Lecture 2

Split Inequalities and Gomory Mixed Integer Cuts

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Mixed Integer Cuts   Gomory 1963

Consider a single constraint : $S := \{x \in \mathbb{Z}_+^p \times \mathbb{R}_+^{n-p} : \sum_{j=1}^n a_j x_j = b\}$.

Let $b = \lfloor b \rfloor + f_0$ where $0 < f_0 < 1$,
and $a_j = \lfloor a_j \rfloor + f_j$ where $0 \leq f_j < 1$.

THEOREM

$$\sum_{\substack{j \leq p: \\ f_j \leq f_0}} \frac{f_j}{f_0} x_j + \sum_{\substack{j \leq p: \\ f_j > f_0}} \frac{1 - f_j}{1 - f_0} x_j + \sum_{\substack{j \geq p+1: \\ a_j > 0}} \frac{a_j}{f_0} x_j - \sum_{\substack{j \geq p+1: \\ a_j < 0}} \frac{a_j}{1 - f_0} x_j \geq 1$$

is a valid inequality for S.

NOTE   The mixed integer cuts dominate the fractional cuts.

Experiments of Bonami and Minoux 2005 on MIPLIB 3 instances give the amount of duality gap $= \min_{x \in S} cx - \min_{x \in P} cx$ closed by strengthening P with mixed integer cuts from the optimal basis :

gap closed : 24 %

Derivation of Gomory's Mixed Integer Cut

Consider a single constraint : $S := \{x \in \mathbb{Z}_+^p \times \mathbb{R}_+^{n-p} : \sum_{j=1}^n a_j x_j = b\}$.

Let $b = \lfloor b \rfloor + f_0$ where $0 < f_0 < 1$,
and $a_j = \lfloor a_j \rfloor + f_j$ where $0 \leq f_j < 1$.

Thus $\sum_{f_j \leq f_0}^{j \leq p:} f_j x_j + \sum_{f_j > f_0}^{j \leq p:} (f_j - 1) x_j + \sum_{j \geq p+1:} a_j x_j = f_0 + k$
for some integer k.

We have $k \geq 0$ or $k \leq -1$. We get, respectively,

$$\sum_{f_j \leq f_0}^{j \leq p:} \frac{f_j}{f_0} x_j - \sum_{f_j > f_0}^{j \leq p:} \frac{1-f_j}{f_0} x_j + \sum_{j \geq p+1:} \frac{a_j}{f_0} x_j \geq 1$$

$$-\sum_{f_j \leq f_0}^{j \leq p:} \frac{f_j}{1-f_0} x_j + \sum_{f_j > f_0}^{j \leq p:} \frac{1-f_j}{1-f_0} x_j - \sum_{j \geq p+1:} \frac{a_j}{1-f_0} x_j \geq 1$$

To get an inequality that is valid for both $k \geq 0$ and $k \leq -1$, we take the maximum coefficient on the left hand side.

Thus

$$\sum_{f_j \leq f_0}^{j \leq p:} \frac{f_j}{f_0} x_j + \sum_{f_j > f_0}^{j \leq p:} \frac{1-f_j}{1-f_0} x_j + \sum_{a_j > 0}^{j \geq p+1:} \frac{a_j}{f_0} x_j - \sum_{a_j < 0}^{j \geq p+1:} \frac{a_j}{1-f_0} x_j \geq 1$$

is a valid inequality for S.

Yet, for thirty years, fractional cuts and mixed integer cuts were not used in MILP solvers.

In 1991, Gomory remembered his experience with fractional cuts as follows : In the summer of 1959, I joined IBM research and was able to compute in earnest... We started to experience the unpredictability of the computational results rather steadily.

In 1991, Padberg and Rinaldi made the following comments : These cutting planes have poor convergence properties... classical cutting planes furnish weak cuts... A marriage of classical cutting planes and tree search is out of the question as far as the solution of large-scale combinatorial optimization problems is concerned.

In 1989, Nemhauser and Wolsey had this to say : They do not work well in practice. They fail because an extremely large number of these cuts frequently are required for convergence.

In 1985, Williams says : Although cutting plane methods may appear mathematically elegant, they have not proved very successful on large problems.

In 1988, Parker and Rardin give the following explanation for this lack of success : The main difficulty has come, not from the number of iterations, but from numerical errors in computer arithmetic.

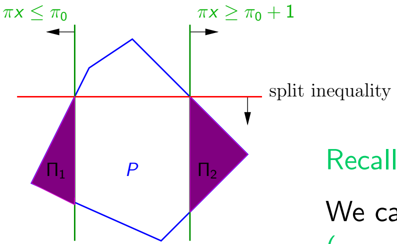
GOMORY CUTS REVISITED

Balas, Ceria, Cornuéjols, Natraj 1996

Split Inequalities Cook-Kannan-Schrijver 1990

$P := \{x \in \mathbb{R}^n : \; Ax \geq b\}$
$S := P \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$.

For $\pi \in \mathbb{Z}^n$ such that $\pi_{p+1} = \ldots = \pi_n = 0$ and $\pi_0 \in \mathbb{Z}$, define



$$\Pi_1 := P \cap \{x : \; \pi x \leq \pi_0\}$$

$$\Pi_2 := P \cap \{x : \; \pi x \geq \pi_0 + 1\}$$

Recall : $\text{conv}(\Pi_1 \cup \Pi_2)$ is a polyhedron.

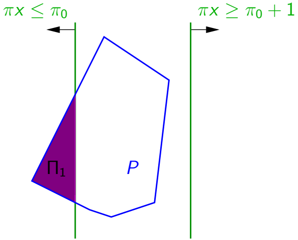
We call $cx \leq c_0$ a split inequality if there exists $(\pi, \pi_0) \in \mathbb{Z}^p \times \mathbb{Z}$ such that $cx \leq c_0$ is valid for $\Pi_1 \cup \Pi_2$.

The split closure is the intersection of all split inequalities.

THEOREM Cook, Kannan, Schrijver 1990
The split closure is a polyhedron.

An Aside : Chvátal Inequalities   Chvátal 1973

A Chvátal inequality is a split inequality where $\Pi_2 = \emptyset$.



REMARK Chvátal defined this concept in 1973 in the context of pure integer programs.

GOMORY CLOSURE

For the mixed integer linear set $Ax \geq b \quad x \in \mathbb{Z}_+^p \times \mathbb{R}_+^{n-p}$

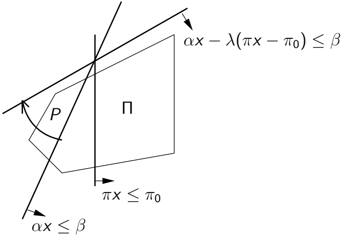
- Every valid linear inequality for $P := \{x \geq 0 : \; Ax \geq b\}$ $(\neq \emptyset)$ is of the form $uAx + vx \geq ub - t$, where $u, v, t \geq 0$.
- Subtract a nonnegative surplus variable $\alpha x - s = \beta$.
- Generate a Gomory inequality.
- Eliminate $s = \alpha x - \beta$ to get the inequality in the x-space.
- The convex set obtained by intersecting all these inequalities with P is called the Gomory closure.

THEOREM Nemhauser-Wolsey 1990, Cornuéjols-Li 2002
The Gomory closure is identical to the split closure.

Proof Outline

GMI cuts are split inequalities. We now prove the converse.

LEMMA Let
$P := \{x \in \mathbb{R}^n : \ Ax \leq b\}$ and let
$\Pi := P \cap \{x : \ \pi x \leq \pi_0\}$. If $\Pi \neq \emptyset$
and $\alpha x \leq \beta$ is a valid inequality for
Π, then there exists a scalar $\lambda \in \mathbb{R}_+$
such that $\alpha x - \lambda(\pi x - \pi_0) \leq \beta$ is
valid for P.



Let $\alpha x \leq \beta$ be a split inequality. There exist λ and μ such that
$\alpha x - \lambda(\pi x - \pi_0) \leq \beta$ and
$\alpha x + \mu(\pi x - (\pi_0 + 1)) \leq \beta$
are both valid for P. Combining, we get
$\pi x + \frac{s_2}{\lambda+\mu} - \frac{s_1}{\lambda+\mu} = \pi_0 + \frac{\mu}{\lambda+\mu}$.

The GMI cut for this equation happens to be $\alpha x \leq \beta$ (Left as exercise).

THEOREM Caprara, Letchford 2002 et Cornuéjols, Li 2002
It is NP-hard to optimize a linear function over the Gomory closure.

Nevertheless,
Balas and Saxena 2006 and Dash, Günlück and Lodi 2007
were able to optimize over the Gomory closure by solving a sequence of parametric MILPs.

DUALITY GAP CLOSED BY GOMORY CUTS
MIPLIB 3

Gomory cuts (optimal basis)	Gomory closure = split closure
24 %	80 %

Reduce-and-split cuts

Andersen, Cornuéjols, Li 2005

Perform linear combinaisons of the constraints $\sum_{j=1}^{n} a_j x_j = b$ in order to reduce the coefficients of the continuous variables, and generate the corresponding Gomory cuts.

Why ?

Remember the Gomory cut formula :

$$\sum_{\substack{j\le p:\\ f_j\le f_0}} \frac{f_j}{f_0}x_j + \sum_{\substack{j\le p:\\ f_j> f_0}} \frac{1-f_j}{1-f_0}x_j + \sum_{\substack{j\ge p+1:\\ a_j>0}} \frac{a_j}{f_0}x_j - \sum_{\substack{j\ge p+1:\\ a_j<0}} \frac{a_j}{1-f_0}x_j \ge 1$$

ALGORITHM Consider the lines L of the optimal simplex tableau for the basic variables x_i such that $i \le p$.

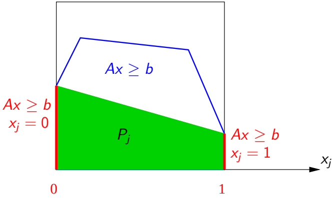
For every line $\ell \in L$, reduce the norm $\|(a^\ell_{p+1}, \ldots, a^\ell_n)\|$ by performing integer combinaisons of the other lines of L.

Computational experiments (Cornuéjols and Nannicini 2010) show that one can gain about 6% on top of Gomory mixed integer cuts from the optimal tableau. Gap closed : 30%

Lift-and-Project

Sherali-Adams 1990
Lovász-Schrijver 1991
Balas-Ceria-Cornuéjols 1993

Let $S := \{x \in \{0,1\}^p \times \mathbb{R}_+^{n-p} : \; Ax \geq b\}$
$P := \{x \in \mathbb{R}_+^n : \; Ax \geq b\}$



$$P_j = \text{Conv}\{\begin{pmatrix} Ax \geq b \\ x_j = 0 \end{pmatrix} \cup \begin{pmatrix} Ax \geq b \\ x_j = 1 \end{pmatrix}\}.$$

THEOREM Balas 1979 $\;\text{Conv}(S) = P_p(\dots P_2(P_1)\dots)$.

Lift-and-project cut

Given a fractional solution $\bar{x}$ of the linear relaxation $Ax \geq b$, find a cutting plane $\alpha x \geq \beta$ (namely $\alpha \bar{x} < \beta$) that is valid for P_j (and therefore for S).

DEEPEST CUT
$$\begin{gathered} \max\ \beta - \alpha \bar{x} \\ \alpha x \geq \beta \text{ valid for } P_j \end{gathered}$$

This can be rewritten as (adding a normalization constraint)

CUT GENERATION LP
$$\begin{gathered} \max\ vb - (uB_j + vD_j)\bar{x} \\ uA_j - vA_j = 0 \\ u \geq 0, v \geq 0 \\ \sum u_i + \sum v_i = 1 \end{gathered}$$

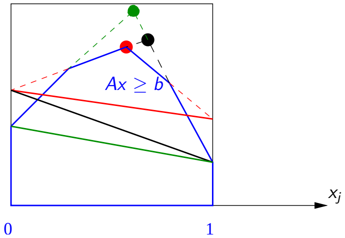
where B_j, D_j and A_j are matrices derived from the data A, b.

SIZE OF THE CUT GENERATION LP

$$\begin{aligned} \max\ vb - (uB_j + vD_j)\bar{x} \\ uA_j - vA_j = 0 \\ \sum u_i + \sum v_i = 1 \\ u \geq 0, v \geq 0 \end{aligned}$$

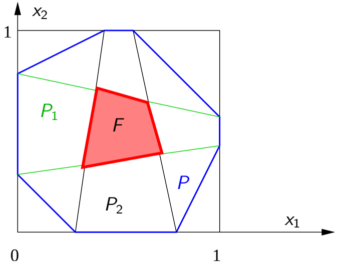
Number of variables : $2m$
Number of constraints : n + nonnegativity

Balas and Perregaard 2003 give a precise correspondance between the basic feasible solutions of the cut generation LP and the basic solutions of the LP $\begin{aligned} \min\ cx \\ Ax \geq b \end{aligned}$



LIFT-AND-PROJECT CLOSURE OF P

$$F := \bigcap_{j=1}^{p} P_j$$



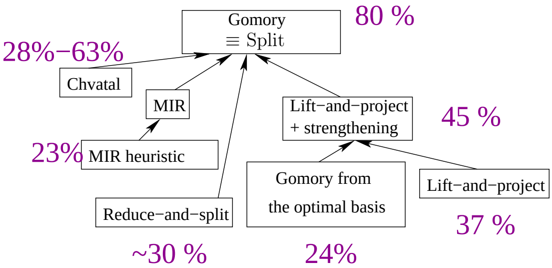
REMARK Balas and Jeroslow 1980 show how to strengthen cutting planes by using the integrality of the other integer variables (lift-and-project only considers the integrality of one x_j at a time).

Experiments of Bonami and Minoux 2005 and Bonami 2010 on MIPLIB03 instances :

Lift-and-project closure	Lift-and-project + strengthening
Gap closed : 37 %	45 %

Duality gap closed by different types of cutting planes

MIPLIB 3 instances



All these cuts are generated from integrality arguments applied to one linear equation. Can we generate deeper cuts by considering several equations ?

Exercises

In "Courses Material" on the webpage

http ://eventos.cmm.uchile.cl/discretas2016/

do the following exercises in Course Notes "Cutting planes in integer programming"

Exercise 2.1

Exercise 2.3

Exercise 2.6

Optional : Exercise 2.10