Problem Set 2 - Examples of pseudorandom graphs

1. Prove that the Paley graph P_q is strongly regular with parameters $(q, (q-1)/2, (q-5)/4, (q-1)/4)$.

2. Suppose that k is an odd integer. Let H_k be the graph with vertex set consisting of all binary vectors of length k with an odd number of ones, except for the all-one vector, where two vertices are adjacent if and only if their scalar product is 1 (mod 2). Show that H_k is strongly regular with parameters $(2^{k-1} - 1, 2^{k-2} - 2, 2^{k-3} - 3, 2^{k-3} - 1)$.

3. Let q be a prime power. Let G_q be a graph whose vertex set is the two-dimensional vector space over the finite field of order q. To define the edge set, partition the $q + 1$ lines through the origin into two sets P and N, where $|P| = k$, and say that x and y are adjacent if and only if $x - y$ is parallel to a line in P. Show that G_q is strongly regular with parameters $(q^2, k(q - 1), (k - 1)(k - 2) + q - 2, k(k - 1))$.

4. Prove that the Erdős–Rényi graph is regular of degree $q + 1$ and C_4-free. Conclude that there are C_4-free graphs with n vertices and $\Omega(n^{3/2})$ edges. Is this best possible?