Efficient computation of equilibria for extensive games

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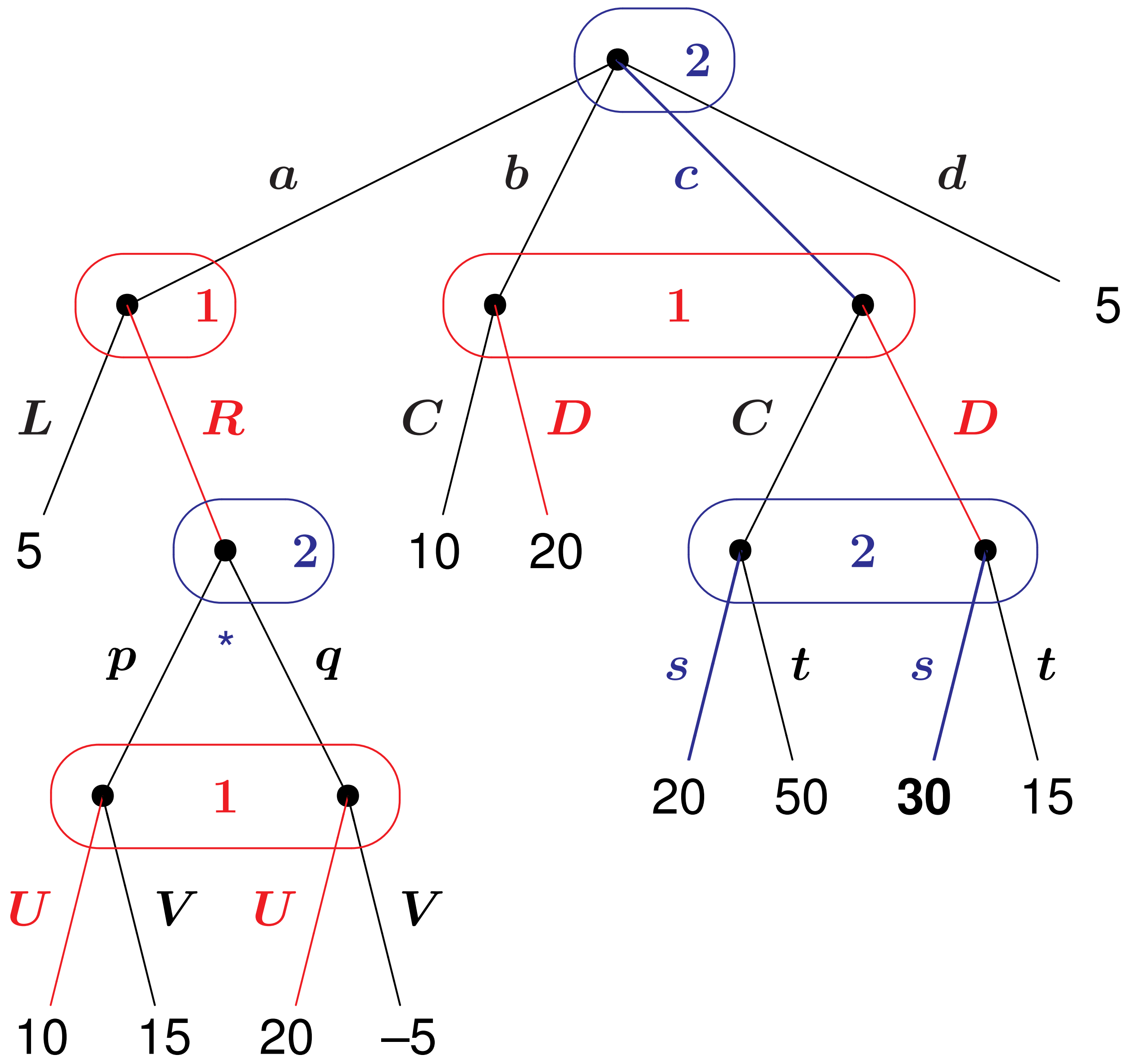
Game tree (game in extensive form)



Strategic (or normal) form

Strategy of a player:

specifies a move for **every** information set of that player.

	a	a	a	a	b	b	b	b	c	c	c	c	d	d	d	d
	p	p	q	q	p	p	q	q	p	p	q	q	p	p	q	q
	s	t	s	t	s	t	s	t	s	t	s	t	s	t	s	t
L, U, C	5	5	5	5	10	10	10	10	20	50	20	50	5	5	5	5
L, V, C	5	5	5	5	10	10	10	10	20	50	20	50	5	5	5	5
L, U, D	5	5	5	5	20	20	20	20	30	15	30	15	5	5	5	5
L, V, D	5	5	5	5	20	20	20	20	30	15	30	15	5	5	5	5
R, U, C	10	10	20	20	10	10	10	10	20	50	20	50	5	5	5	5
R, U, D	10	10	20	20	20	20	20	20	30	15	30	15	5	5	5	5
R, V, C	20	20	–5	–5	10	10	10	10	20	50	20	50	5	5	5	5
R, V, D	10	10	20	20	20	20	20	20	30	15	30	15	5	5	5	5

Reduced strategic form

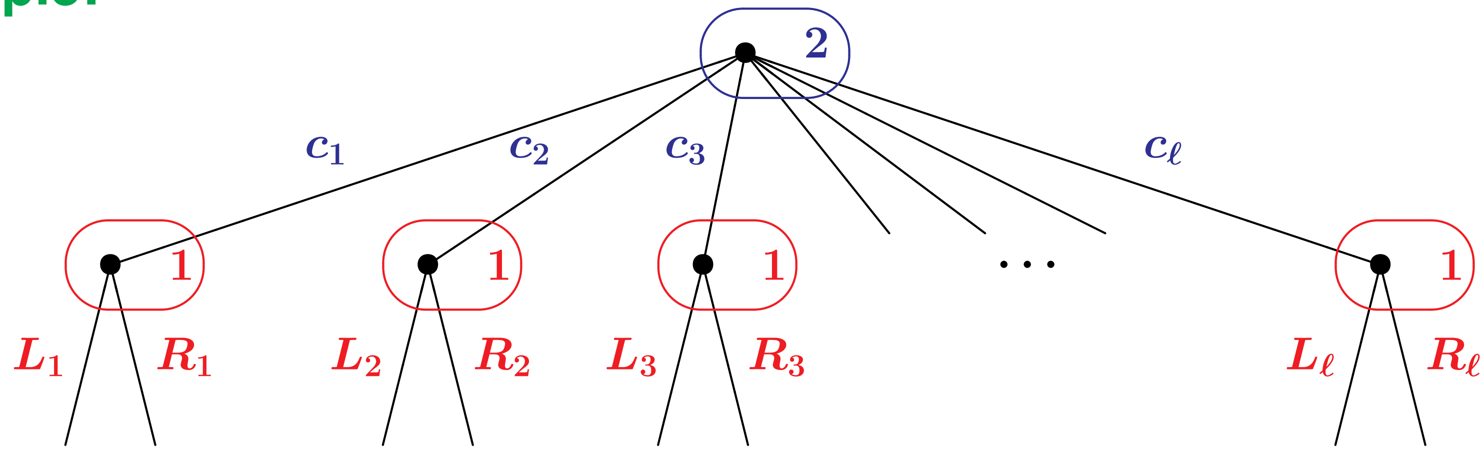
Reduced strategy of a player:
specifies a move for every information set of that player,
except for those information sets unreachable due to an **own** earlier move (where we write $*$ instead of a move).

	$a,p,*$	$a,q,*$	$b,*,*$	$c,*,s$	$c,*,t$	$d,*,*$
$L,*,C$	5	5	10	20	50	5
$L,*,D$	5	5	20	30	15	5
R,U,C	10	20	10	20	50	5
R,U,D	10	20	20	**30**	15	5
R,V,C	15	–5	10	20	50	5
R,V,D	15	–5	20	30	15	5

Exponential blowup of strategic form

number of pure strategies typically
exponential in number of information sets.

Example:



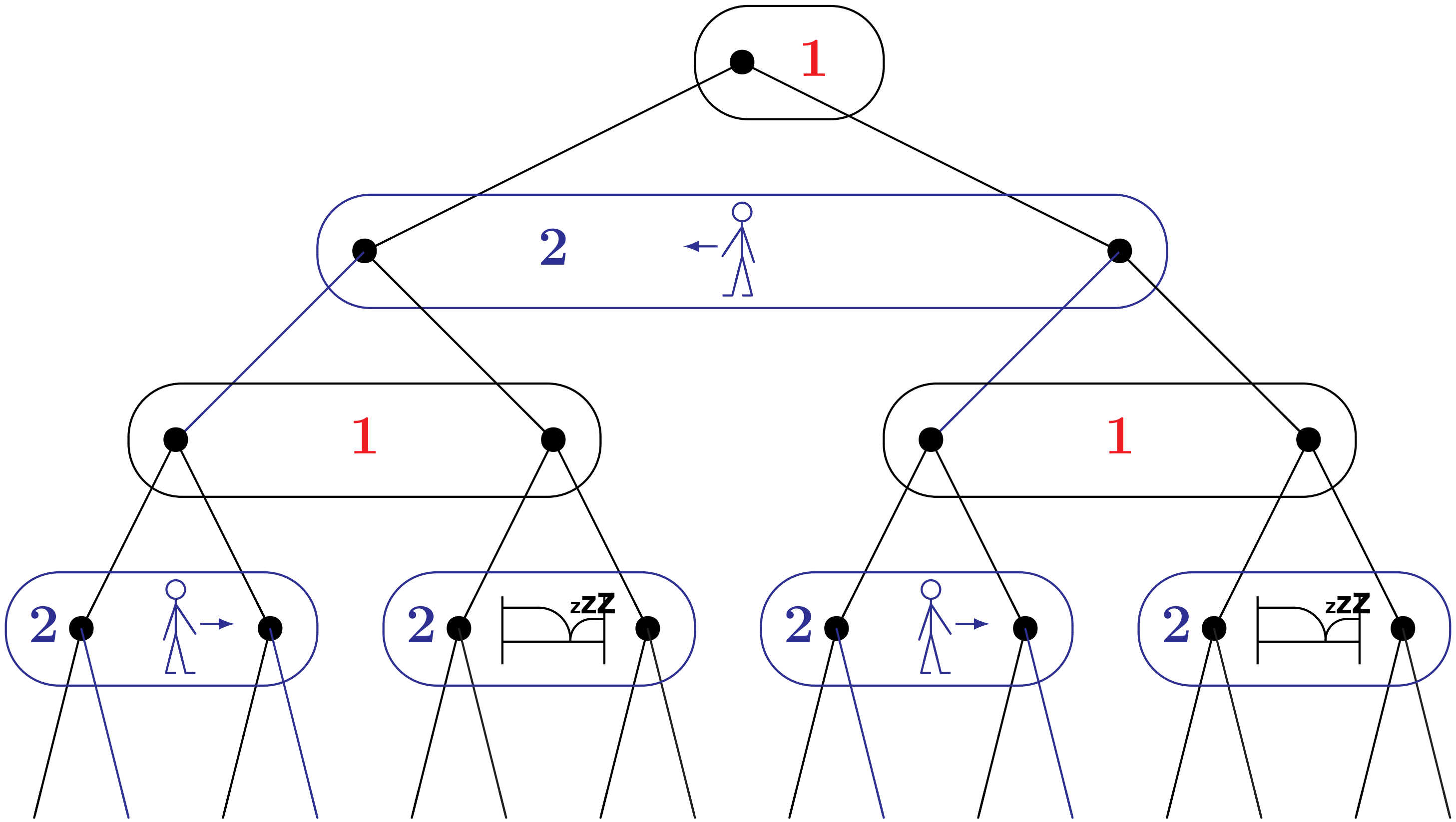
$$\begin{aligned}\text{number of information sets} &= \ell, \\ \text{number of pure strategies} &= 2^\ell.\end{aligned}$$

Example [Kuhn]: simplified poker game,

$$\begin{aligned}\text{number of information sets} &= 13, \\ \text{number of pure strategies} &= 8192.\end{aligned}$$

Exponential blowup of reduced strategic form

Example: Game with (1) **bounded** number of moves per node, (2) no **subgames** (otherwise simplify by solving subgames first).



This tree with n nodes: $\approx 2^{\sqrt{n}/2}$ strategies per player, reduced strategic form still (sub-)**exponential** in **tree** size.

Our result (sneak preview)

The **sequence form** is a strategic description of an extensive game with perfect recall that has the **same** size as the game tree, as opposed to **exponential** size of reduced strategic form.

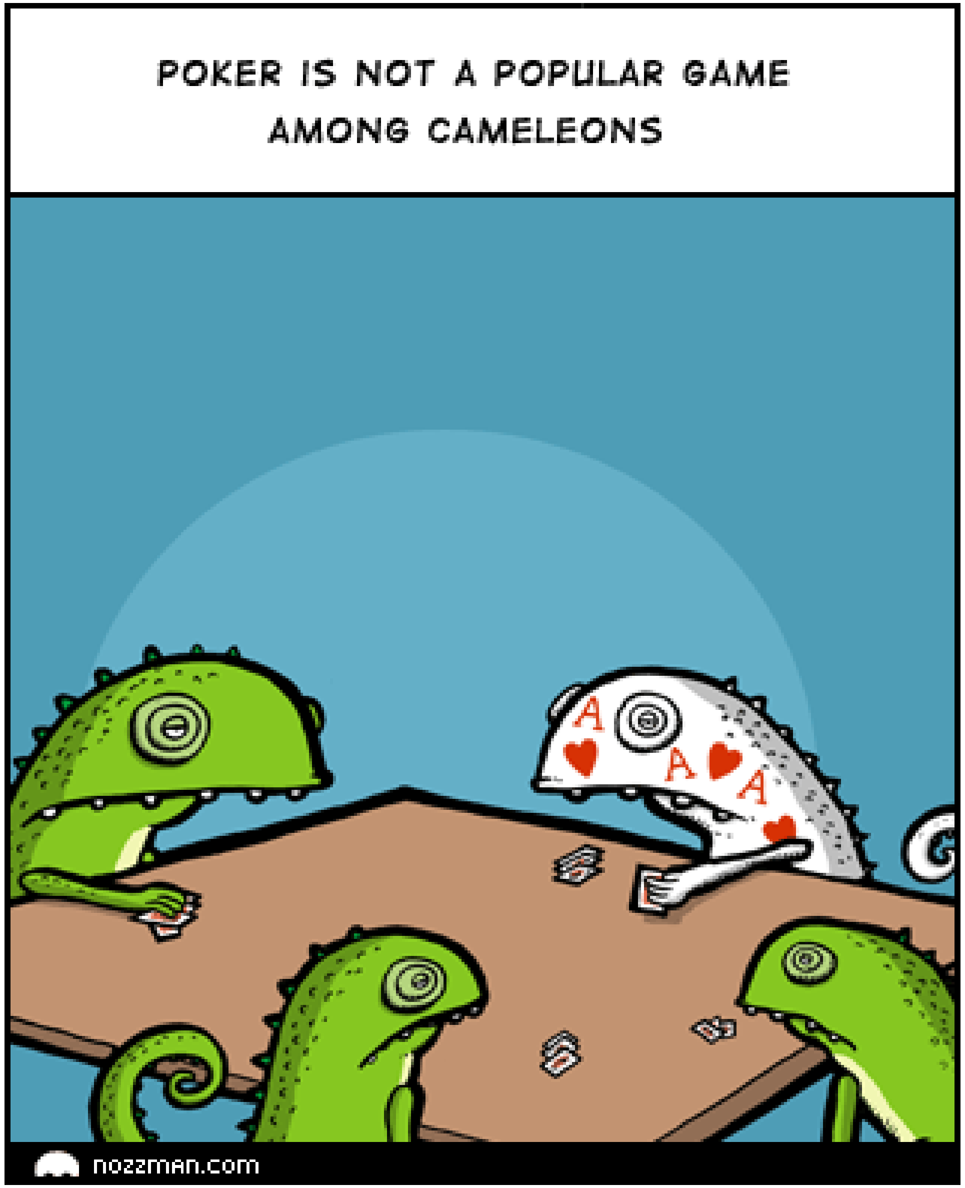
The same known strategic-form algorithms for **finding equilibria** can be applied to the sequence form:
linear programming (LP) for two-player zero-sum games,
linear complementarity (LCP) for general two-player games,

Game tree of size n:
sequence form size $n \times n$,
reduced strategic form: possibly size $2^{\sqrt{n}}$.

Size of reduced strategic form versus sequence form

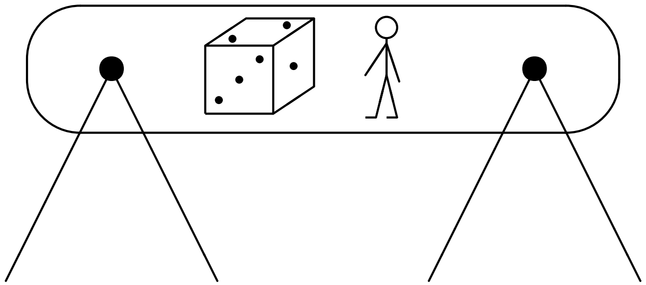
tree depth	tree size (nodes)	number of reduced strategies: player 1	number of reduced strategies: player 2	Reduced Strategic Form size	indep. SF variables: pl. 1	indep. SF variables: pl. 2	SF size
1	3	2	1	2	1		2
2	7		2	4		1	2
3	15	4		8	2		4
4	31		8	32		4	16
5	63	16		128	8		64
6	127		128	2048		16	256
7	255	256		32768	32		1024
8	511		32768	8388608		64	4096
9	1023	65536		2147483648	128		16384
10	2047		2147483648	140737488355328		256	65536

Large game trees (two–person poker) solved with the sequence form



Use behavior strategies

Behavior strategy = **local** randomization



Mixed strategy too redundant, use behavior strategy instead:

- only one variable per **move**:

  player 1 chooses L with probability X_L

  player 1 chooses R with probability X_R . . .

  player 2 chooses a with probability Y_a . . .

- expected payoff = $5\,Y_aX_L + 10\,Y_aX_RY_pX_U + 15\,Y_aX_RY_pX_V + \cdots$

- problem: **nonlinear**!

Variable transformation

For each **sequence** σ of moves of player 1
introduce new variable x_σ

- new variables replace products:
  if $\sigma = PQRS$ then $x_\sigma = X_P X_Q X_R X_S$
- Example:

$$\begin{aligned} x_L &= X_L \\ x_{RU} &= X_R X_U \\ &\cdots \\ y_a &= Y_a \\ y_{ap} &= Y_a Y_p \\ &\cdots \end{aligned}$$

- expected payoff = $5\, x_L\, y_a + 10\, x_{RU}\, y_{ap} + 15\, x_{RV}\, y_{ap} + \cdots$

  is **linear** in variables of one player.

New paradigm: Sequences instead of pure strategies

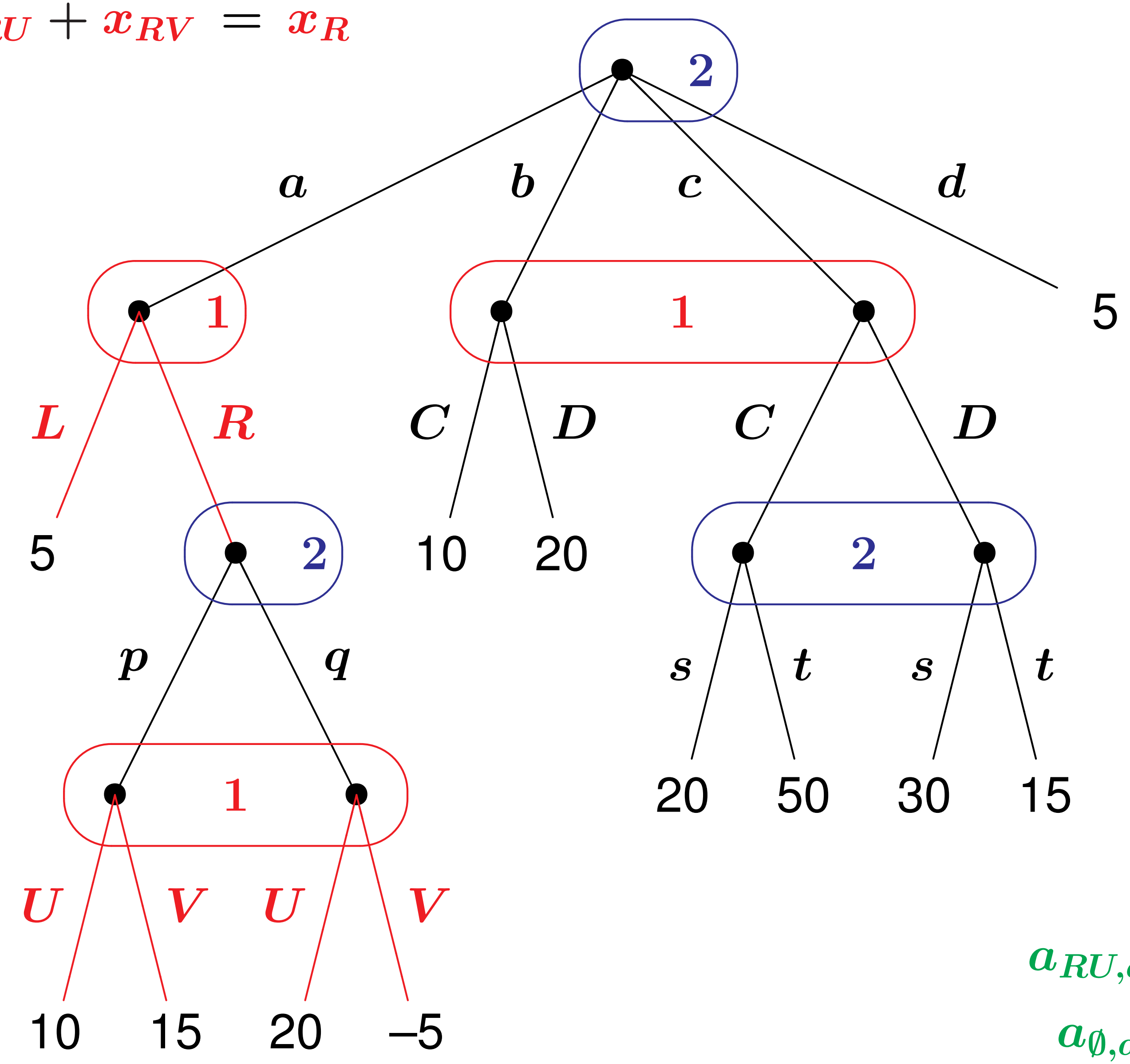
Before:

pure strategy	i
probability	x_i
mixed strategy	x
characterized by	$\mathbf{1}x = 1$
expected payoff	$x^\top A y$

After:

sequence	σ
realization probability	x_σ
realization plan	x
characterized by	$Ex = e$
expected payoff	$x^\top A y$

$$
\begin{aligned}
x_\emptyset &= 1 \\
x_L + x_R &= x_\emptyset \\
x_{RU} + x_{RV} &= x_R
\end{aligned}
$$



$$
\begin{aligned}
a_{RU,ap} &= 10 \\
a_{\emptyset,d} &= 5 \\
a_{RU,b} &= 0
\end{aligned}
$$

Realization plans

Realization plan $x = (x_\emptyset, x_L, x_R, x_C, x_D, x_{RU}, x_{RV})$

(= vector of realization probabilities)

characterized by $x \geq 0$ and **linear** equalities

$$
\begin{aligned}
x_\emptyset &= 1 \\
x_\emptyset &= x_L + x_R \\
x_\emptyset &= x_C + x_D \\
x_R &= x_{RU} + x_{RV}
\end{aligned}
$$

written as $Ex = e$ with

$$
E = \begin{bmatrix}
1 & & & & & & \\
-1 & 1 & 1 & & & & \\
-1 & & & 1 & 1 & & \\
 & & -1 & & & 1 & 1
\end{bmatrix}, \qquad
e = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

The sequence form

Payoff matrix A

	$\emptyset$	a	b	c	d	ap	aq	bs	bt
$\emptyset$					5				
L		5							
R									
RU						10	20		
RV						15	–5		
C			10					20	50
D			20					30	15

expected payoff $x^\top A y$,

rows played with x subject to $x \geq 0, \quad Ex = e$,

columns played with y subject to $y \geq 0, \quad Fy = f$.

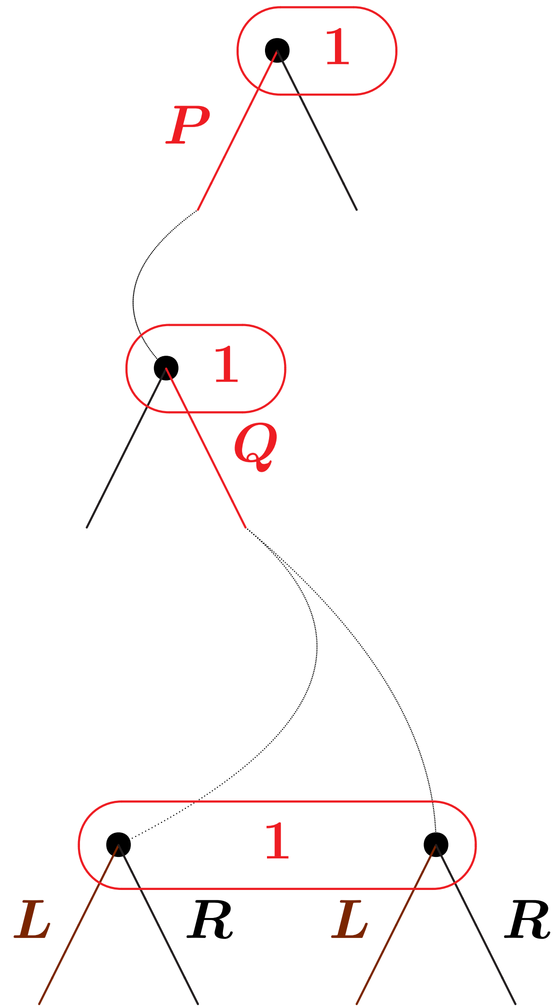
How to play

Given: realization plan x with $Ex = e$.

Move L is last move of **unique** sequence,
say PQL, where $x_{PQL} + x_{PQR} = x_{PQ}$.

$$\Longrightarrow \qquad \text{behavior-probability}(L) = \frac{x_{PQL}}{x_{PQ}}.$$

Required assumption of **perfect recall**
[Kuhn 1953, Selten 1975]:
Each node in an information set is
preceded by same sequence, here PQ,
of the player's **own** earlier moves.



Best responses – LP duality

1) Best response $\boldsymbol{x}$ against fixed $\boldsymbol{y}$ solves LP:

$$\begin{array}{lrl} \max\limits_{\boldsymbol{x}} & \boldsymbol{x}^\top(\boldsymbol{A}\boldsymbol{y}) & \\ \text{subject to} & \boldsymbol{E}\boldsymbol{x} & = \boldsymbol{e} \\ & \boldsymbol{x} & \geq \boldsymbol{0} \end{array}$$

2) Consider the **dual** of this LP:

$$\begin{array}{ll} \min\limits_{\boldsymbol{u}} & \boldsymbol{e}^\top\boldsymbol{u} \\ \text{subject to} & \boldsymbol{E}^\top\boldsymbol{u} \geq \boldsymbol{A}\boldsymbol{y} \end{array}$$

LP duality $\Longrightarrow$ same optimal value (payoff to player 1).

Best responses – LP duality

2) Consider the **dual** of this LP:

$$\begin{array}{ll} \min\limits_{u} & e^\top u \\ \text{subject to} & E^\top u \geq Ay \end{array}$$

LP duality $\Longrightarrow$ same optimal value (payoff to player 1),

3) **minimized** by player 2 if **zero-sum game**, $B = -A$:

$$\begin{array}{ll} \min\limits_{u,\,y} & e^\top u \\ \text{subject to} & E^\top u \geq Ay \\ & \quad\;\; Fy = f \\ & \quad\;\;\;\; y \geq 0 \end{array}$$

Example

1) Best response LP

$$\max_{x} \quad x^\top (Ay)$$

$$\text{subject to} \quad Ex = e$$

$$x \geq 0$$

$$
\begin{array}{rr|ccc|c|c|}
 & x_\emptyset & 1 & -1 & -1 & & 0 \\
 & x_L & & 1 & & & 2 \\
\geq 0 & x_R & & 1 & & & 2 \\
 & x_C & & & 1 & & 1 \\
 & x_D & & & 1 & & 0 \\
\end{array}
$$

$$
\begin{array}{ccc|c}
 & \Vert & & \downarrow \\
1 & 0 & 0 & \text{max}
\end{array}
$$

2) dual LP

$$\min_{u} \quad e^\top u$$

$$\text{subject to} \quad E^\top u \geq Ay$$

$$
\begin{array}{|ccc|c|c|}
u_0 & u_1 & u_2 & & \\
\hline
1 & -1 & -1 & & 0 \\
 & 1 & & & 2 \\
 & 1 & & \geq & 2 \\
 & & 1 & & 1 \\
 & & 1 & & 0 \\
\hline
1 & 0 & 0 & \longrightarrow & \text{min}
\end{array}
$$

Example

2) dual LP

$$\min_{u} \quad e^\top u$$

$$\text{subject to} \quad E^\top u \geq A y$$

$$\begin{array}{ccc} u_0 & u_1 & u_2 \end{array}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ & 1 & \\ & 1 & \\ & & 1 \\ & & 1 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \longrightarrow \text{min}$$

3) Treat y as a variable:

$$\min_{u,\, y} \quad e^\top u$$

$$\text{subject to} \quad E^\top u \geq A y$$

$$F y = f$$

$$y \geq 0$$

$$\begin{array}{ccc} u_0 & u_1 & u_2 \end{array} \qquad \begin{array}{cccc} y_\emptyset & y_a & y_b & y_c \end{array} \geq 0$$

$$\begin{bmatrix} 1 & -1 & -1 \\ & 1 & \\ & 1 & \\ & & 1 \\ & & 1 \end{bmatrix} \geq \begin{bmatrix} & & & \\ & 6 & 0 & \\ & 2 & 4 & \\ & & & 3 \\ & & & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & \\ -1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & & & & \end{bmatrix} \longrightarrow \text{min}$$

Results

Input:

Two-person game tree with perfect recall.

Theorem:

A zero-sum game is solved via a Linear Program (LP) of **linear** size.

Theorem:

A non-zero-sum game is solved via a Linear Complementarity Problem (LCP) of **linear** size.

A sample equilibrium is found by Lemke's algorithm.

This algorithm mimicks the Harsanyi–Selten tracing procedure and finds a **normal form perfect** equilibrium.

LCP – Lemke's algorithm

Consider a **prior** $(\overline{x}, \overline{y})$, and a new variable z_0 in the system

$$
\begin{array}{rcl}
 & & \boxed{\begin{array}{rr} Ex & \\ & Fy \\ E^\top u & -Ay \\ F^\top v - B^\top x & \end{array}} \begin{array}{c} + \\ + \\ - \\ - \end{array} \boxed{\begin{array}{r} e \\ f \\ A\overline{y} \\ B^\top\overline{x} \end{array}} \begin{array}{c} z_0 \\ z_0 \\ z_0 \\ z_0 \end{array} \begin{array}{c} = \\ = \\ \geq \\ \geq \end{array} \boxed{\begin{array}{c} e \\ f \\ 0 \\ 0 \end{array}}
\end{array}
$$

$$\boxed{r} = E^\top u - Ay - A\overline{y} z_0 \geq 0, \qquad \boxed{s} = F^\top v - B^\top x - B^\top \overline{x} z_0 \geq 0$$

$$\boxed{x, \quad y,} \quad \boxed{z_0} \geq 0$$

Equilibrium condition $x^\top r = 0, \quad y^\top s = 0, \quad [\, z_0 = 0 \,].$

Initial solution $z_0 = 1,\ x = 0,\ y = 0.$

Complementary pivoting:

$x_\sigma \leftrightarrow r_\sigma,\ y_\tau \leftrightarrow s_\tau$, until z_0 leaves the basis.

History of sequence form

- I. V. Romanovskii (1962), Reduction of a game with complete memory to a matrix game. *Soviet Mathematics* **3**, 678-681 (Russian orig.: *Doklady Akademii Nauk SSSR* **144**, 62-64). [told to BvS by IR in 1996]

- R. Selten (1988), Evolutionary stability in extensive two-person games - correction and further development. *Math. Social Sciences* **16**, 223-266. [told to BvS by RS in 1996]

- D. Koller and N. Megiddo (1992), The complexity of two-person zero-sum games in extensive form. *Games and Economic Behavior* **4**, 528-552.

Related own work

- BvS (1996), Efficient computation of behavior strategies. *Games Econ. Behavior* **14**, 220-246.

- D. Koller, N. Megiddo, and BvS (1996), Efficient computation of equilibria for extensive two-person games. *GEB* **14**, 247-259.

- BvS, A. van den Elzen, A. J. J. Talman (2002), Computing Normal-Form Perfect Equilibria for Extensive Two-Person Games. *Econometrica* **70**, 693-715.

- F. Forges and BvS (2002), Computationally efficient coordination in game trees. *LSE-CDAM*-2002-02. [Extensive-form correlated equilibria for 2 players and no chance moves.]

Summary

- Sequence form: # of variables = game tree size.

- Algorithm = **Lemke**'s complementary pvioting.

- Efficient **normal form** computation.

- Mimic tracing procedure for normal form.

- Computed equilibrium is normal form **perfect**
  if prior is completely mixed.
  Relative mistake probabilities are as in prior.

- Computing several equilibria possible,
  but **no** guarantee to find them **all**.