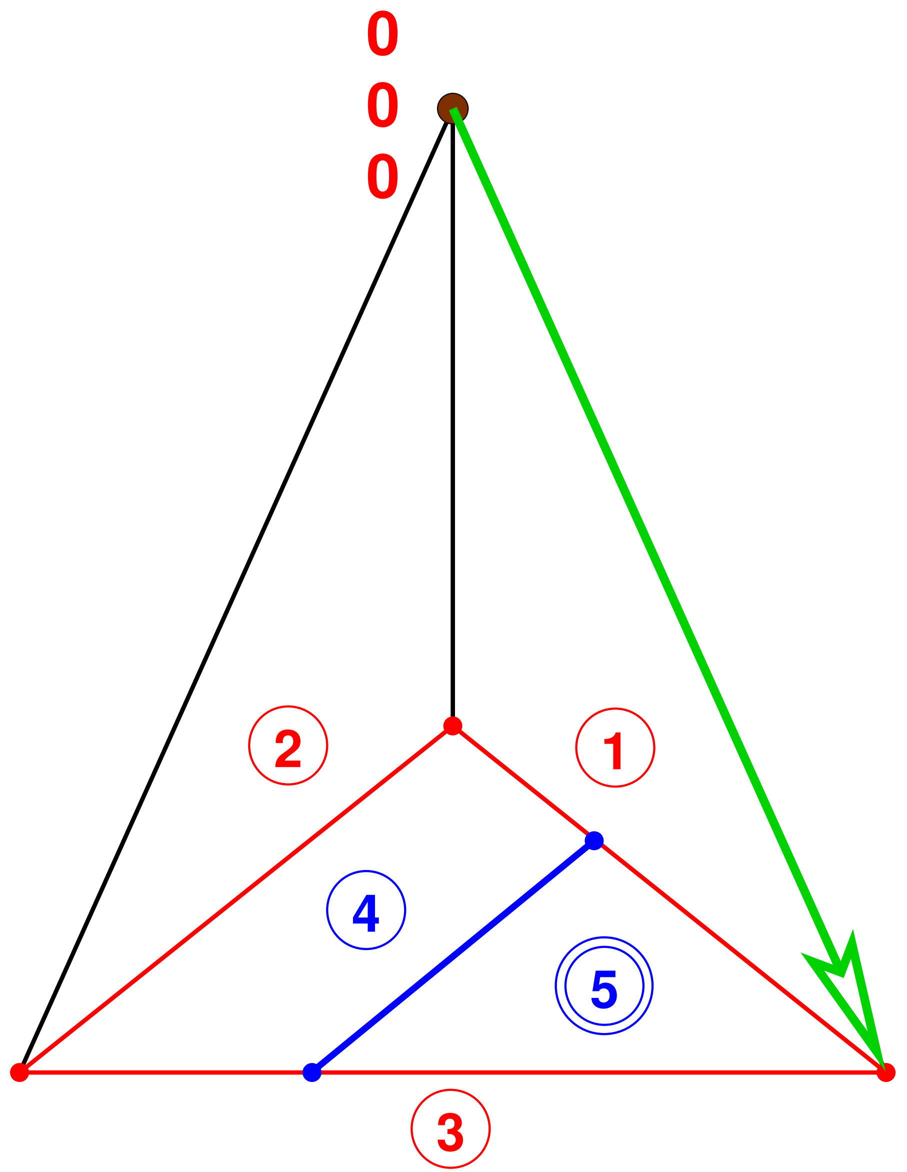
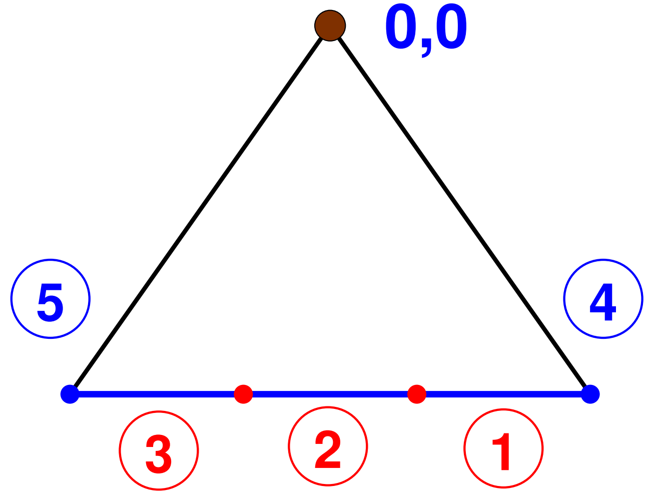
The Lemke–Howson algorithm





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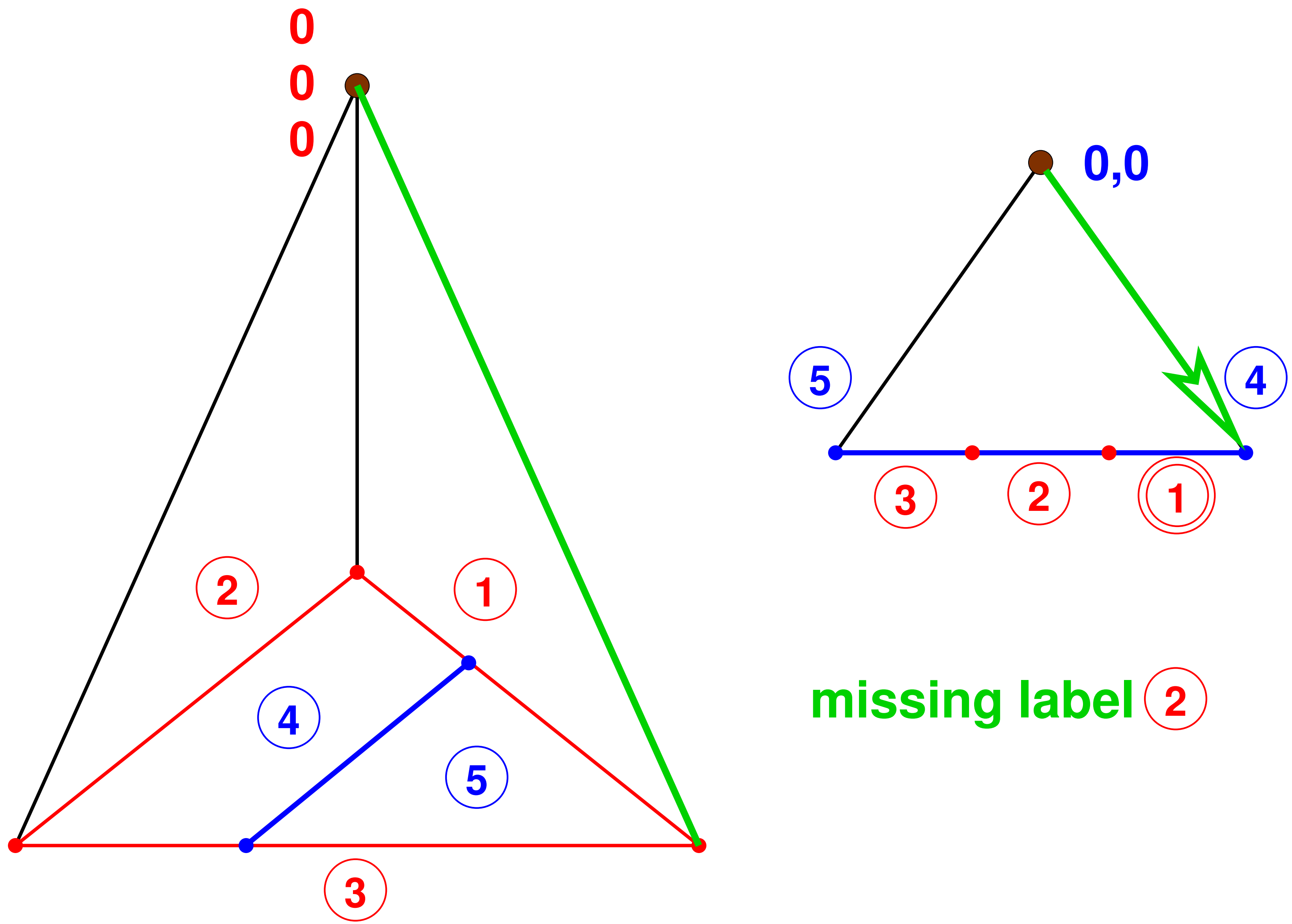

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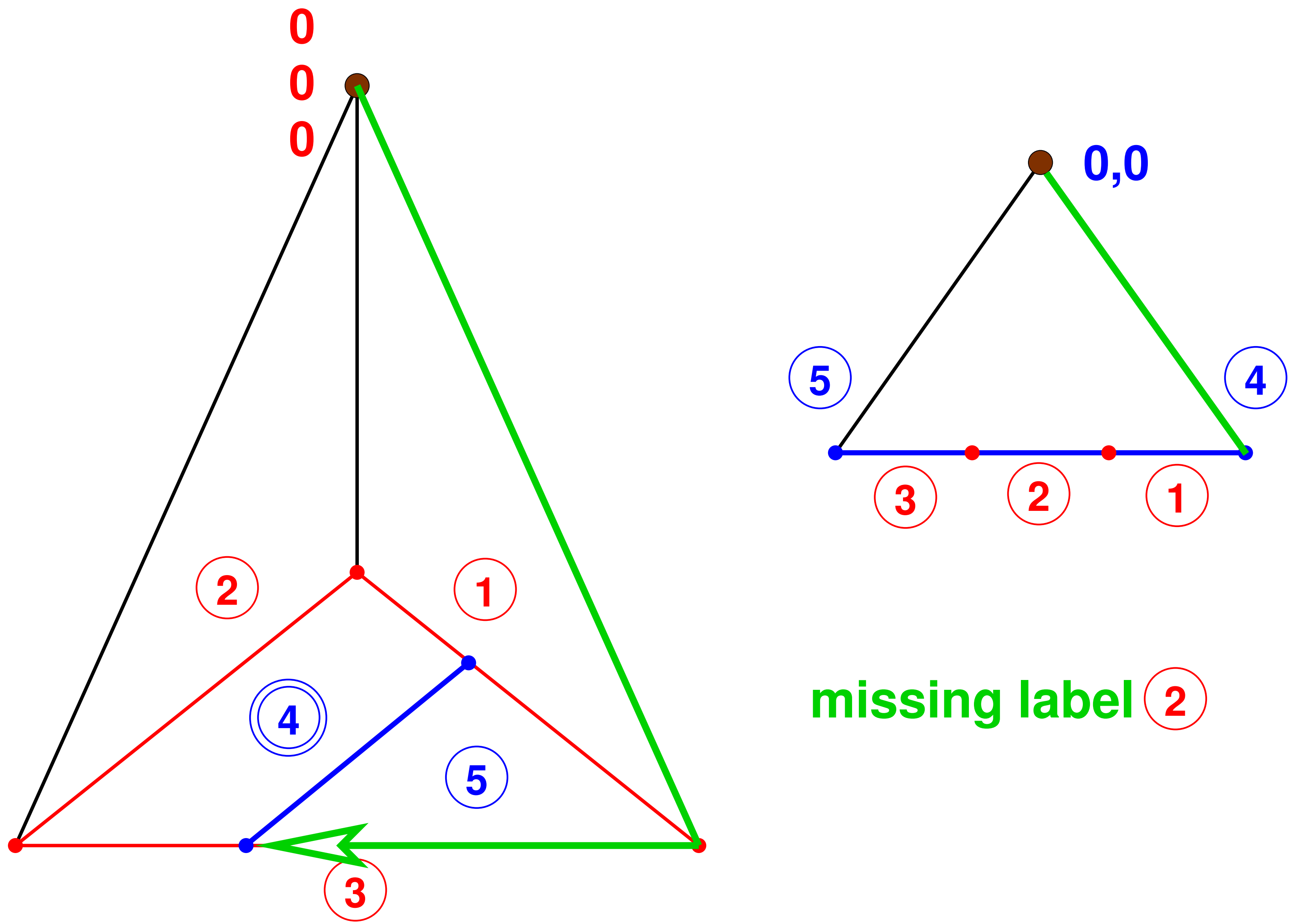
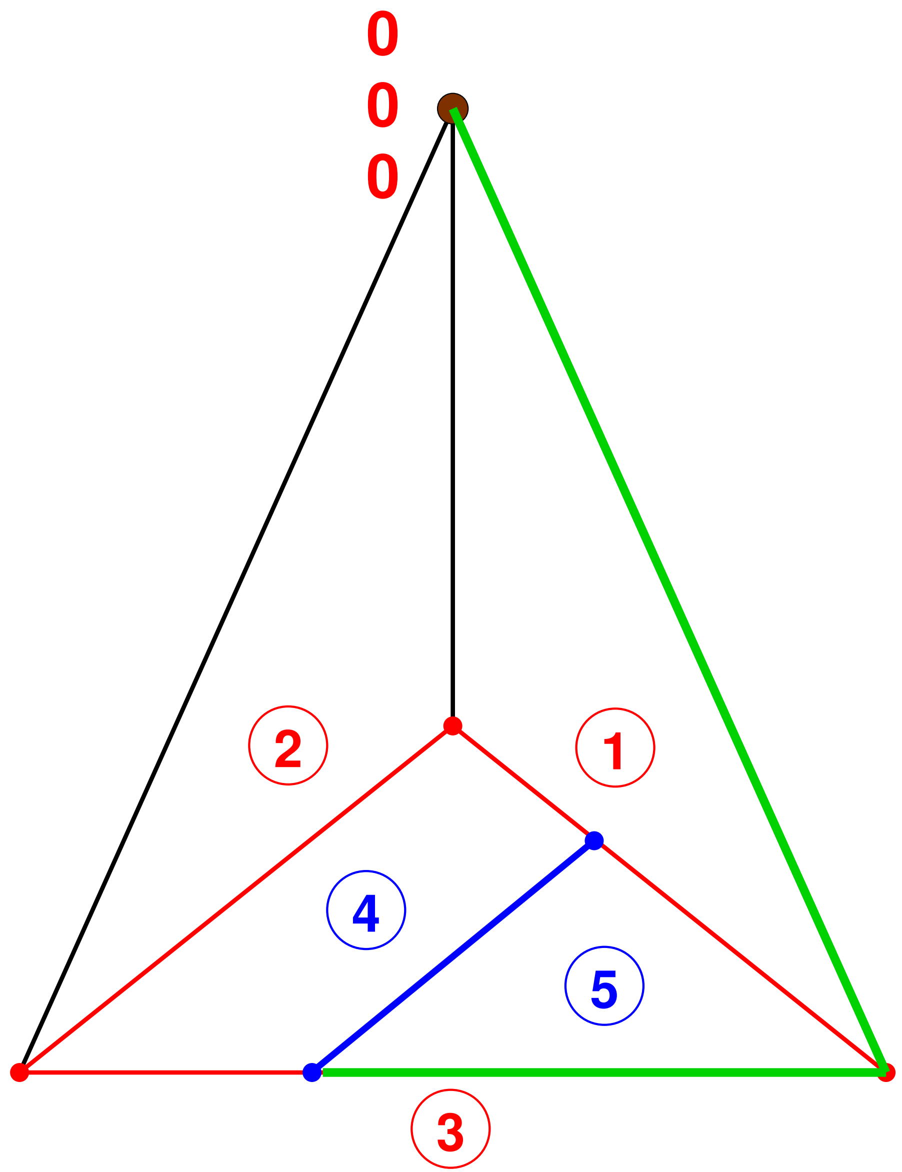
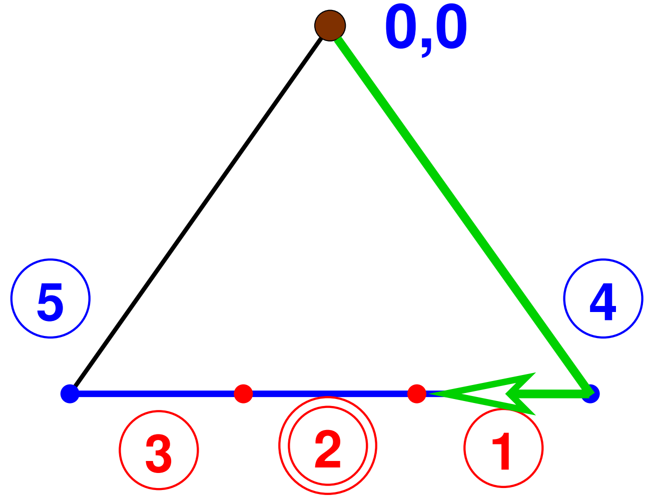


The Lemke–Howson algorithm





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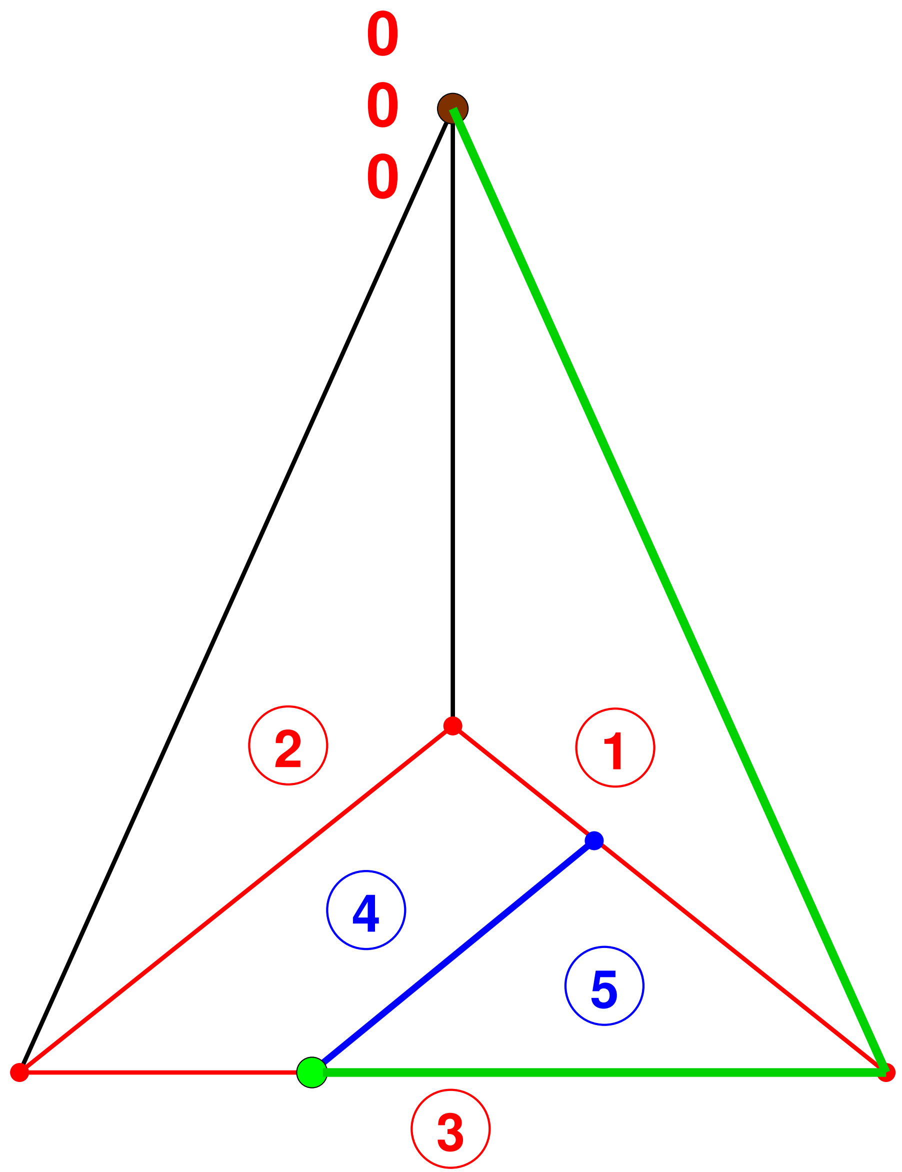

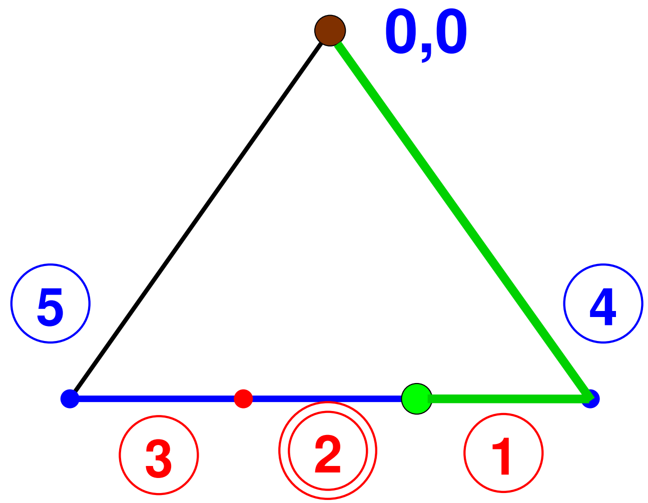

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Why Lemke-Howson works

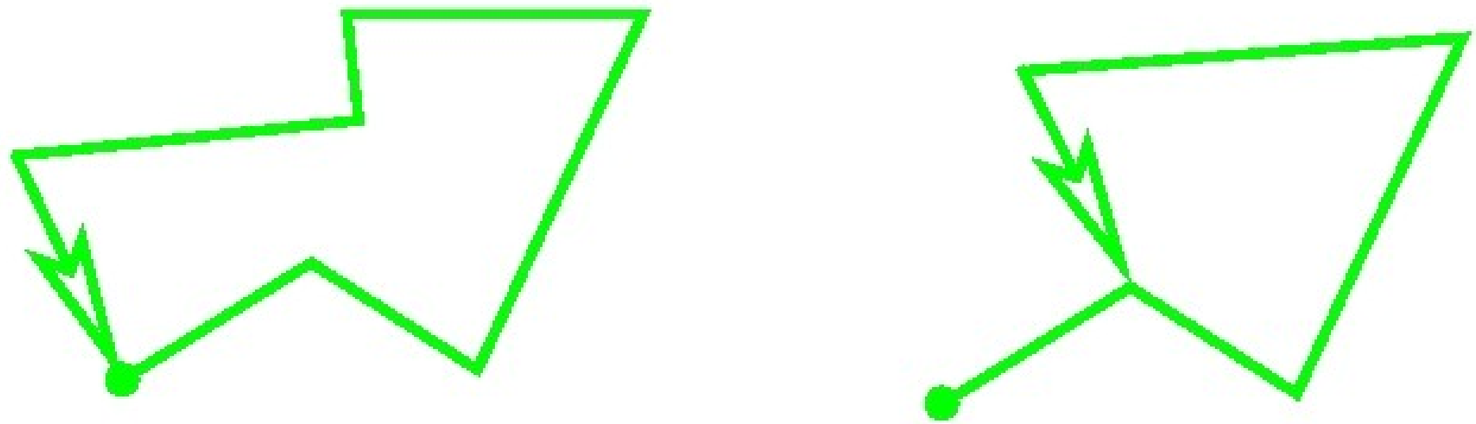
LH finds at least one Nash equilibrium because

- finitely many "vertices"

for nondegenerate (generic) games:

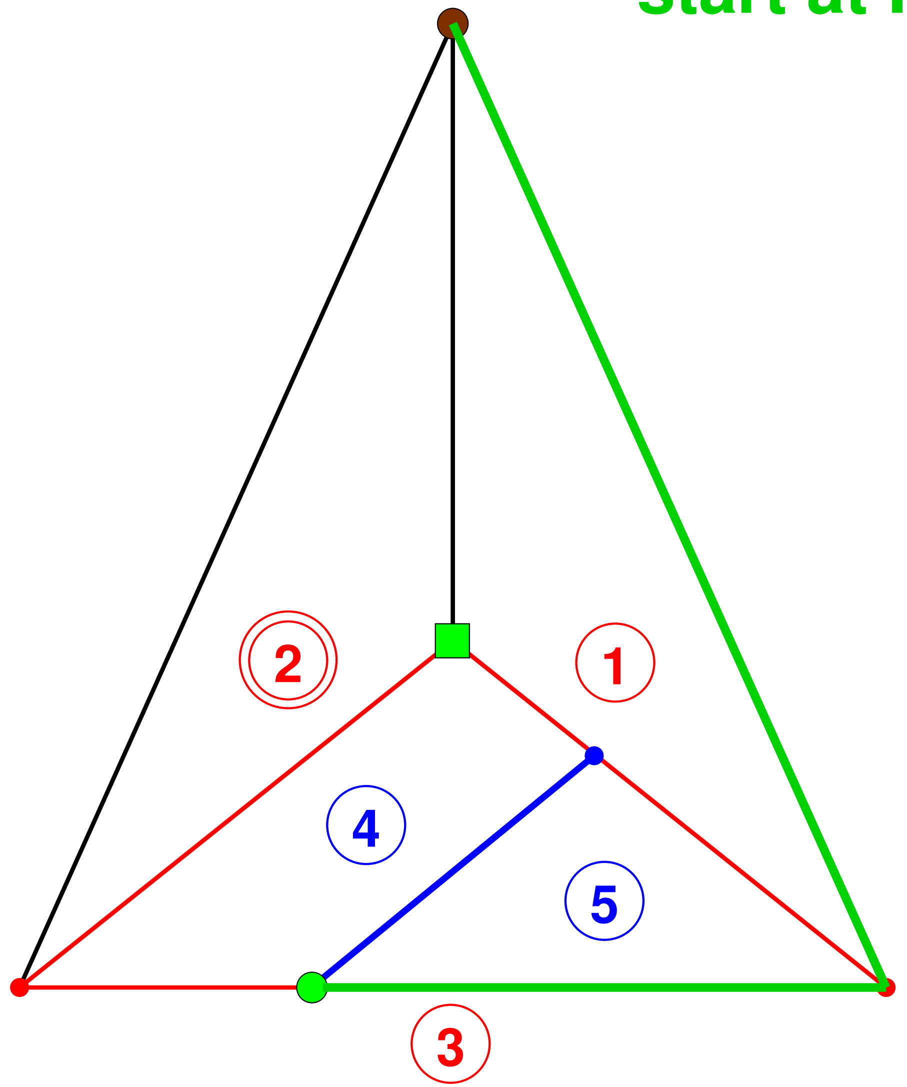
- **unique** starting edge given missing label
- **unique** continuation

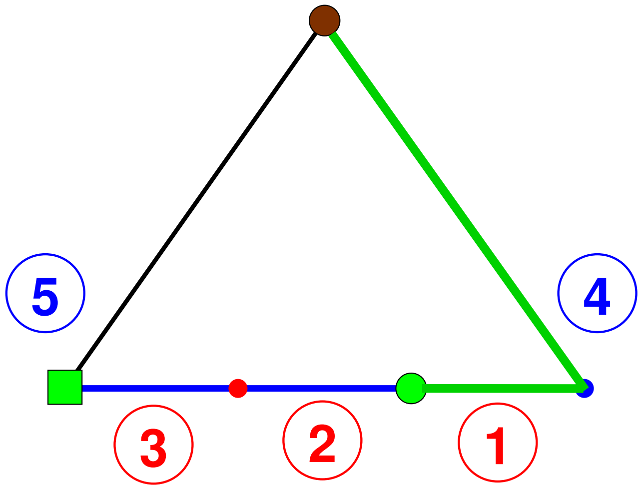
$\Rightarrow$ precludes "coming back" like here:



The Lemke–Howson algorithm

start at Nash equilibrium ■





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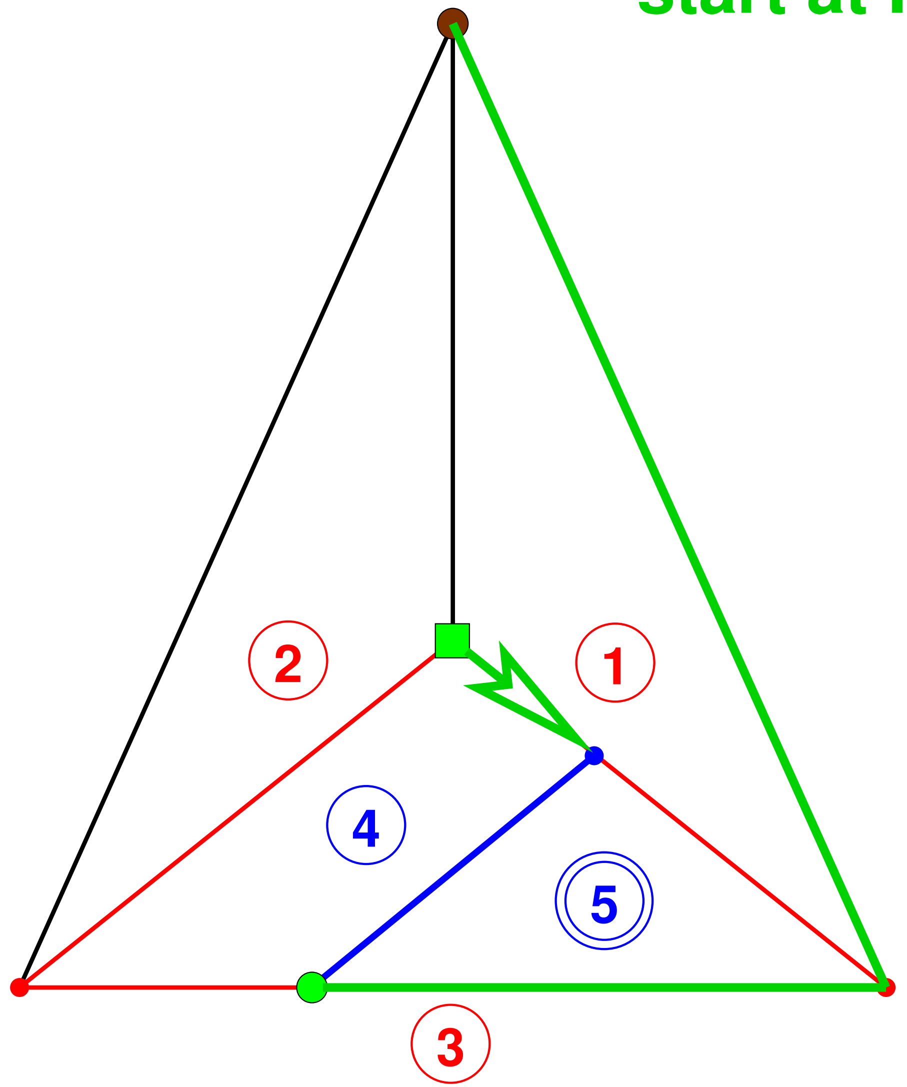
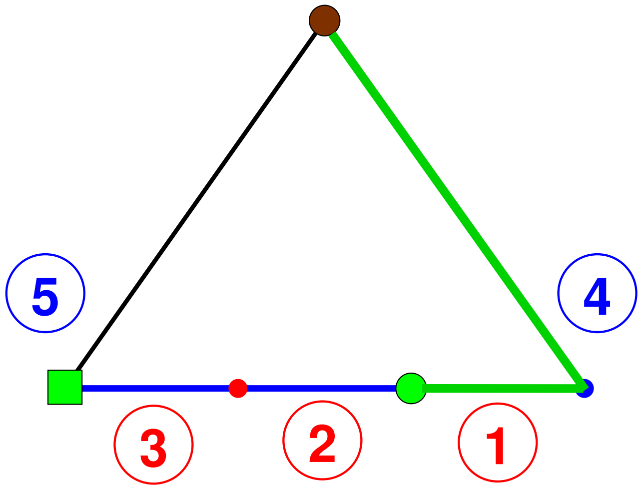
The Lemke–Howson algorithm

start at Nash equilibrium ■





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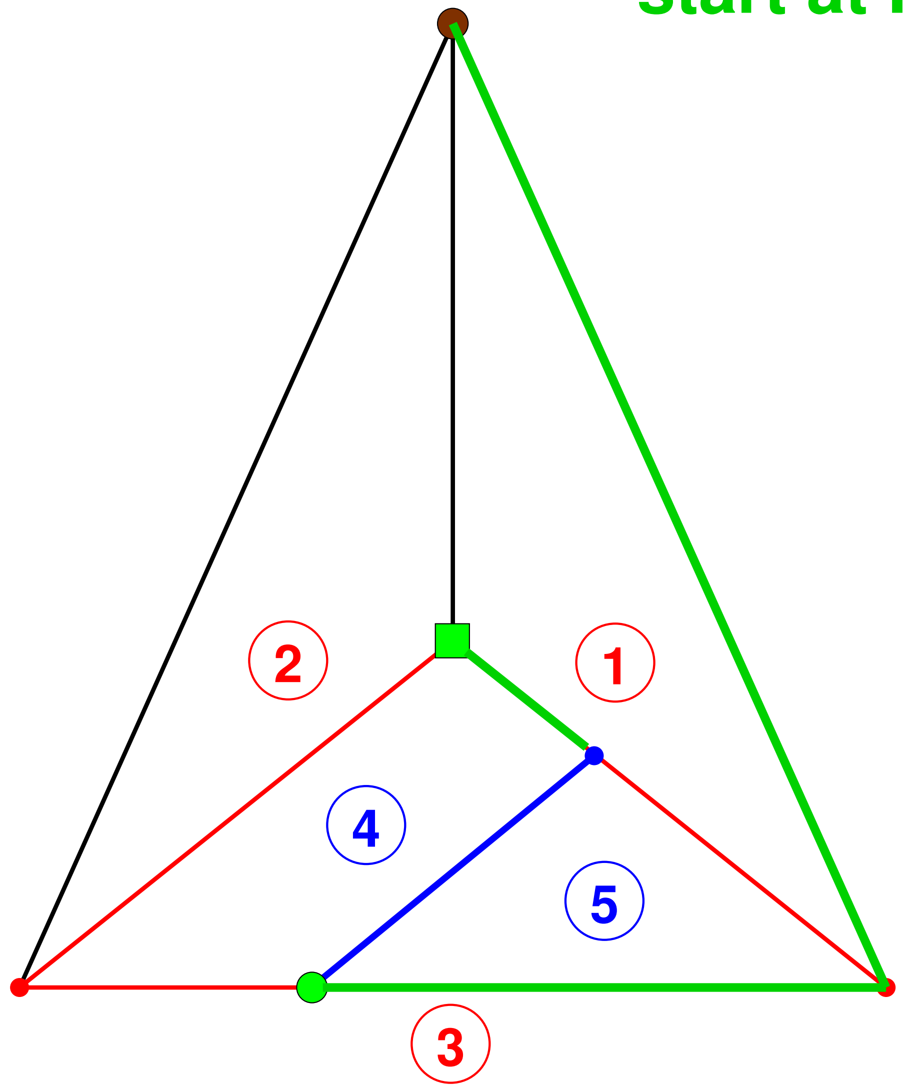
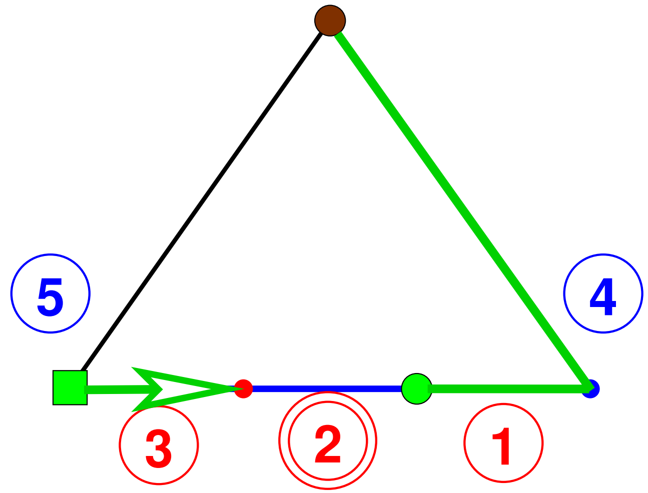
The Lemke–Howson algorithm

start at Nash equilibrium ■

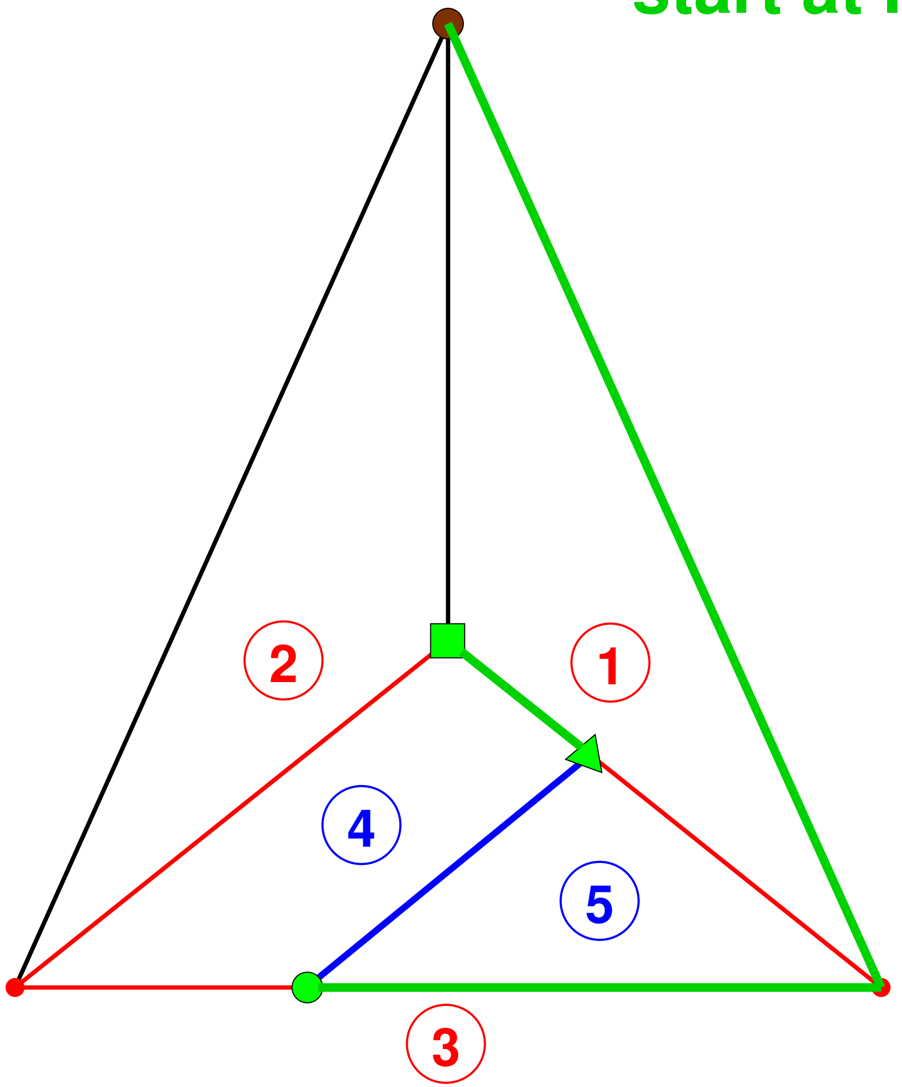


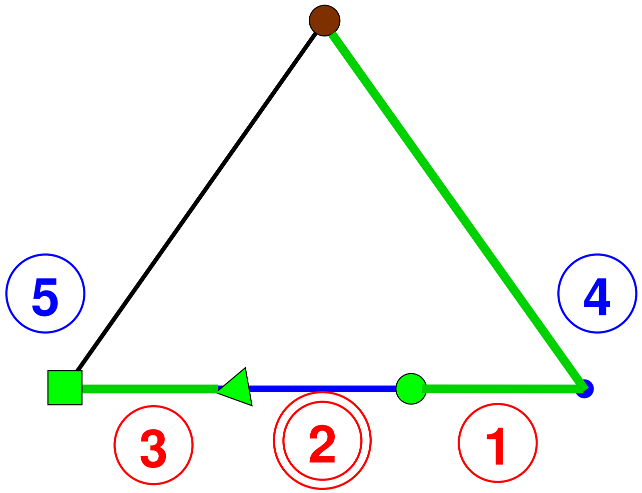


missing label 2

Odd number of Nash equilibria!

start at Nash equilibrium ■





found label 2

Nondegenerate bimatrix games

Given: $m \times n$ bimatrix game (A,B)

$X = \{ x \in \mathbf{R}^m \mid x \geq \mathbf{0},\ x_1 + \ldots + x_m = 1 \}$

$Y = \{ y \in \mathbf{R}^n \mid y \geq \mathbf{0},\ y_1 + \ldots + y_n = 1 \}$

$\mathrm{supp}(x) = \{ i \mid x_i > 0 \}$

$\mathrm{supp}(y) = \{ j \mid y_j > 0 \}$

(A,B) **nondegenerate** $\Leftrightarrow$ $\forall\ x \in X,\ y \in Y$:

$| \{ j \mid j \text{ best response to } x \} | \leq | \mathrm{supp}(x) |,$

$| \{ i \mid i \text{ best response to } y \} | \leq | \mathrm{supp}(y) |.$

Nondegeneracy via labels

$m \times n$ bimatrix game (A,B) **nondegenerate**

$\Leftrightarrow$ no $x \in X$ has more than m labels,
no $y \in Y$ has more than n labels.

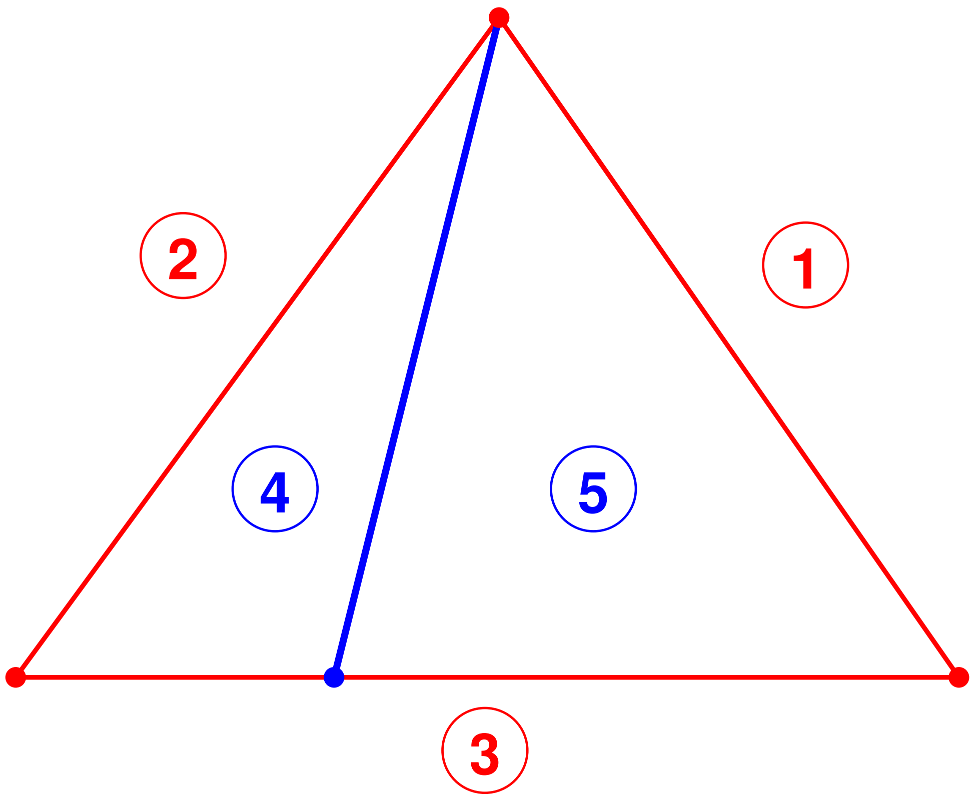
E.g. x with $> m$ labels,
s labels from $\{ 1, \ldots, m \}$,

$\Rightarrow$ $> m-s$ labels from $\{ m+1, \ldots, m+n \}$

$\Leftrightarrow$ $> |\mathrm{supp}(x)|$ best responses to x.

$\Rightarrow$ degenerate.

Example of a degenerate game



	4	5
1	2	1
2	1	3
3	4	4

= B

payoffs to
player II

Handling degenerate games

Lemke–Howson implemented by pivoting, i.e., changing from one *basic feasible solution* of a linear system to another by choosing an entering and a leaving variable.

Choice of entering variable via complementarity (only difference to simplex algorithm for linear programming).

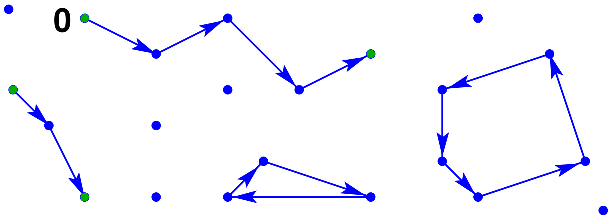
Leaving variable is *unique* in nondegenerate games.

In degenerate games: *perturb* system by adding $(\varepsilon,\ldots,\varepsilon^n)^\top$, creates nondegenerate system.
Implemented *symbolically* by lexicographic rule.

2-player game: find one Nash equilibrium

2-NASH $\in$ PPAD (Polynomial Parity Argument with Direction)

Implicit digraph with indegrees and outdegrees $\leq$ **1** is a set of [nodes], paths and cycles:



Parity argument: number of **sources** of paths = number of **sinks**

Comput. problem: given one source **0**, find another source or sink

[Chen/Deng 2006] 2-NASH is PPAD-complete.

Symmetric Nash equilibria of symmetric games

square game matrix A = payoffs to row player

$$A = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{bmatrix}$$

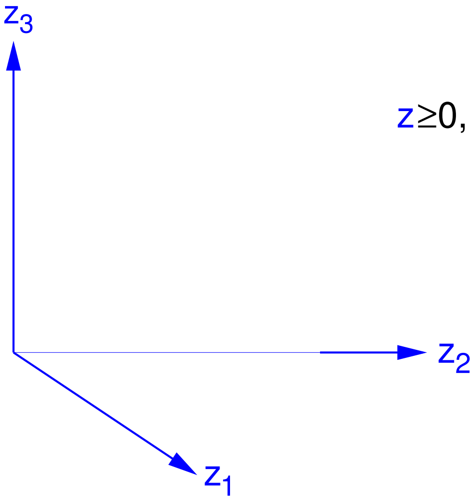
Symmetric Nash equilibria of symmetric games

equilibrium: only optimal strategies are played

	1/3	2/3	**0**	
A =	0	3	0	**2**
	2	2	2	**2**
	3	0	0	1

Symmetric Nash equilibria of symmetric games

plot polytope with strategy weights z_1, z_2, z_3

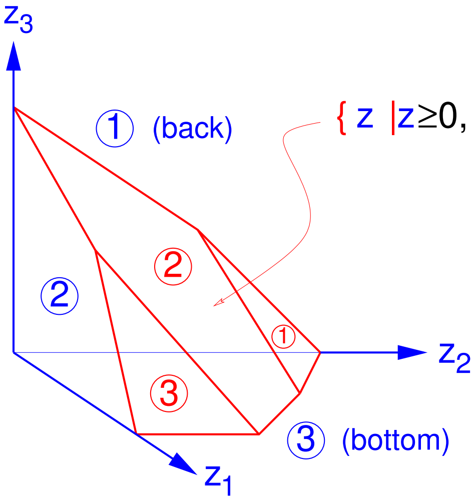


$z \geq 0,$

$$A = \begin{array}{c} \\ \end{array}\begin{array}{ccc} z_1 & z_2 & z_3 \\ \hline 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{array}$$

Symmetric Nash equilibria of symmetric games

with payoffs (scaled to 1) and **labels** for binding inequalities

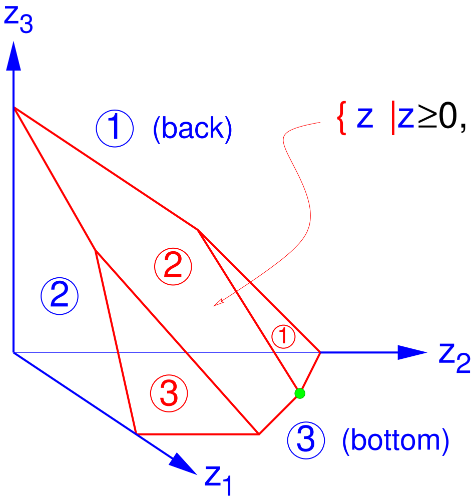


$\{ z \mid z \geq 0,\ Az \leq 1 \}$

$$A = \begin{array}{c} \\ \end{array} \begin{array}{ccc} z_1 & z_2 & z_3 \\ \hline 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{array}$$

Symmetric Nash equilibria of symmetric games

equilibrium = **completely labeled** point

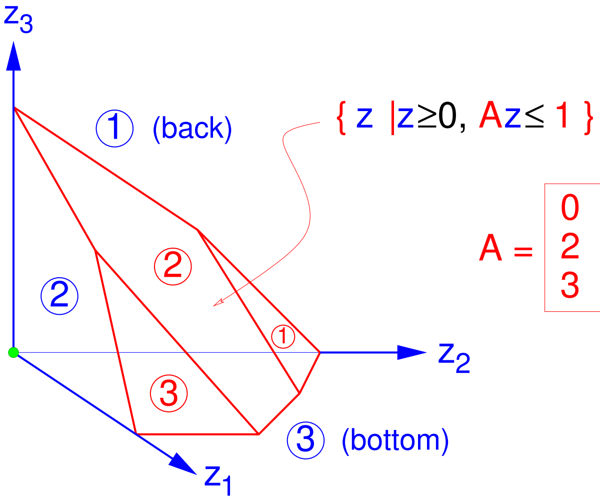


$\{ z \mid z \geq 0, Az \leq 1 \}$

	1/6	1/3	**0**	
$A =$	0	3	0	**1**
	2	2	2	**1**
	3	0	0	1/2

Symmetric Nash equilibria of symmetric games

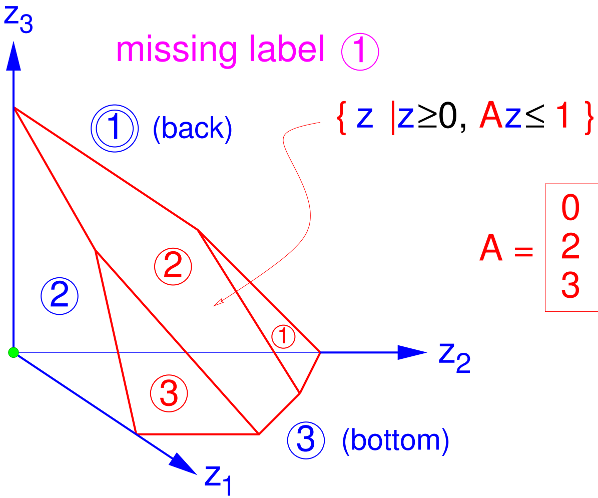
start path with **artificial equilibrium** z=**0**



$$A = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{bmatrix}$$

Symmetric Nash equilibria of symmetric games

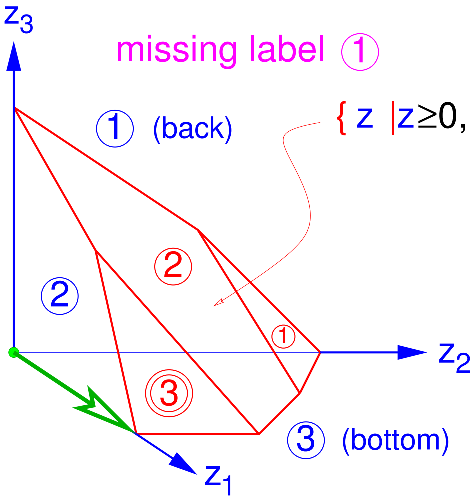
start path with **artificial equilibrium** z=**0**, choose e.g.



$$A = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{bmatrix}$$

Symmetric Nash equilibria of symmetric games

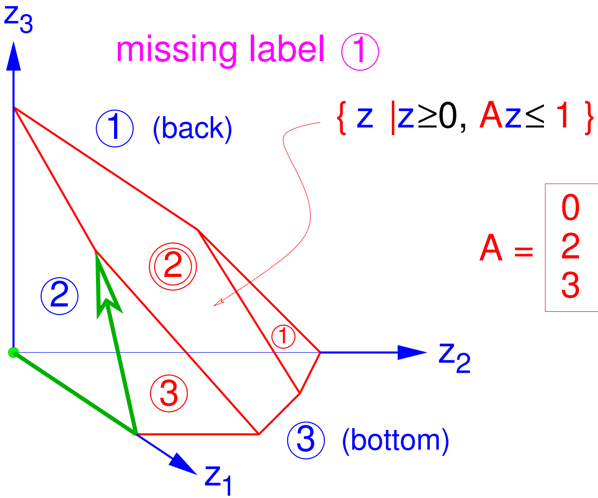
leave facet with label **1**, find duplicate label **3**



$$A = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{bmatrix}$$

Symmetric Nash equilibria of symmetric games

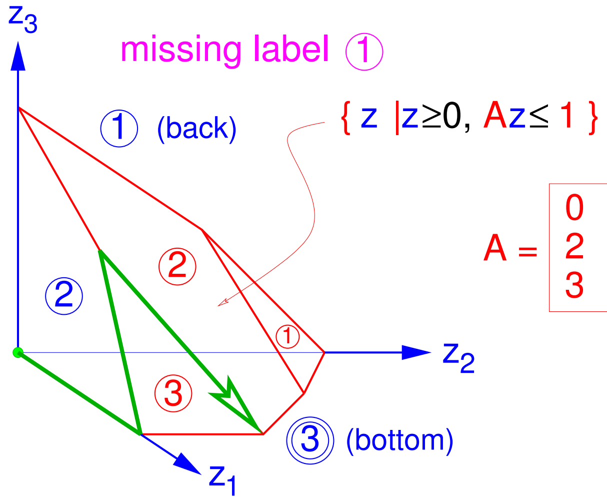
leave facet with old label **3**, find duplicate label **2**



$$A = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{bmatrix}$$

Symmetric Nash equilibria of symmetric games

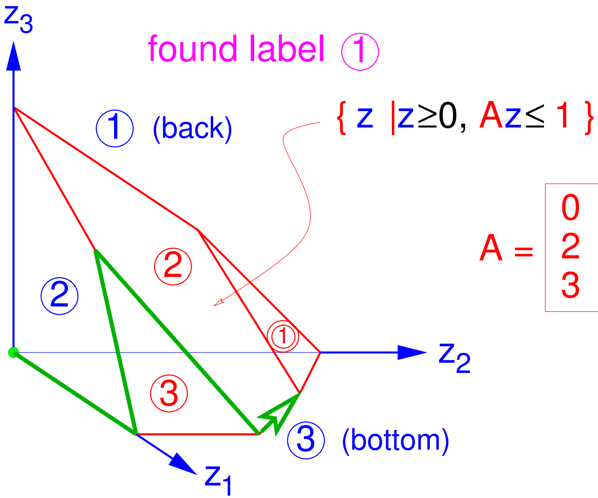
leave facet with old label **2**, find duplicate label **3**



$$A = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{bmatrix}$$

Symmetric Nash equilibria of symmetric games

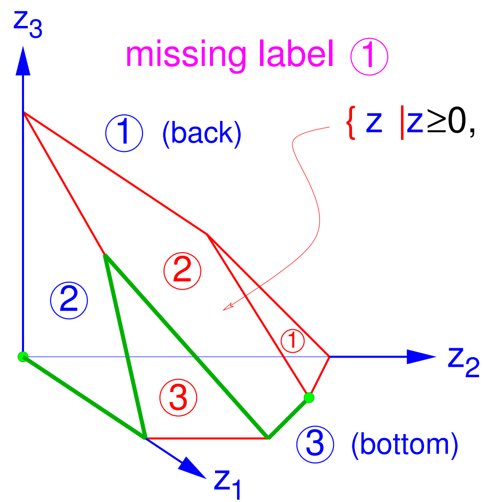
leave facet with old label **3**, find missing label **1**



$$A = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{bmatrix}$$

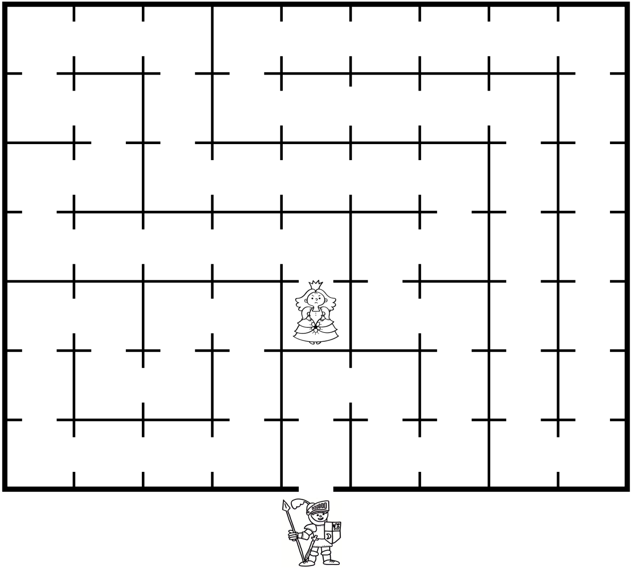
Symmetric Nash equilibria of symmetric games

equilibria (including artificial equilibrium) = endpoints of paths

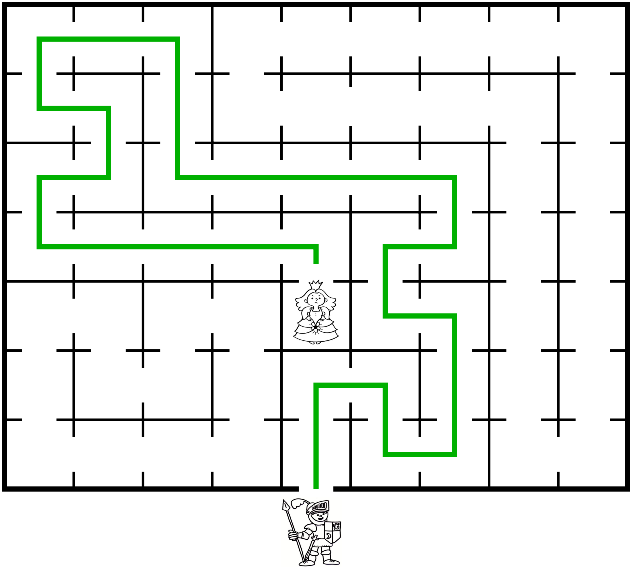


	1/6	1/3	**0**	
A =	0	3	0	**1**
	2	2	2	**1**
	3	0	0	1/2

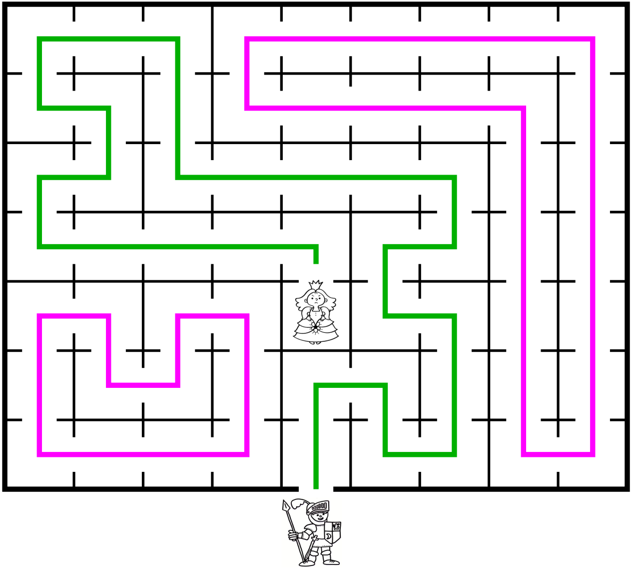
The castle where each room has at most two doors



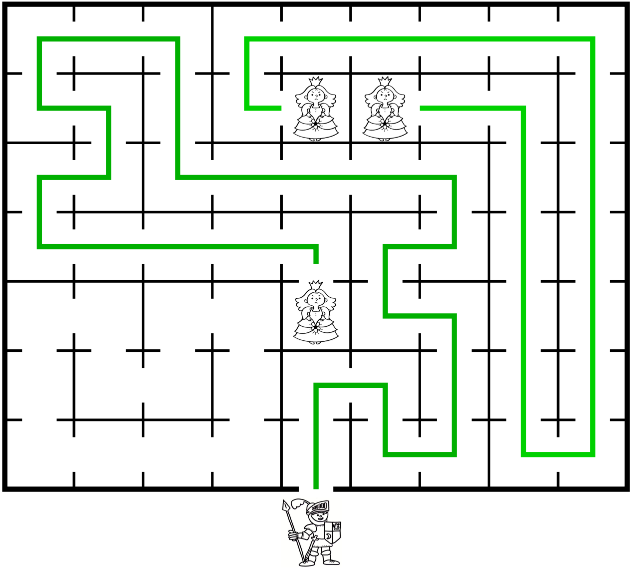
The castle where each room has at most two doors



The castle where each room has at most two doors

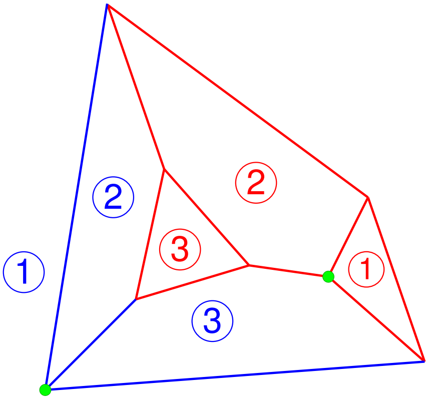


The castle where each room has at most two doors



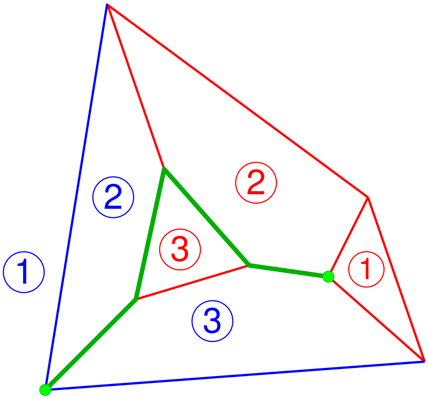
Path of “almost completely labeled” edges

two completely labeled vertices



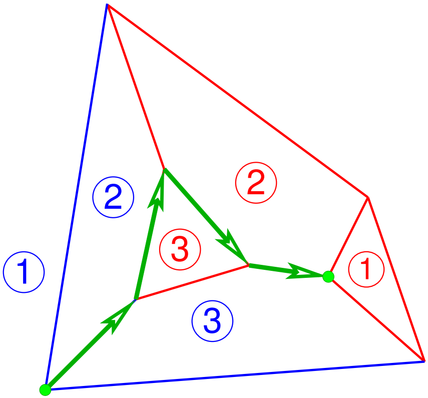
Path of “almost completely labeled” edges

path because at most two neighbours (“doors” in castle)



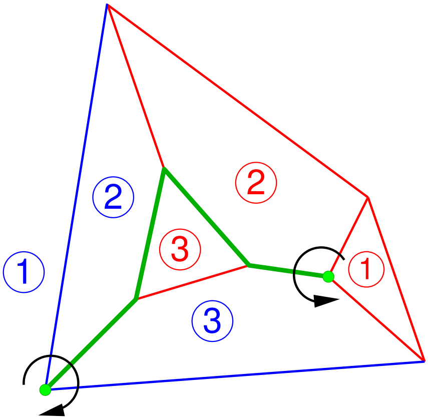
Path of "almost completely labeled" edges

orientation of edges: **2** on left, **3** on right



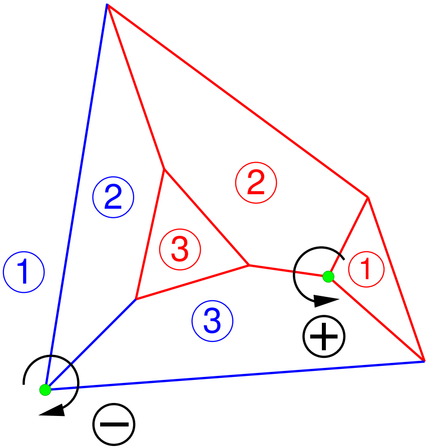
Path of “almost completely labeled” edges

opposite orientation (“sign”) of endpoints



Path of “almost completely labeled” edges

equilibrium **sign** ⊖ or ⊕ does not depend on path



Path of "almost completely labeled" edges

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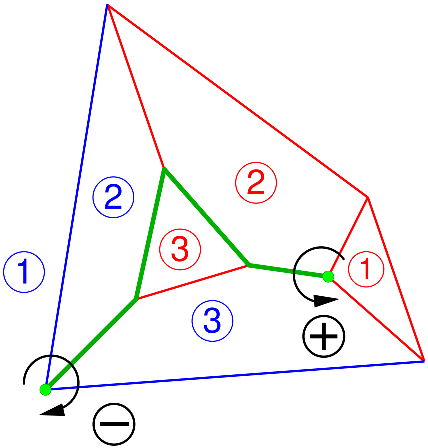

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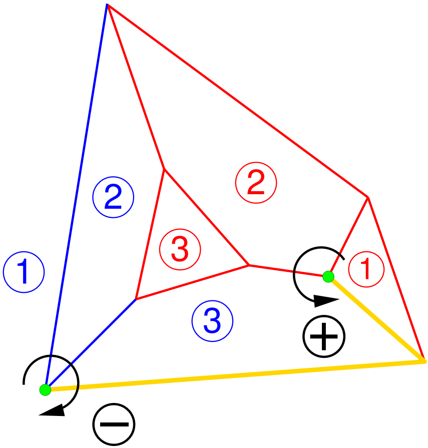

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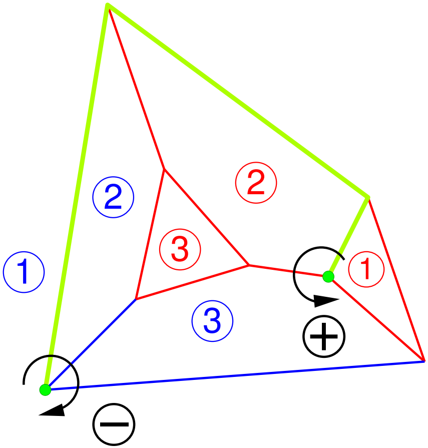
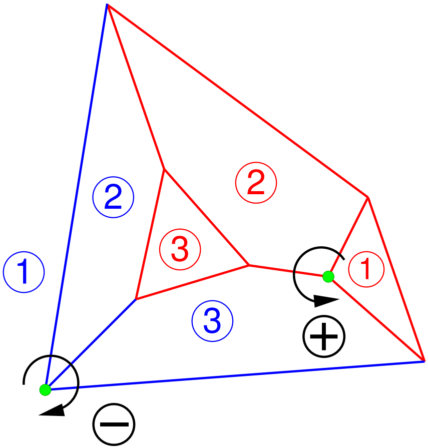


Path of “almost completely labeled” edges

equilibrium **sign** ⊖ or ⊕ does not depend on path



Labeled polytope P

Let $a_j \in \mathbb{R}^m$, $\beta_j \in \mathbb{R}$,

$$P = \{x \in \mathbb{R}^m \mid a_j x \le \beta_j, \;\; 1 \le j \le n\},$$

let **facet** $F_j = \{ x \in P \mid a_j x = \beta_j \}$ have

label $l(j) \in \{1, \ldots, m\}$.

Assume P is a **simple** polytope (no $x \in P$ on $> m$ facets)

$\Rightarrow$ each vertex x on m facets = m linearly independent equations.

x **completely labeled** $\Leftrightarrow \{l(j) \mid x \in F_j\} = \{1, \ldots, m\}$.

Completely labeled points come in pairs

Theorem [Parity Argument]

Let $\boldsymbol{P}$ be a labeled polytope.

Then $\boldsymbol{P}$ has an **even** number of completely labeled vertices.

Completely labeled points come in pairs of opposite sign

Theorem [Parity Argument with Direction]

Let P be a labeled polytope.

Then P has an **even** number of completely labeled vertices.
Half of these have **sign** $\ominus$, half have sign $\oplus$.

Completely labeled points come in pairs of opposite sign

Theorem [Parity Argument with Direction]

Let $\boldsymbol{P}$ be a labeled polytope.

Then $\boldsymbol{P}$ has an **even** number of completely labeled vertices.
Half of these have **sign** ⊖, half have sign ⊕.

---

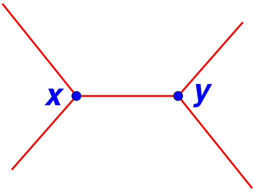
sign of completely labeled $\boldsymbol{x}$ is **sign of determinant** of facet normal vectors in order of their labels: if (e.g.) facet $\boldsymbol{a_i x} = \beta_i$ has label $\boldsymbol{i} = \mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{m}$, then

$$sign(x) = sign\,|a_1\ a_2 \cdots a_m|$$

Pivoting changes signs

Lemma

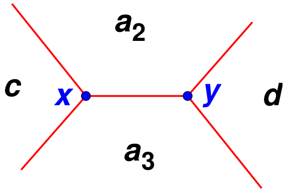
Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{\boldsymbol{m}}$ be adjacent vertices of a simple polytope $\boldsymbol{P}$



Pivoting changes signs

Lemma

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{\boldsymbol{m}}$ be adjacent vertices of a simple polytope $\boldsymbol{P}$ with facet normals $\boldsymbol{c}, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m$ for $\boldsymbol{x}$ and $\boldsymbol{d}, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m$ for $\boldsymbol{y}$.



Pivoting changes signs

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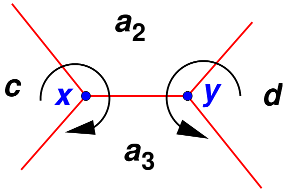
Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{\boldsymbol{m}}$ be adjacent vertices of a simple polytope $\boldsymbol{P}$ with facet normals $\boldsymbol{c}, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m$ for $\boldsymbol{x}$ and $\boldsymbol{d}, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m$ for $\boldsymbol{y}$.

Then $|\boldsymbol{c}\ \boldsymbol{a}_2 \cdots \boldsymbol{a}_m|$ and $|\boldsymbol{d}\ \boldsymbol{a}_2 \cdots \boldsymbol{a}_m|$ have opposite sign.



Pivoting changes signs

Proof :

$$
\begin{array}{cc}
\boldsymbol{c}\boldsymbol{x} = \beta_0 & \\
 & \boldsymbol{d}\boldsymbol{y} = \beta_1 \\
\boldsymbol{a}_2\boldsymbol{x} = \beta_2 & \boldsymbol{a}_2\boldsymbol{y} = \beta_2 \\
\vdots & \vdots \\
\boldsymbol{a}_m\boldsymbol{x} = \beta_m & \boldsymbol{a}_m\boldsymbol{y} = \beta_m
\end{array}
$$

Pivoting changes signs

Proof :

$$
\begin{array}{rcl}
\mathbf{c}\mathbf{x} = \beta_0 & & \\
& & \mathbf{d}\mathbf{y} = \beta_1 \\
\mathbf{a}_2\mathbf{x} = \beta_2 & & \mathbf{a}_2\mathbf{y} = \beta_2 \\
\vdots & & \vdots \\
\mathbf{a}_m\mathbf{x} = \beta_m & & \mathbf{a}_m\mathbf{y} = \beta_m
\end{array}
$$

Let $(\gamma, \delta, \alpha_2, \ldots, \alpha_m) \neq (0, 0, 0, \ldots, 0)$ with

$$\gamma \mathbf{c} + \delta \mathbf{d} + \alpha_2 \mathbf{a}_2 + \cdots + \alpha_m \mathbf{a}_m = \mathbf{0}$$

Pivoting changes signs

Proof :

$$\begin{array}{rcl} cx = \beta_0 & & \\ & & dy = \beta_1 \\ a_2 x = \beta_2 & & a_2 y = \beta_2 \\ \vdots & & \vdots \\ a_m x = \beta_m & & a_m y = \beta_m \end{array}$$

Let $(\gamma, \delta, \alpha_2, \ldots, \alpha_m) \neq (0, 0, 0, \ldots, 0)$ with

$$\gamma c + \delta d + \alpha_2 a_2 + \cdots + \alpha_m a_m = 0$$

$\Rightarrow \quad \gamma \neq 0, \quad \delta \neq 0,$

$$(\gamma c + \delta d) x = (\gamma c + \delta d) y$$

Pivoting changes signs

Proof :

$$\begin{array}{ll} \mathbf{c}\mathbf{x} = \beta_0 & \boxed{\mathbf{c}\mathbf{y} < \beta_0} \\ \boxed{\mathbf{d}\mathbf{x} < \beta_1} & \mathbf{d}\mathbf{y} = \beta_1 \\ \mathbf{a}_2\mathbf{x} = \beta_2 & \mathbf{a}_2\mathbf{y} = \beta_2 \\ \vdots & \vdots \\ \mathbf{a}_m\mathbf{x} = \beta_m & \mathbf{a}_m\mathbf{y} = \beta_m \end{array}$$

Let $(\gamma, \delta, \alpha_2, \ldots, \alpha_m) \neq (0, 0, 0, \ldots, 0)$ with

$$\gamma \mathbf{c} + \delta \mathbf{d} + \alpha_2 \mathbf{a}_2 + \cdots + \alpha_m \mathbf{a}_m = 0$$

$\Rightarrow \quad \gamma \neq 0, \quad \delta \neq 0,$

$$(\gamma \mathbf{c} + \delta \mathbf{d})\mathbf{x} = (\gamma \mathbf{c} + \delta \mathbf{d})\mathbf{y}, \qquad \gamma(\mathbf{c}\mathbf{x} - \mathbf{c}\mathbf{y}) = \delta(\mathbf{d}\mathbf{y} - \mathbf{d}\mathbf{x})$$

Pivoting changes signs

Proof :

$$\begin{array}{cc} cx = \beta_0 & \boxed{cy < \beta_0} \\ \boxed{dx < \beta_1} & dy = \beta_1 \\ a_2x = \beta_2 & a_2y = \beta_2 \\ \vdots & \vdots \\ a_mx = \beta_m & a_my = \beta_m \end{array}$$

Let $(\gamma, \delta, \alpha_2, \ldots, \alpha_m) \neq (0, 0, 0, \ldots, 0)$ with

$$\gamma c + \delta d + \alpha_2 a_2 + \cdots + \alpha_m a_m = 0$$

$\Rightarrow \quad \gamma \neq 0, \quad \delta \neq 0,$

$$(\gamma c + \delta d)x = (\gamma c + \delta d)y, \qquad \gamma(cx - cy) = \delta(dy - dx)$$

$\Rightarrow \quad \gamma$ and δ have same sign

Pivoting changes signs

Proof :

$$\begin{array}{ll} cx = \beta_0 & \boxed{cy < \beta_0} \\ \boxed{dx < \beta_1} & dy = \beta_1 \\ a_2 x = \beta_2 & a_2 y = \beta_2 \\ \vdots & \vdots \\ a_m x = \beta_m & a_m y = \beta_m \end{array}$$

Let $(\gamma, \delta, \alpha_2, \ldots, \alpha_m) \neq (0, 0, 0, \ldots, 0)$ with

$$\gamma c + \delta d + \alpha_2 a_2 + \cdots + \alpha_m a_m = 0$$

$\Rightarrow \quad \gamma \neq 0, \quad \delta \neq 0,$

$$(\gamma c + \delta d) x = (\gamma c + \delta d) y, \qquad \gamma (cx - cy) = \delta (dy - dx)$$

$\Rightarrow \quad \gamma$ and δ have same sign,

$$|(\gamma c + \delta d)\ a_2 \cdots a_m| = \gamma\ |c\ a_2 \cdots a_m| + \delta\ |d\ a_2 \cdots a_m| = 0$$

Pivoting changes signs

Proof :

$$\begin{array}{cc} c\mathbf{x} = \beta_0 & \boxed{c\mathbf{y} < \beta_0} \\ \boxed{d\mathbf{x} < \beta_1} & d\mathbf{y} = \beta_1 \\ a_2\mathbf{x} = \beta_2 & a_2\mathbf{y} = \beta_2 \\ \vdots & \vdots \\ a_m\mathbf{x} = \beta_m & a_m\mathbf{y} = \beta_m \end{array}$$

Let $(\gamma, \delta, \alpha_2, \ldots, \alpha_m) \neq (0, 0, 0, \ldots, 0)$ with

$$\gamma c + \delta d + \alpha_2 a_2 + \cdots + \alpha_m a_m = 0$$

$\Rightarrow \quad \gamma \neq 0, \quad \delta \neq 0,$

$$(\gamma c + \delta d)\mathbf{x} = (\gamma c + \delta d)\mathbf{y}, \qquad \gamma(c\mathbf{x} - c\mathbf{y}) = \delta(d\mathbf{y} - d\mathbf{x})$$

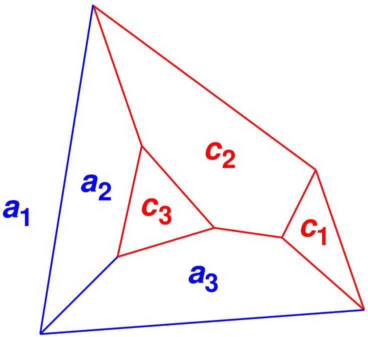
$\Rightarrow \quad \gamma$ and δ have same sign,

$$|(\gamma c + \delta d)\ a_2 \cdots a_m| = \gamma\ |c\ a_2 \cdots a_m| + \delta\ |d\ a_2 \cdots a_m| = 0$$

$\Rightarrow \quad |c\ a_2 \cdots a_m|$ and $|d\ a_2 \cdots a_m|$ have opposite sign, QED.

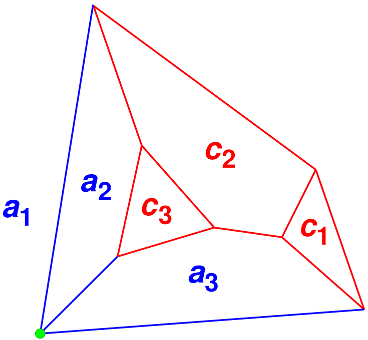
General Parity Argument with Direction

Facet normal vectors a_1 a_2 a_3 c_1 c_2 c_3, labels **1 2 3 1 2 3**



General Parity Argument with Direction

Start with a_1 a_2 a_3, sign ⊖

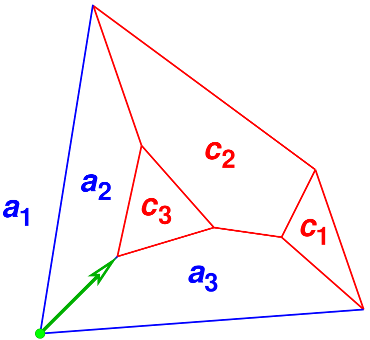


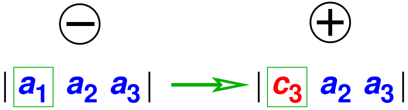
⊖

$| \boxed{a_1}\ a_2\ a_3 |$

General Parity Argument with Direction

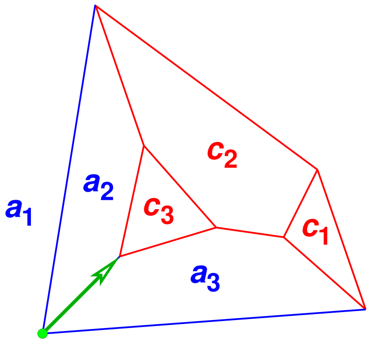
Start with a_1 a_2 a_3, sign ⊖, label **1** missing, $a_1 \to c_3$ gives sign ⊕

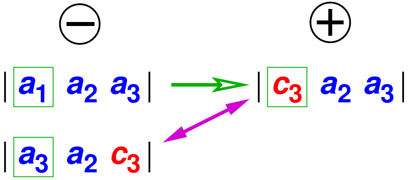




General Parity Argument with Direction

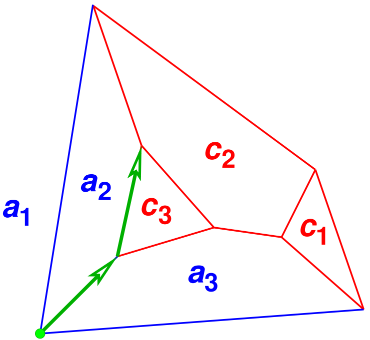
Switch columns c_3 and a_3 in determinant: back to sign ⊖

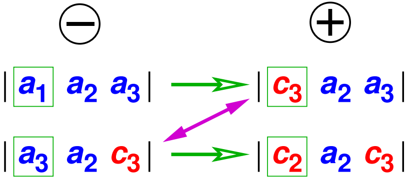




General Parity Argument with Direction

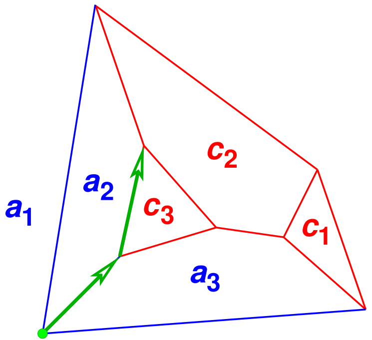
next pivot $a_3 \rightarrow c_2$ gives sign $\oplus$

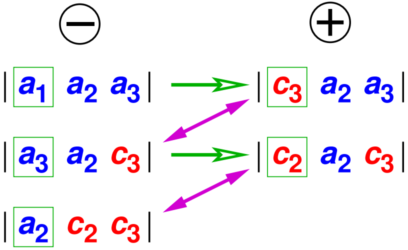




General Parity Argument with Direction

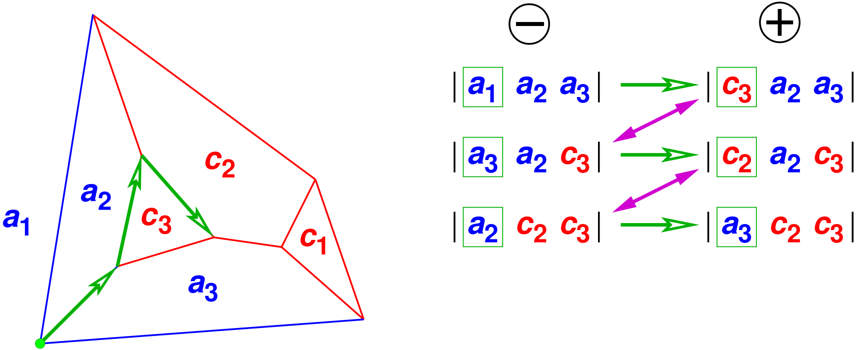
Switch columns c_2 and a_2 in determinant: back to sign ⊖





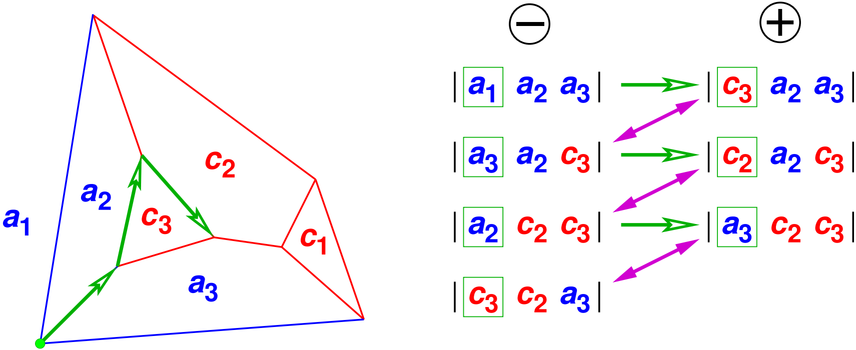
General Parity Argument with Direction

next pivot $a_2 \rightarrow a_3$ gives sign $\oplus$



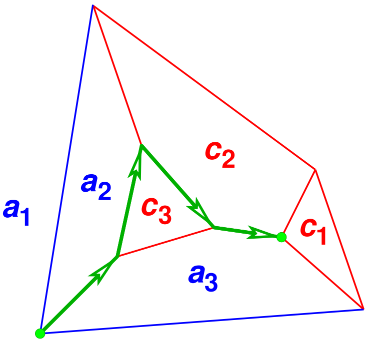
General Parity Argument with Direction

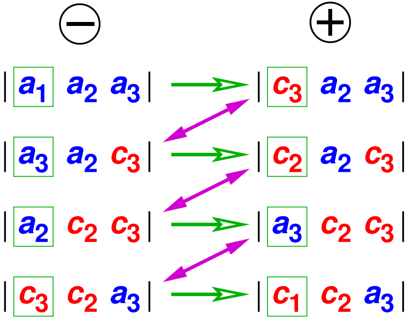
Switch columns a_3 and c_3 in determinant: back to sign ⊖



General Parity Argument with Direction

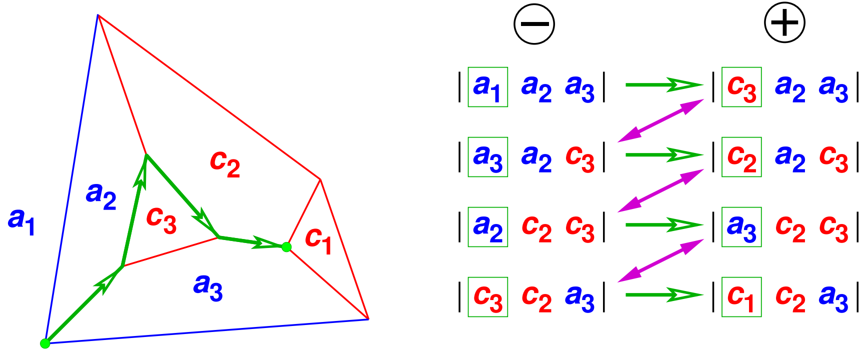
Last pivot $c_3 \rightarrow c_1$ gives sign $\oplus$, opposite to starting sign $\ominus$.





General Parity Argument with Direction

Only need: sign-switching of **pivots** and **column exchanges**



A more abstract example

$\ominus$

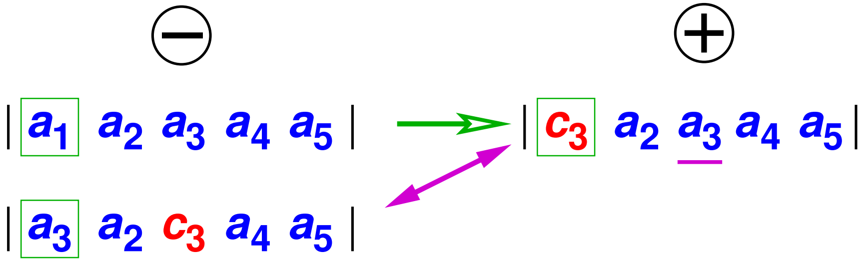
$| \boxed{a_1}\ a_2\ a_3\ a_4\ a_5 |$

A more abstract example

$\ominus$ $\oplus$

$$| \boxed{a_1}\ a_2\ a_3\ a_4\ a_5 | \longrightarrow | \boxed{c_3}\ a_2\ \underline{a_3}\ a_4\ a_5 |$$

A more abstract example



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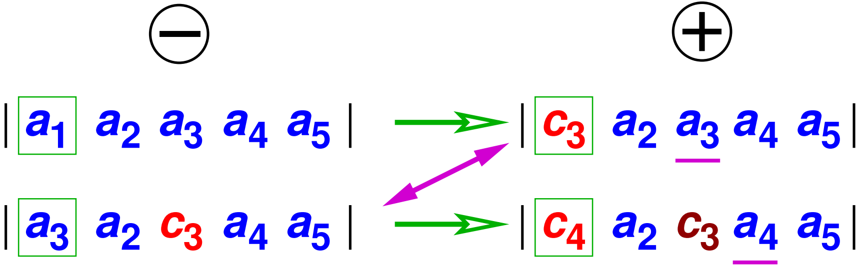
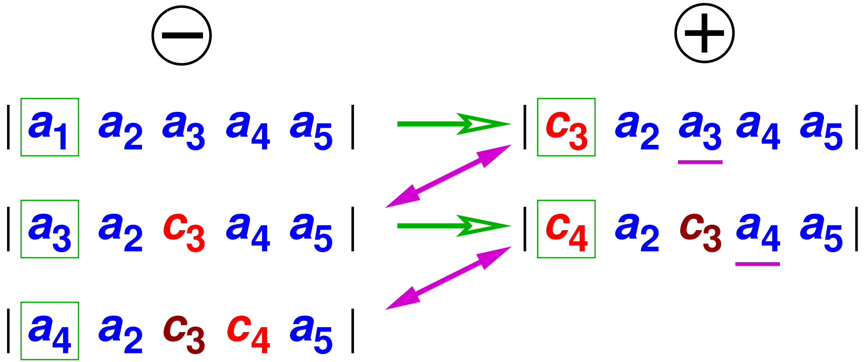
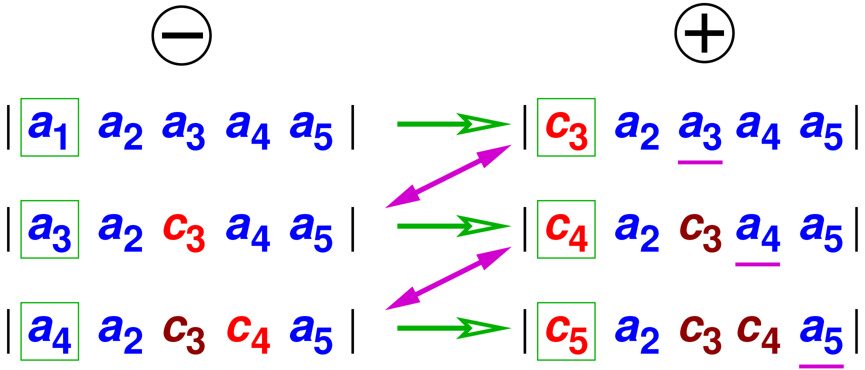


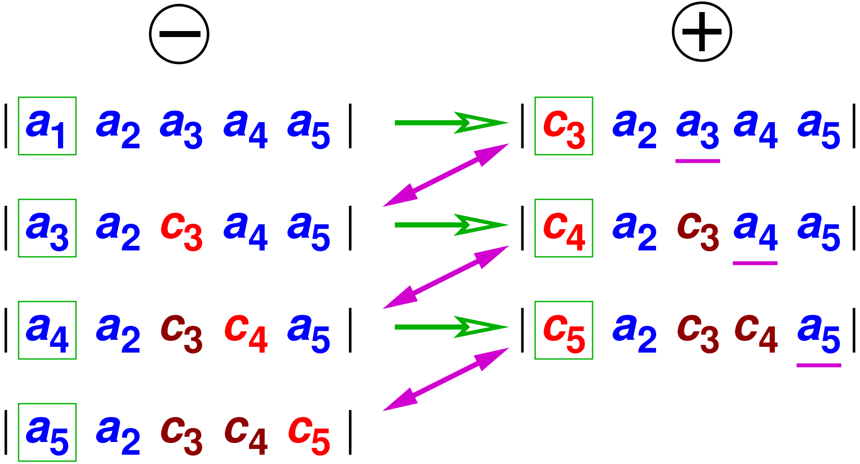
A more abstract example



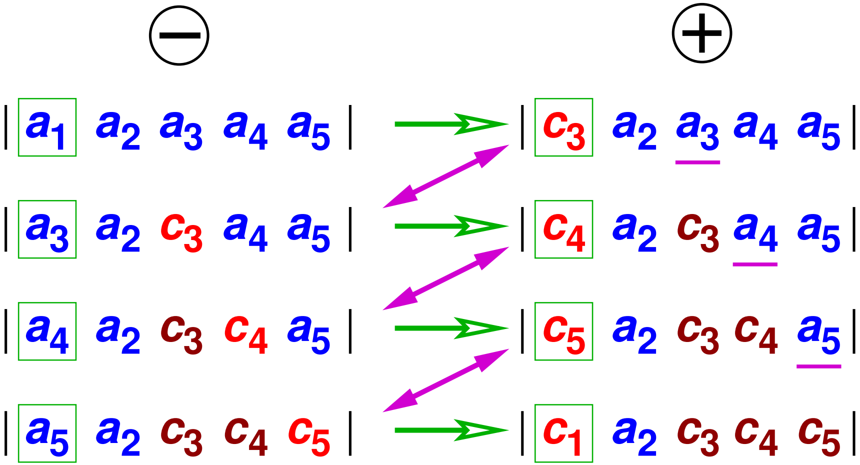
A more abstract example



A more abstract example



A more abstract example



Nash equilibria of bimatrix games

Recall: $m \times m$ matrix C,

$$P = \{ z \in \mathbb{R}^m \mid -z \leq \mathbf{0}, \quad Cz \leq \mathbf{1} \}$$

with $2m$ inequalities labeled $1, \ldots, m, 1, \ldots, m$.

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Completely labeled $z \neq \mathbf{0} \quad \Leftrightarrow$

Nash equilibrium (z, z) of game $(C, C^\top)$

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Nash equilibrium (z, z) of game $(C, C^\top)$

Normalize sign of "artificial equilibrium" **0** to ⊖, in general

$$index(z) = sign(z) \cdot (-1)^{m+1}$$

Nash equilibria of bimatrix games

Recall: $m \times m$ matrix C,

$$P = \{z \in \mathbb{R}^m \mid -z \le \mathbf{0}, \;\; Cz \le \mathbf{1}\}$$

with $2m$ inequalities labeled $1, \ldots, m, 1, \ldots, m$.

bimatrix game (A, B):

$$C = \begin{pmatrix} 0 & A \\ B^\top & 0 \end{pmatrix}, \quad z = (x, y):$$

Completely labeled $(x, y) \neq (\mathbf{0}, \mathbf{0}) \quad \Leftrightarrow$

Nash equilibrium (x, y) of game (A, B)

Index of an equilibrium

Theorem [Shapley 1974]

A nondegenerate bimatrix game (A, B) has an odd number of equilibria, one more of index $\oplus$ than of index $\ominus$.

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[*Proof:* Endpoints of pivoting paths have opposite index $\ominus$ and $\oplus$.]

Index of an equilibrium

Theorem [Shapley 1974]

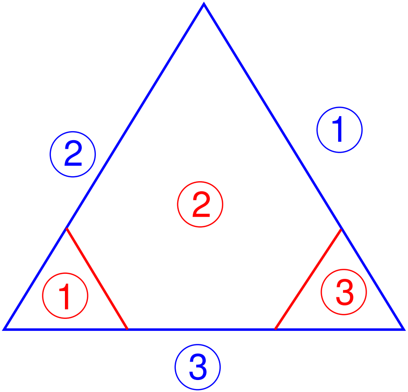
A nondegenerate bimatrix game **(*A*, *B*)** has an odd number of equilibria, one more of index ⊕ than of index ⊖.

[*Proof:* Endpoints of pivoting paths have opposite index ⊖ and ⊕.]

Equilibria of index ⊕ include every

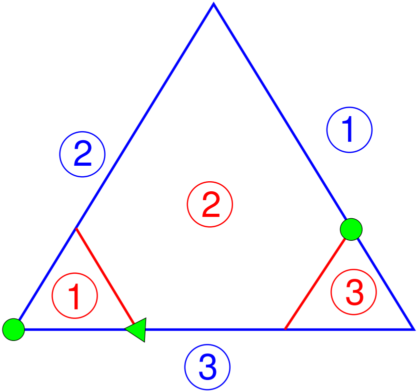
- pure-strategy equilibrium
- unique equilibrium
- **dynamically stable** equilibrium [Hofbauer 2003]

Dynamically stable equilibrium: only if index ⊕



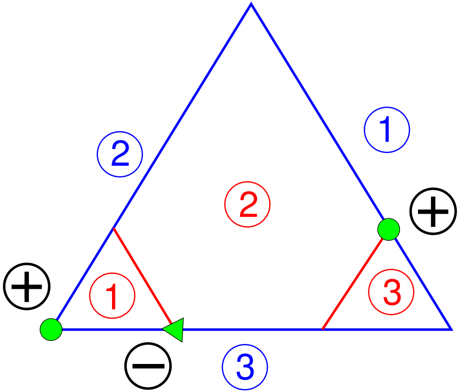
	1	2	3
1	**3**	**0**	**0**
2	**2**	**2**	**2**
3	**0**	**3**	**0**

Dynamically stable equilibrium: only if index ⊕



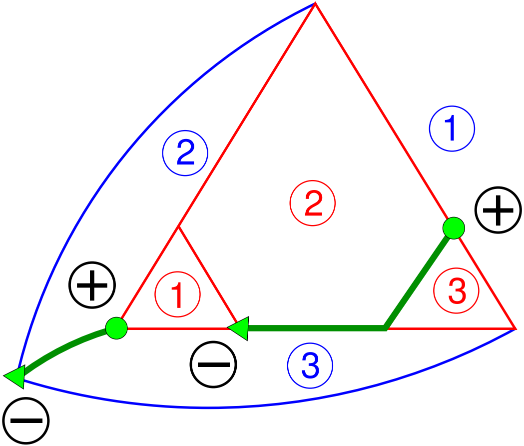
	1	2	3
1	**3**	**0**	**0**
2	**2**	**2**	**2**
3	**0**	**3**	**0**

Dynamically stable equilibrium: only if index $\oplus$



	1	2	3
1	**3**	**0**	**0**
2	**2**	**2**	**2**
3	**0**	**3**	**0**

Dynamically stable equilibrium: only if index ⊕



	1	2	3
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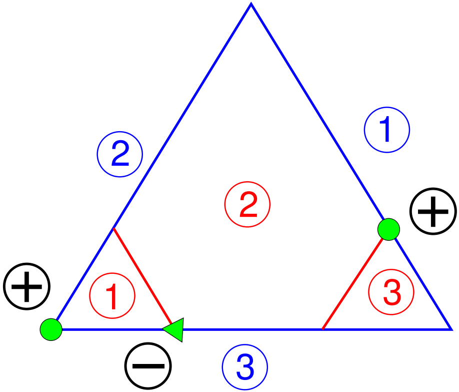

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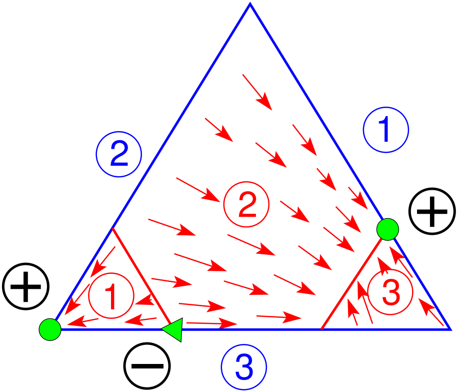

	1	2	3
1	**3**	**0**	**0**
2	**2**	**2**	**2**
3	**0**	**3**	**0**

Strategic characterization of the index

Theorem [von Schemde / von Stengel 2004]

An equilibrium of a nondegenerate bimatrix game has index $\oplus$

$\Leftrightarrow$ it is the **unique** equilibrium in a larger game that has suitable additional strategies for one player.

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Theorem [Balthasar / von Stengel 2009]

A *symmetric* equilibrium of a nondegenerate *symmetric* bimatrix game has *symmetric* index $\oplus$

$\Leftrightarrow$ it is the **unique** equilibrium in a larger *symmetric* game that has suitable additional strategies for both players.