Strategic stability

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Outline

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- The collection of games is denoted by Γ

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A strategy profile is completely mixed when $\sigma_i(a_i) > 0$ for all $i \in N$ and all $a_i \in A_i$.

Nash equilibrium

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A strategy $\tau_i \in \Delta(A_i)$ is a best response against σ when

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Definition (Nash) A strategy profile $\sigma \in \Delta$ is a Nash equilibrium if for every player i, σ_i is a best response against σ.

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Solutions

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An element of $C(G)$ is called a solution, or stable set, of G.

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Only then they started a search for a solution concept that satisfied all requirements.

The, slightly adjusted and amended, list looks as follows.

Requirements

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For every game G, $C(G) \neq \phi$.

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For every game G, every element of $C(G)$ is (topologically) connected.

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For every game G, every stable set of G consists only of perfect equilibria of G.

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For every game G, every stable set of G contains a proper equilibrium.

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A strategy τ_i is admissible against a strategy σ,

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A strategy τ_i is admissible against a strategy σ,
if there is a sequence $(\sigma^k)_{k \in \mathbb{N}}$ of completely mixed strategy profiles converging to σ

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A strategy τ_i is admissible against a strategy σ,
if there is a sequence $(\sigma^k)_{k \in \mathbb{N}}$ of completely mixed strategy profiles converging to σ
such that τ_i is a best response against every σ^k.

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Let S be a solution.

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Let S be a solution.
A strategy τ_i is admissible against S,
if τ_i is admissible against some $\sigma \in S$.

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Let S be a solution of the game G.

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Let S be a solution of the game G.
Suppose that a_i is not admissible against S.

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Let S be a solution of the game G.
Suppose that a_i is not admissible against S.
Then S contains a solution of the game G' where a_i is not available.

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Ordinality is implied by invariance (INV) and admissible best reply invariance (ABR-I).

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ABR-I: two games with the same ABR-s have the same solutions.

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INV: payoff equivalent games have essentially the same solutions.

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For any game, any solution for a group of insiders can be extended to a solution for the entire game.

Why set valued: argument 1

Example I (Kohlberg and Mertens 86)

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Example I (Kohlberg and Mertens 86) IIS and ADM

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$$
\begin{array}{c c}
 & \begin{array}{cc} L & R \end{array} \\
\begin{array}{c} U \\ M \\ D \end{array} & \begin{bmatrix} 3,2 & 3,2 \\ 1,1 & 0,0 \\ 0,0 & 1,1 \end{bmatrix}
\end{array}
$$

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Example I (Kohlberg and Mertens 86) IIS and ADM

	L	R
U	3, 2	3, 2
M	1, 1	0, 0
D	0, 0	1, 1

	L	R
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$$\begin{array}{c} \\ U \\ M \\ D \\ S \end{array} \begin{array}{c} \begin{array}{cc} \quad L \quad & \quad R \quad \end{array} \\ \begin{bmatrix} 6,0 & 6,0 \\ 8,0 & 0,8 \\ 0,8 & 8,0 \\ 6\lambda, 8(1-\lambda) & 8-2\lambda, 0 \end{bmatrix} \end{array}$$

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Outline

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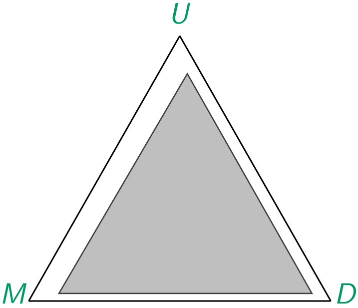

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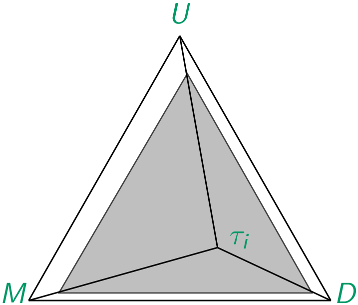

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Zorn

The proof of the existence of KM stable sets is based on the Lemma of Zorn.

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Let X be any non-empty set. A binary relation on X is a subset of $X \times X$. A binary relation $\preceq$ is a partial order on X that satisfies

[1] (reflexivity) $x \preceq x$, and

[2] (transitivity) $x \preceq y$ and $y \preceq z$ imply that $x \preceq z$.

Let $\preceq$ be a partial order on X. A subset C of X is called a chain if for any two elements x and y of C we have at least one of the two inequalities $x \preceq y$ and $y \preceq x$.

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Theorem (ZORN) Suppose that every chain of X has a lower bound. Then X has a minimal element.

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The result now follows from the Lemma of Zorn.

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For every k, σ^k is η^k-perfect, where $\eta^k = \|\varepsilon^k\|$.

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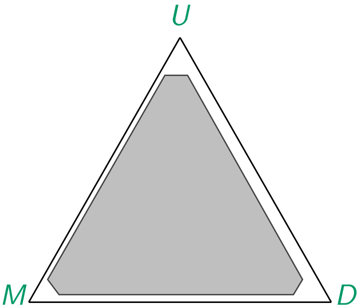

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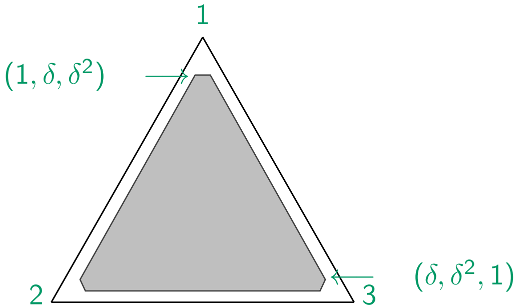

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Q-sets violate ORD

Consider the bimatrix game

$$\begin{array}{c} \\ U \\ M \\ D \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{bmatrix} 6,0 & 6,0 \\ 8,0 & 0,8 \\ 0,8 & 8,0 \end{bmatrix} \end{array}$$

From Example II, we know that, when BI and ORD are satisfied, any solution S has to contain the line segment

$$(U,(\mu,1-\mu)) \quad \text{for } \frac{1}{2} \leq \mu \leq \frac{3}{4}.$$

However, Q stable sets for bimatrix games are finite.
Hence, Q-stable sets do not satisfy ORD.

Outline

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Write

$$\mathrm{fix}(\varphi) = \{x \in \Delta(A) \mid x \in \varphi(x)\}.$$

Note that

[1] the best response correspondence BR is an element of $\mathcal{H}$,

[2] $\mathrm{fix}(\mathrm{BR}) = NE(G)$.

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Note that

[1] the best response correspondence BR is an element of $\mathcal{H}$,

[2] $\text{fix}(\text{BR}) = NE(G)$.

Let $\varphi, \psi \in \mathcal{H}$. Define

$$d(\varphi, \psi) = \sup\{d_H(\varphi(x), \psi(x)) \mid x \in \Delta(A)\}.$$

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Definition A compact set $S \subseteq \Delta(A)$ is BR-stable if

[a] S is a BR-set, and

[b] S is a connected set of perfect equilibria of G.

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A slight adaptation of the definition of BR stable sets also renders ORD.

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So, S is a BR stable set.

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Since F is not a BR set we find a similar open set $V \supseteq F$. We assume wlog that U and V are disjoint.

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Clearly, S is a subset of $W = U \cup V$. So, since S is a BR set, there is $\eta > 0$ such that

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This is a contradiction.

Intermezzo: homology groups

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We write

$$\partial P(\delta) = \{(\tau, \eta) \in P(\delta) \mid \text{ for some } i, a_i, \eta_i \in \{0, \delta\} \text{ or } \tau_i(a_i) = 0\}.$$

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[2] $\pi^* \colon H(S(\delta), \partial S(\delta)) \to H(P(\delta), \partial P(\delta))$ for sufficiently small δ.

The main result

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Theorem (Mertens) Stable sets satisfy EX, CON, ADM, BI, IIS, ORD, and SW.

Outline

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fully stable sets (K & M) $\xrightarrow{\text{K \& M}}$ K & M $\xleftarrow{\text{K \& M}}$ essential sets

$\uparrow$ (H90)

Q sets

$\uparrow$

BR sets

$\uparrow$

stable sets (Mertens)