Approximation algorithms for discrete stochastic optimization problems

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Stochastic Optimization

- Way of modeling uncertainty.
- Exact data is unavailable or expensive – data is uncertain, specified by a probability distribution.

Want to make the best decisions given this uncertainty in the data.

- Dates back to 1950's and the work of Dantzig.
- Applications in logistics, transportation models, financial instruments, network design, production planning, …

A priori optimization (no recourse)

Given: Probability distribution over inputs.

In advance: Compute master plan.

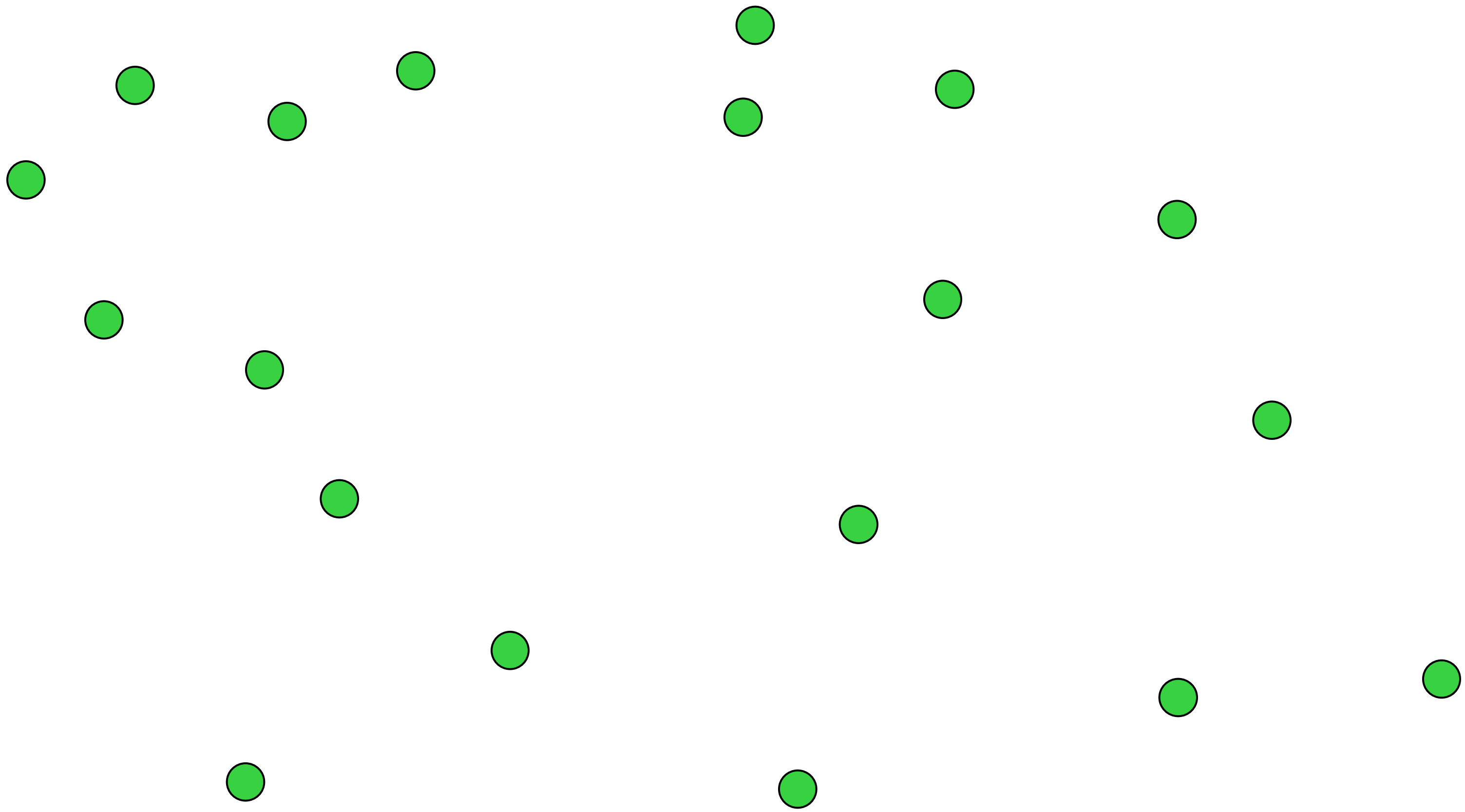
Observe the actual input scenario.

In real time: Adapt master plan to scenario.

Compute master plan to minimize

expected real time cost.

The Traveling Salesman Problem (TSP)



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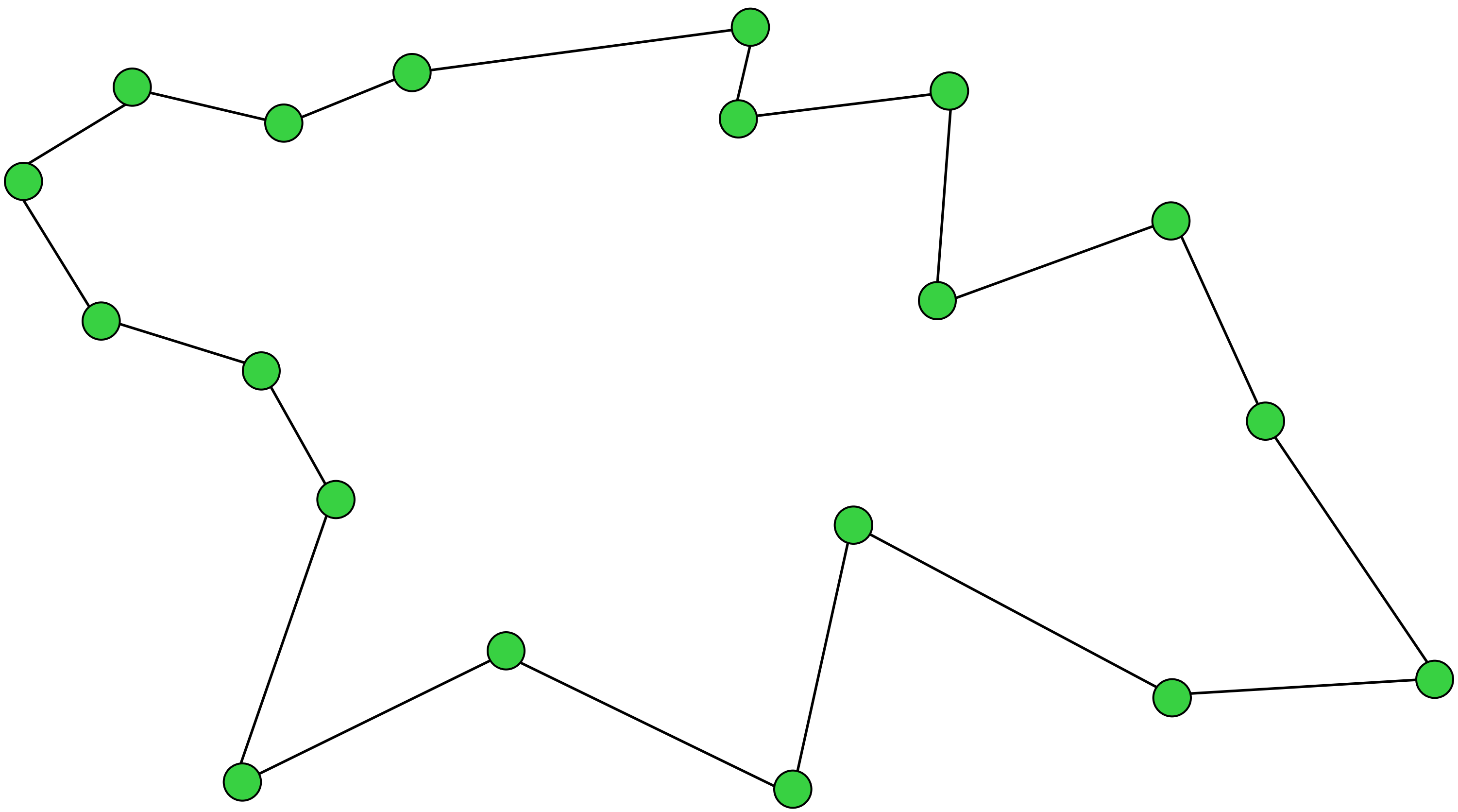

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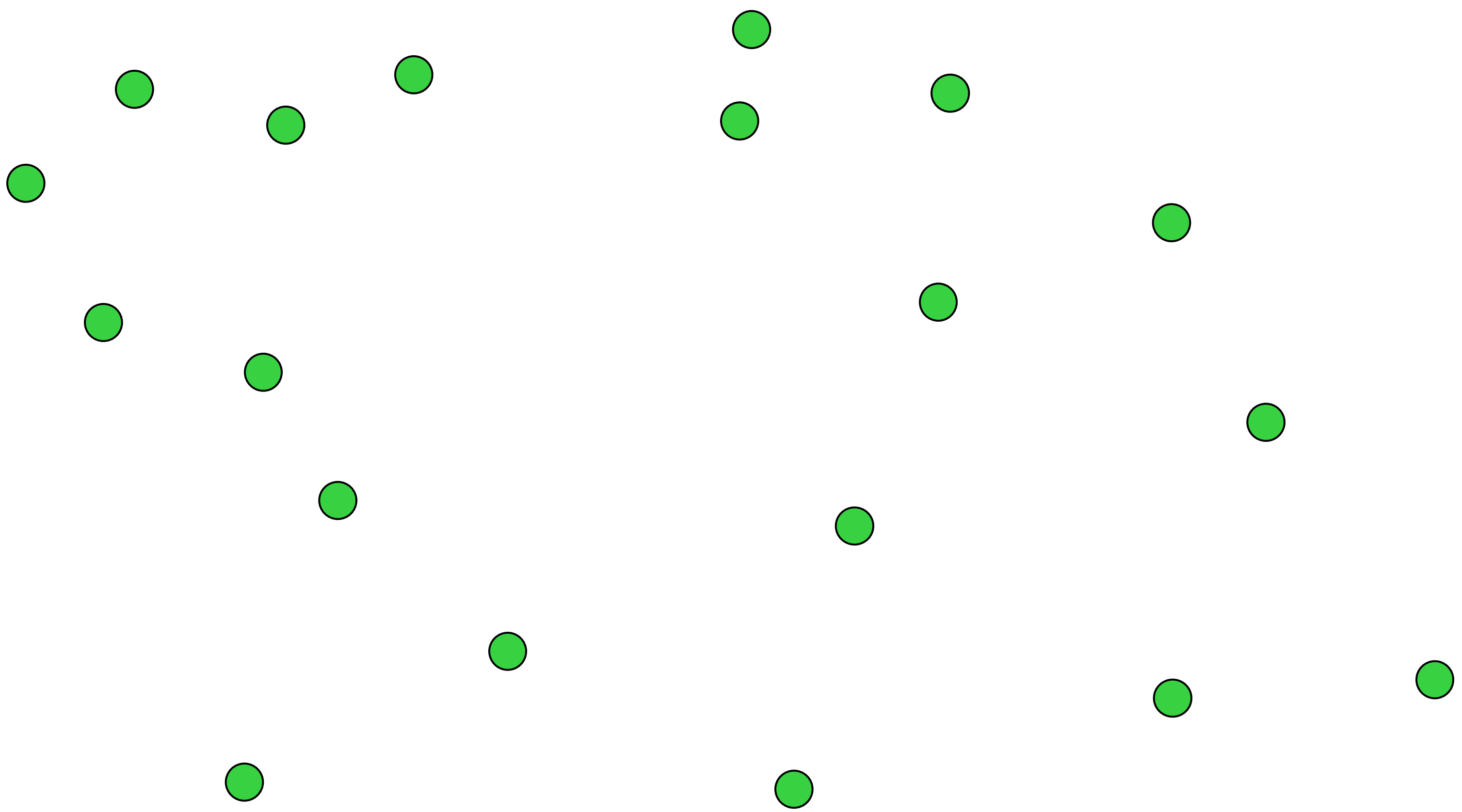
The *A Priori* TSP



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Need to specify the probability that a given set A is active

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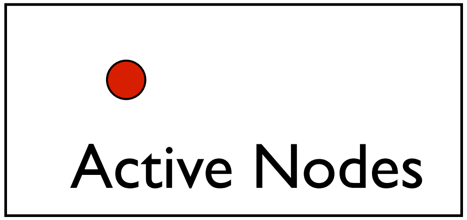

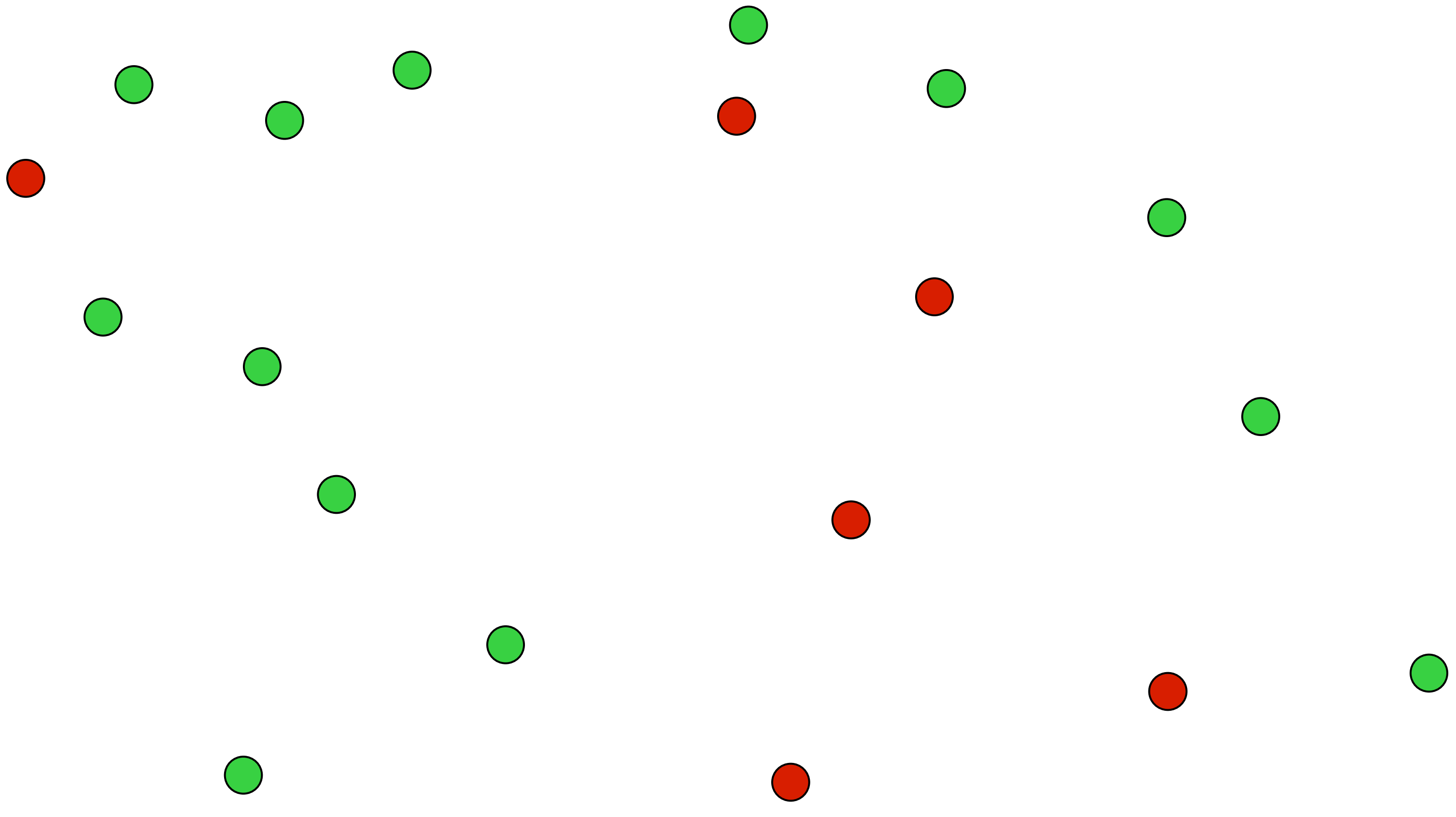

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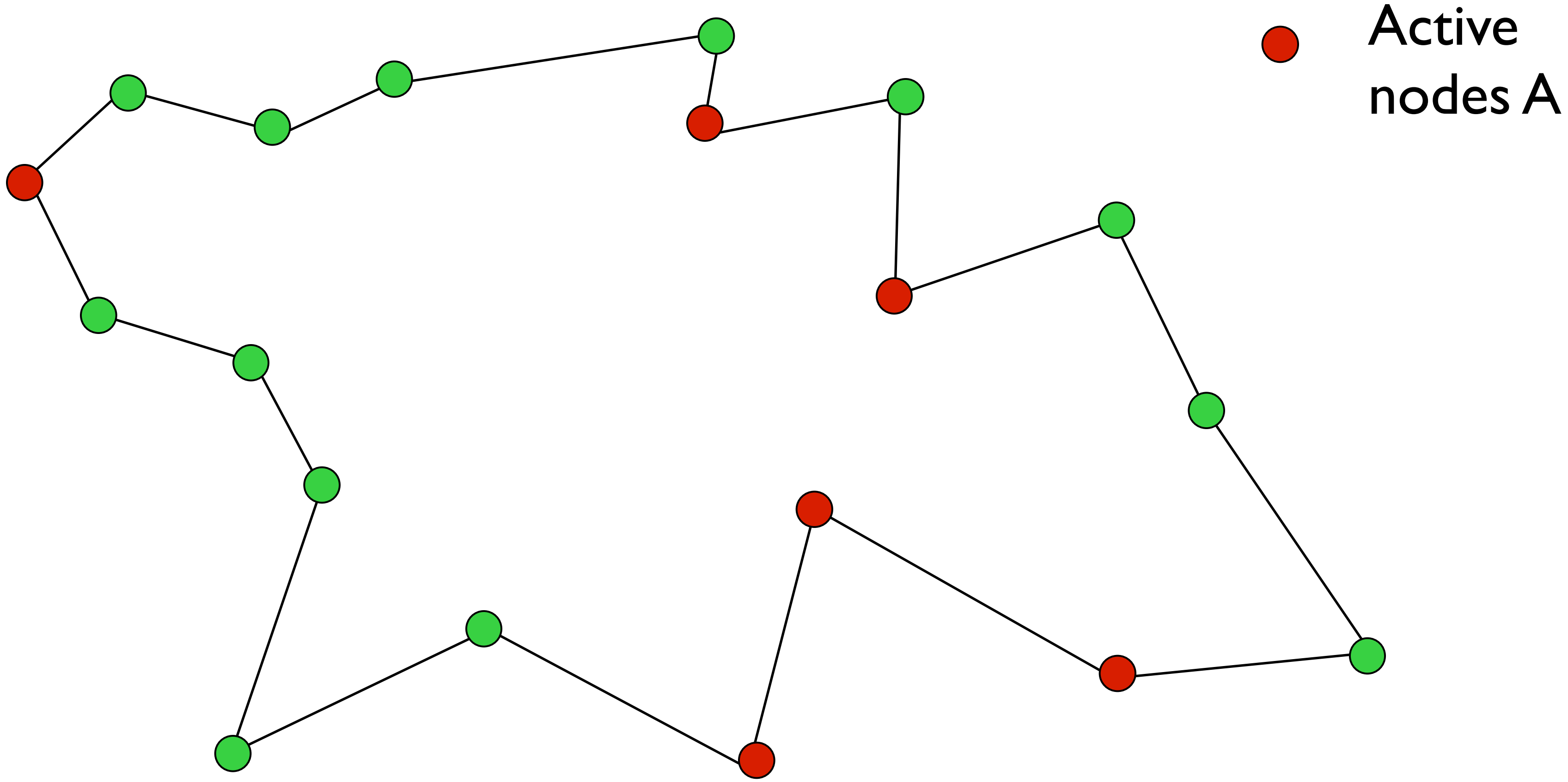

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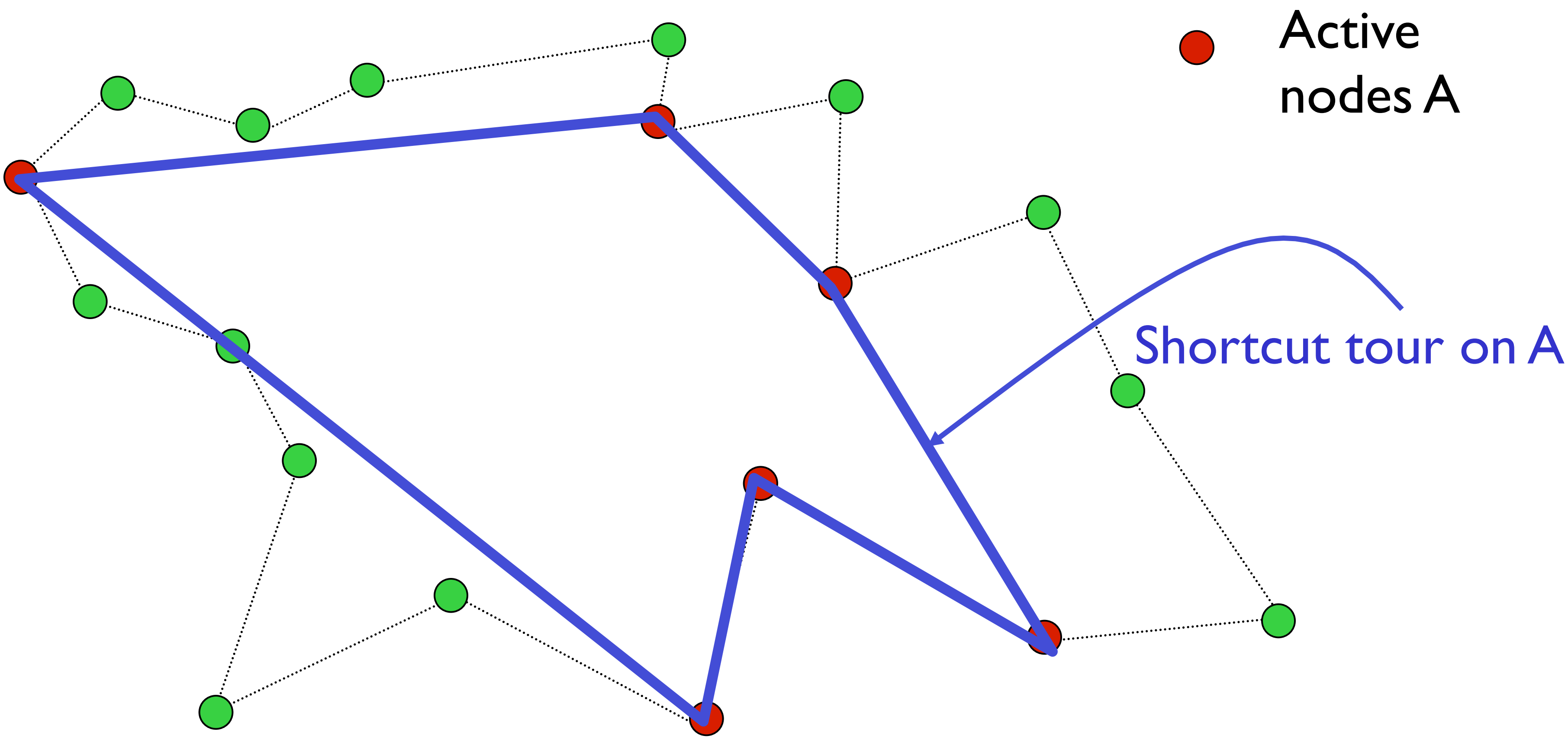

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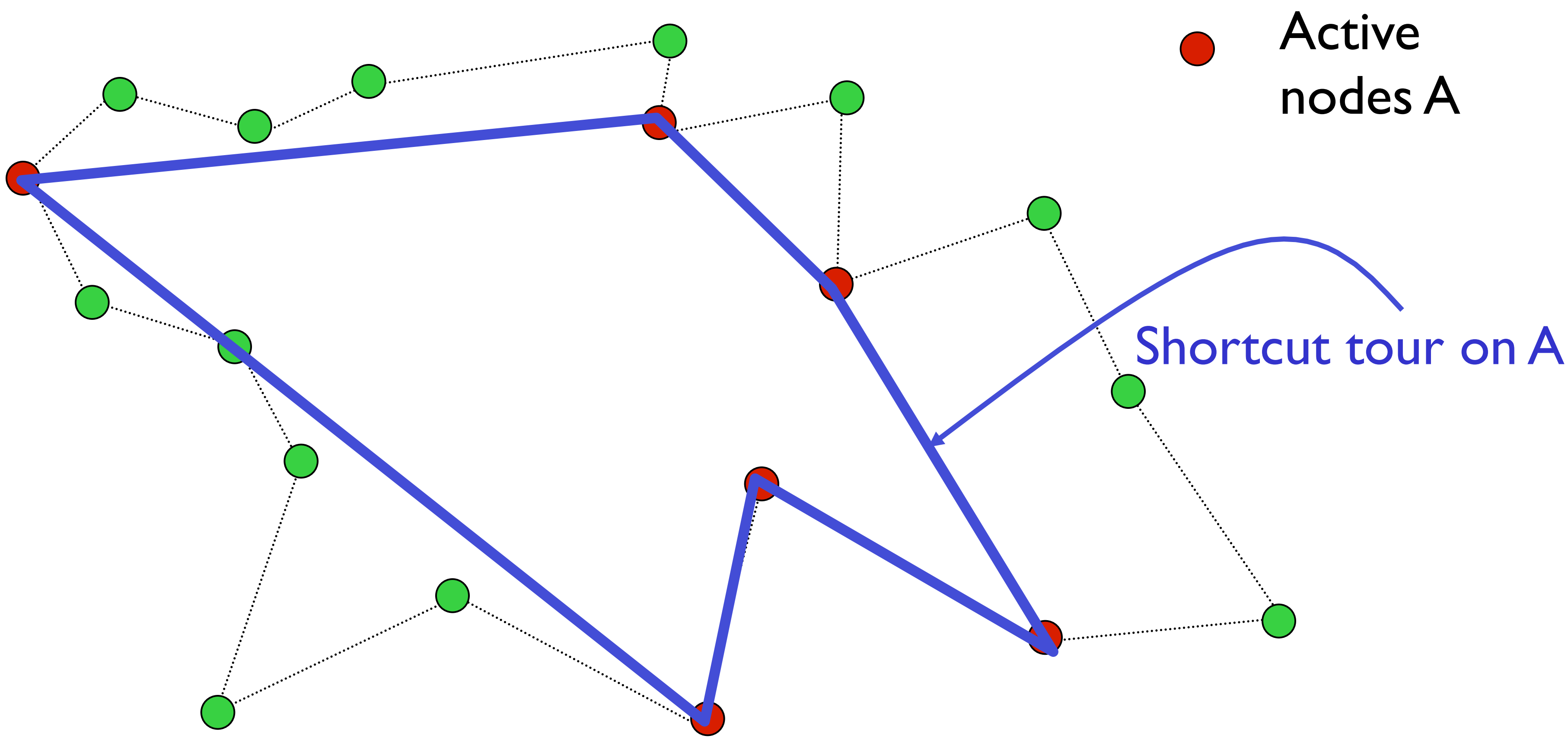

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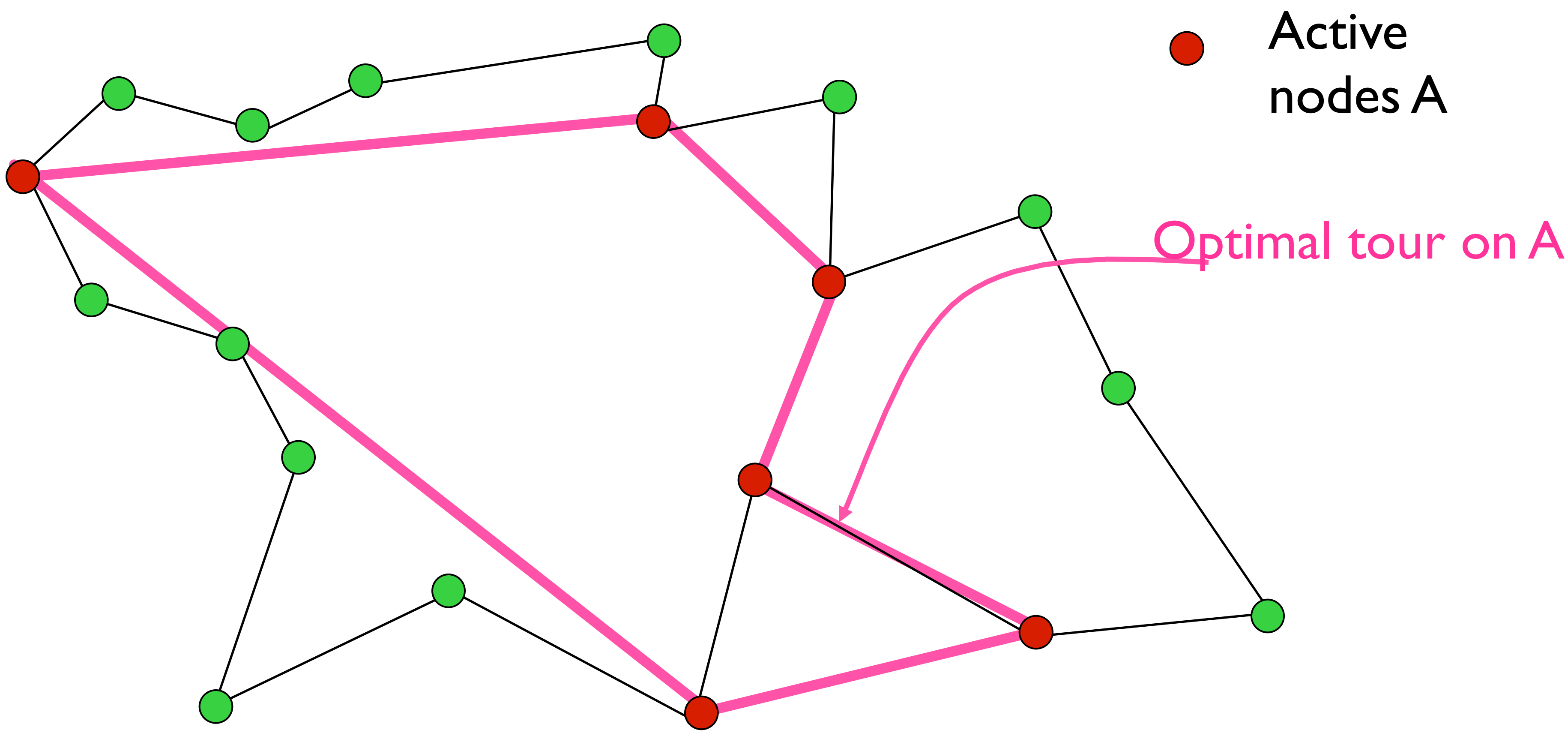

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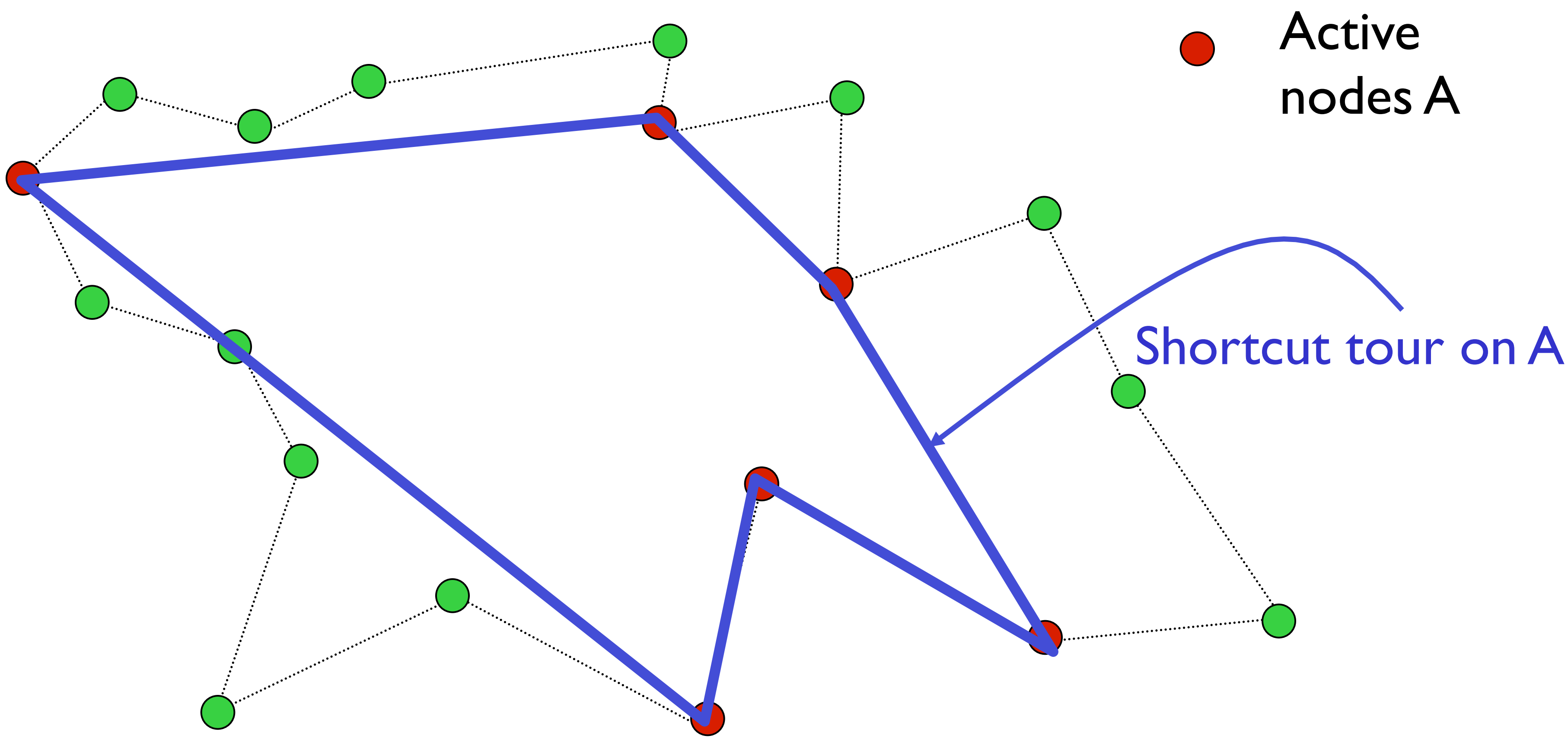

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The *A Priori* TSP

Given input points N and a distribution Π of active sets A 2 2^N, compute tour ¿ to minimize expected length E_A [c(τ_A)], where τ_A is the tour τ shortcut to serve only A $\Rightarrow$ τ * (optimal solution)

Goal: Find tour τ such that E_A [c(τ_A)] $\leq$ ®E_A [c(τ *$_A$)] $\Rightarrow$ ®OPT

(This is an ®-approximation algorithm for the *a priori* TSP.)

How is the probability distribution on active set specified?

- A short (polynomial) list of possibile scenarios;
- Independent probabilities that each point is active;
- A black box that can be sampled.

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Some relevant history for *a priori* TSP

- Jaillet (1985, 1988), Bertsimas (1988), Jaillet, Bertsimas, & Odoni (1990) introduce problem – analyze with probabilistic assumptions on distances
- Schalekamp & S (2007) randomized O(log n)-approximation
- Maybecast problem Karger & Minkoff (2000)
- Rent-or-buy problem Gupta, Kumar, Pál, Roughgarden (2007)
- Stochastic Steiner Tree variants Gupta, Pál, Ravi, Sinha (2004)
  Gupta, Ravi, Sinha ('04), Hayraptian, Swamy, Tardos ('05)
  Garg, Gupta, Leonardi, Sankowski (2008)
- Universal TSP Bartholdi & Plazman (1989), Jia, Lin, Noubir, Rajaraman & Sundaram, (2005), Hajiaghayi, Kleinberg & Leighton (2006), Gupta, Hajiaghayi, Räcke (2006)

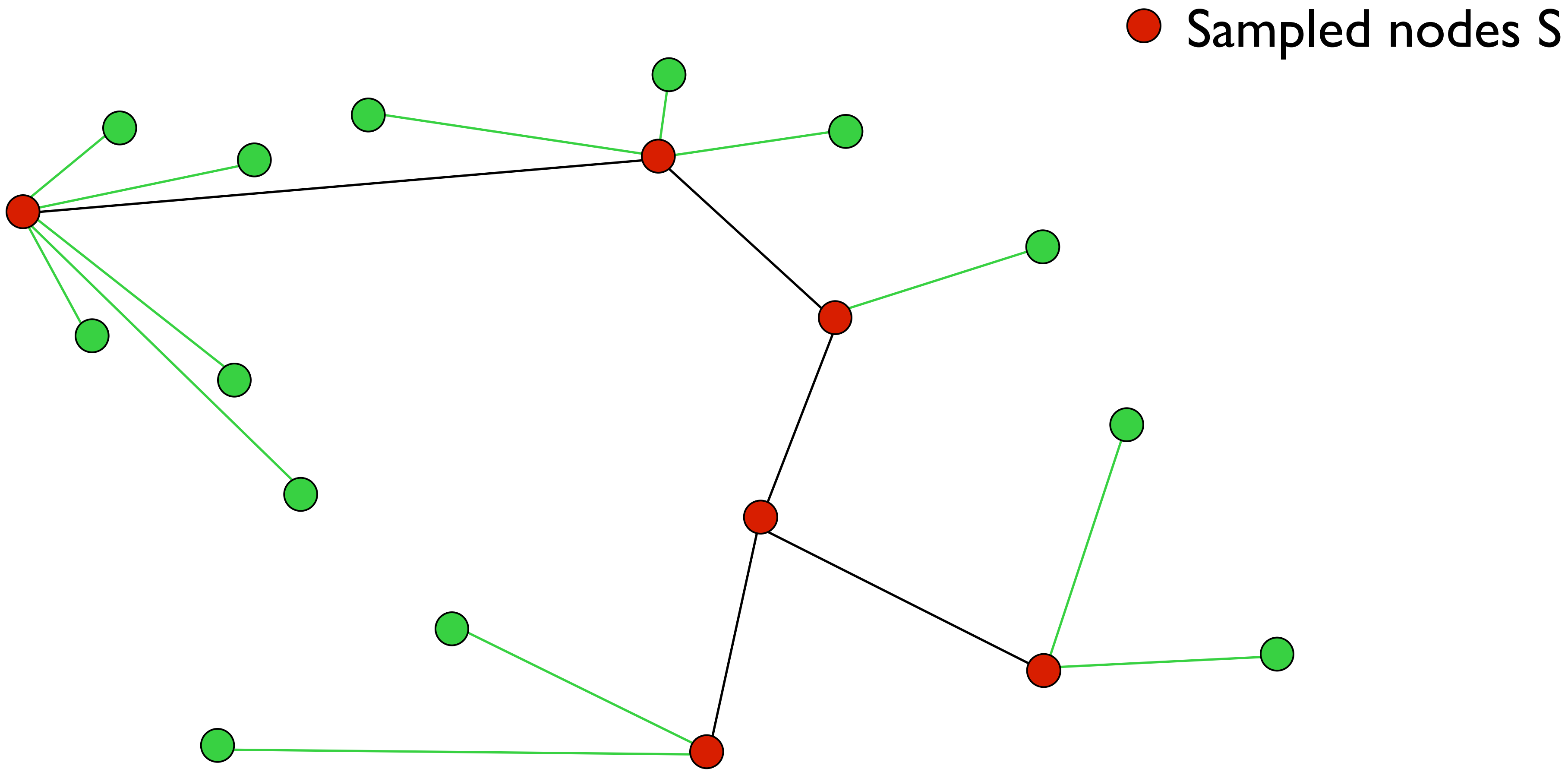
The One Random Sample Algorithm

1. Draw sample $S \subseteq N$ according to Π (i.e., pick each point j independently with probability p_j)
2. Build minimum spanning tree on S
3. For each $j \notin S$, connect j to its nearest neighbor in S
4. Build "double tree" tour of this tree $\Rightarrow \tau$

Simplifying Assumption: $\exists$ node r with $p_r = 1$ (wlog)

Theorem The one random sample algorithm is a 4-approximation algorithm for the *a priori* TSP.

Running the Algorithm



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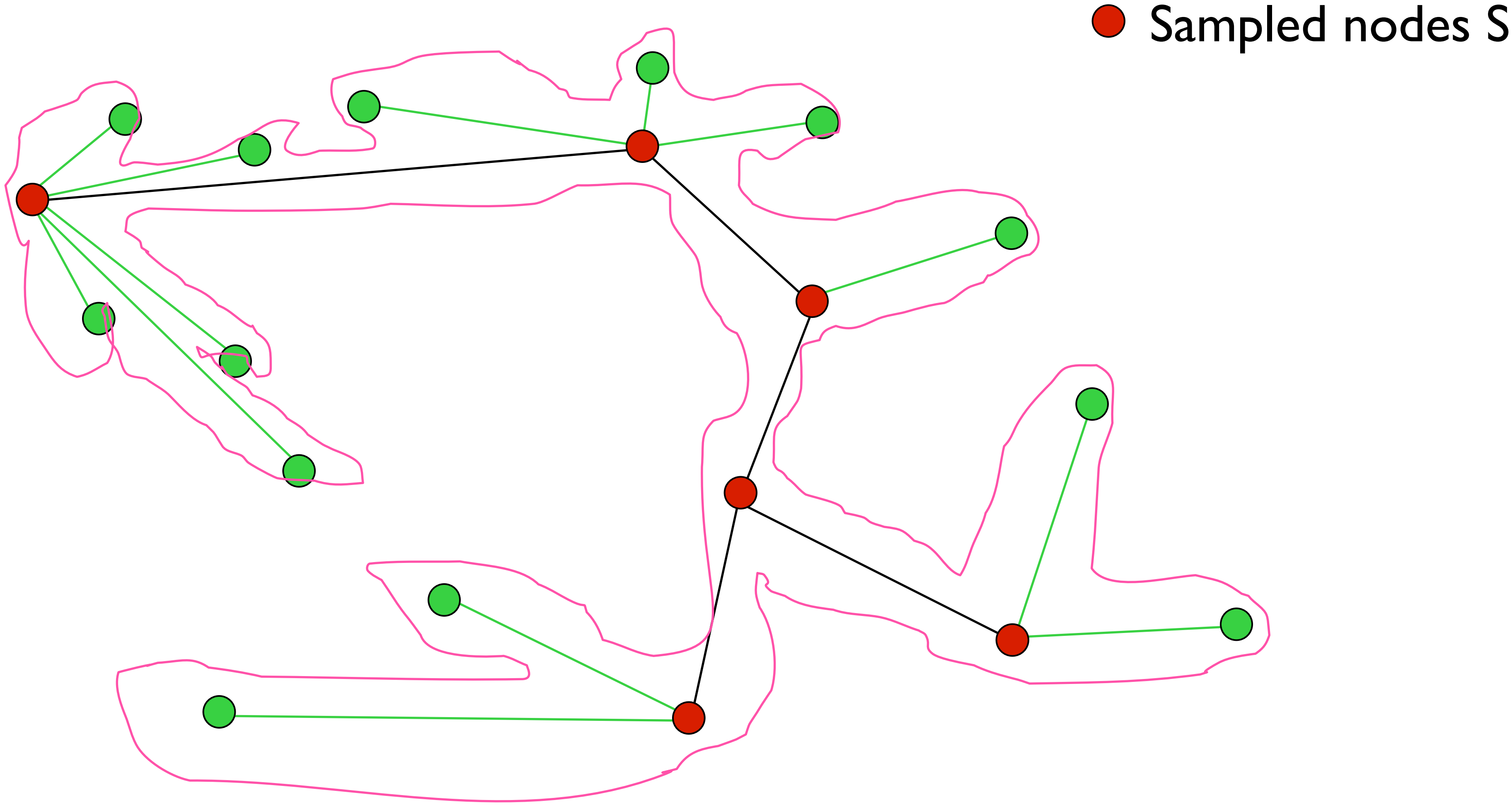

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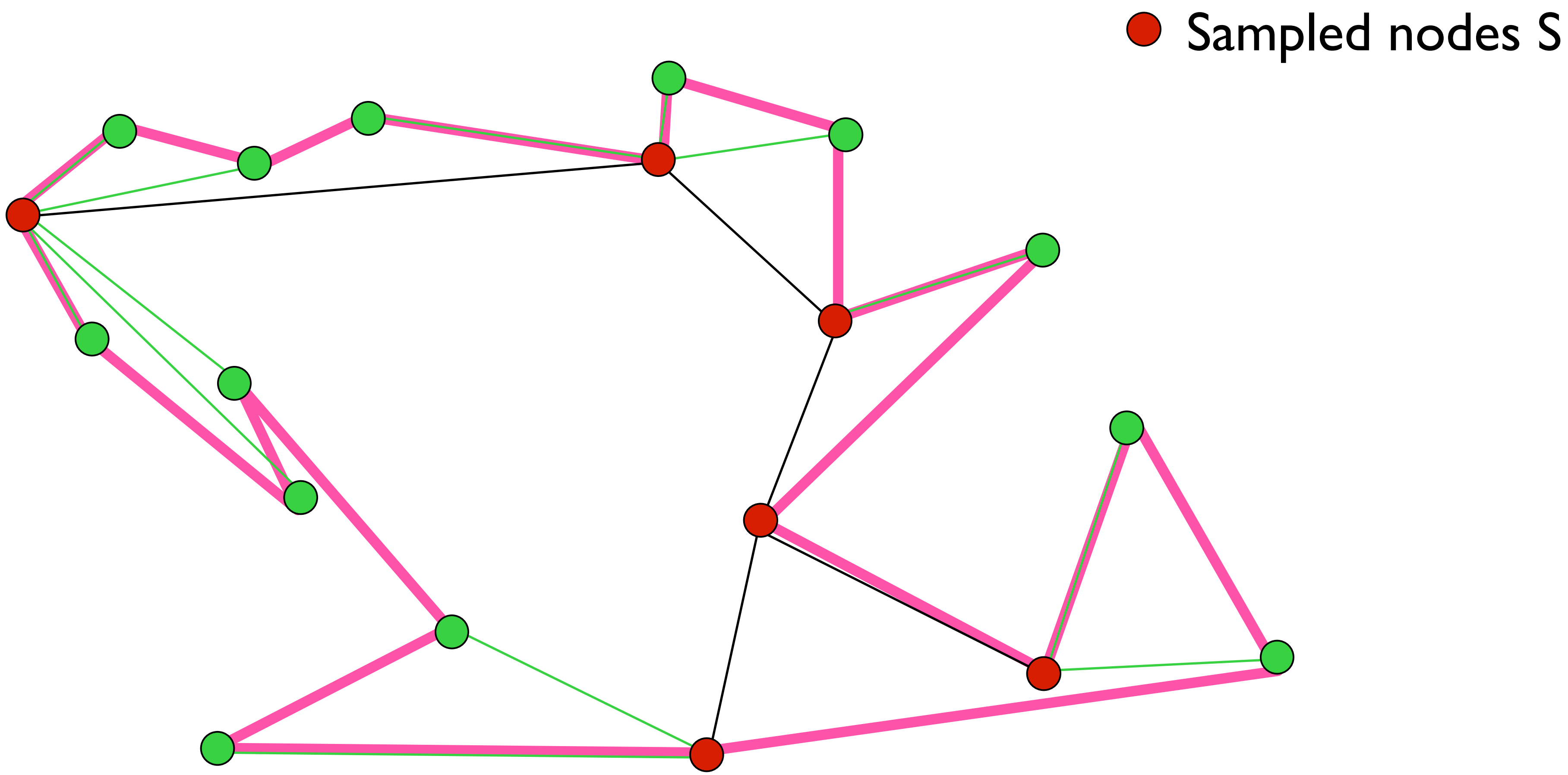

Analyzing the Algorithm

Let $D_j(S)$ be the distance from j to its nearest neighbor in S-{j}

Let MST(S) be the length of the minimum spanning tree on S

Goal: Analyze $E_S [E_A [c(\tau_A)]]$

Fact 1. $E_S [D_j(S)] = E_S [D_j | j \notin S] = E_S [D_j \mid j \in S] = E_A [D_j(A) | j \in A]$

Why? Choice of S-{j} is independent of whether j 2 S, and
S and A are independent draws from same distribution

Fact 2. MST(A) ≤ c(¿ $^*_A$) for each A µ N

Why? Tour ¿* shortcut to A still contains spanning tree

Fact 3. $\sum_{j \neq r} 1(j \in A)\, D_j(A) \le c(\tau^*_A)$ for all A

Why? Any tour on A "leaves" each node i by some edge

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Key Idea: always pay for backbone built on S (for any active A)

$$E_S[E_A[c(\tau_A)]] \leq E_S[2MST(S)]+E_S [E_A [\sum_{j \neq r} 1(j \in A)\, 1(j \notin S)\, 2D_j(S)]]$$

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$$= E_S [2MST(S)] + 2\sum_{j \neq r} p_j (1-p_j)\, E_S[D_j(S)]$$

$$\leq 2(E_S [MST(S)] + \sum_{j \neq r} p_j\, E_S[D_j(S)])$$

$$\leq 2 (OPT+OPT)$$

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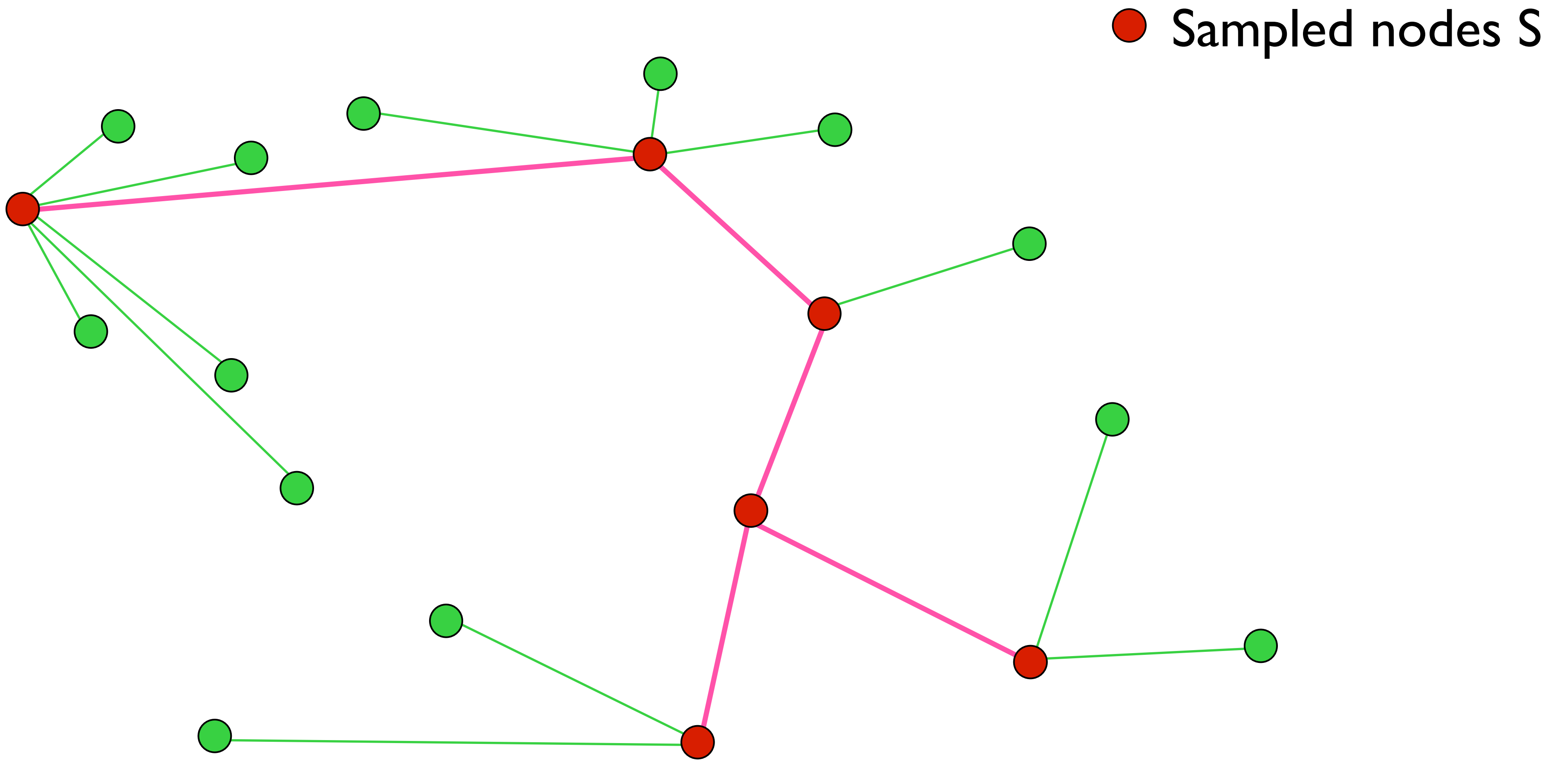

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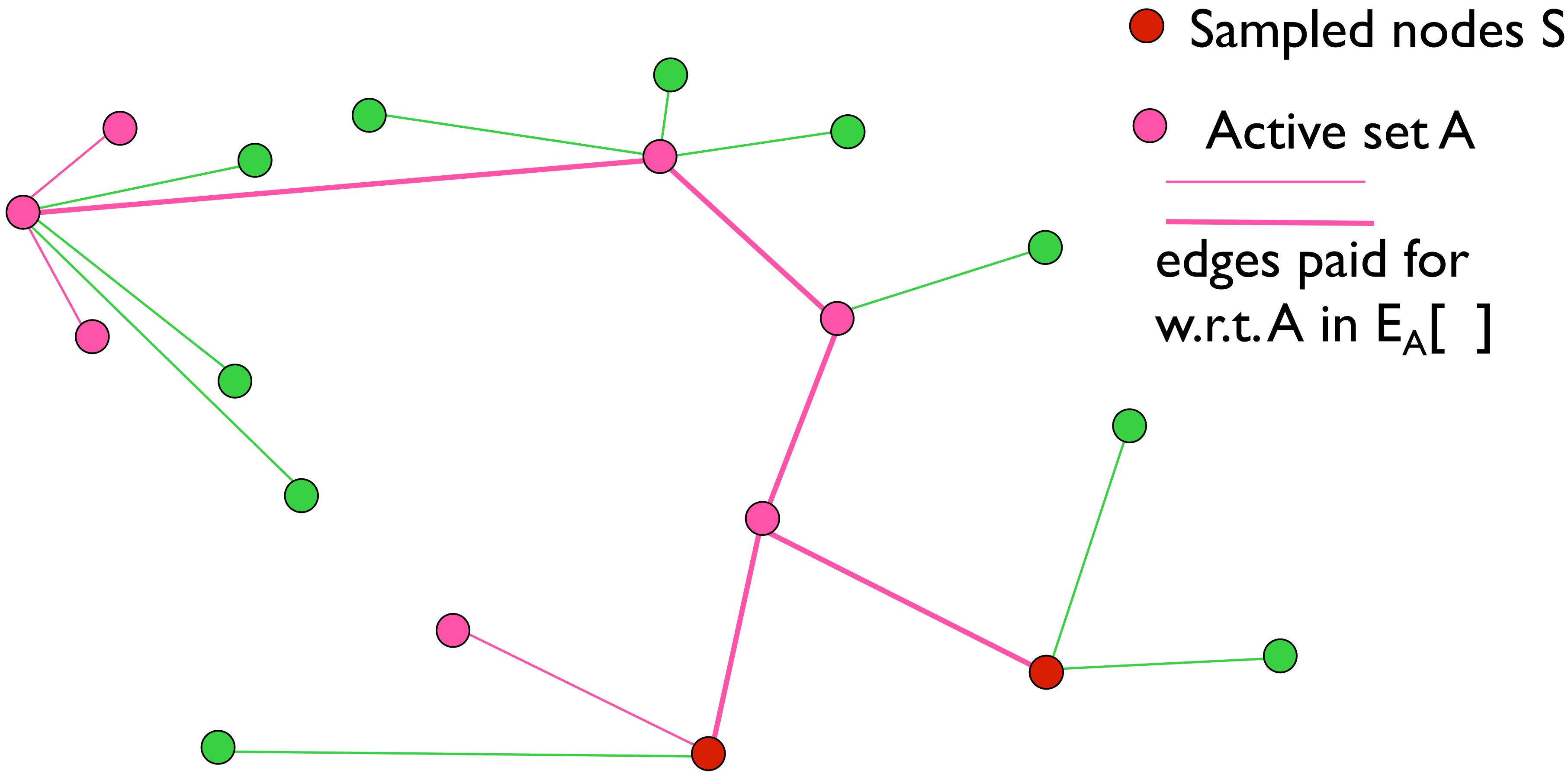

Always pay for all of backbone and just those attached leaves you need

Cost of shortcut tour for A is at most twice the cost of these edges

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Two Footnotes

Can be derandomized -

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Two Footnotes

Can be derandomized – Williamson & van Zuylen (2007) show how to deterministically achieve twice guarantee for rent-or-buy/connected facility location problem by the method of conditional probabilities (by an LP estimate)

Assumption that $p_r = 1$ is not needed;

Need only that $D_j(S)$ is well defined.

Modify Π to condition on that each set has cardinality ≥ 2

Can sample according to this new distribution also, and this just rescales things (any tour has cost 0 restricted to 0 or 1 points) but must be careful about dependence

Theorem (S & Talwar) There is a deterministic 8-approximation algorithm for the a priori TSP in the independent activation model

What about the black box model?

Recent work of Gorodezky, R. Kleinberg, S, & Spencer shows that for a (slightly) restricted class of algorithms can embed a universal computation in an a priori one, and thereby show a non-constant lower bound on performance guarantees possible with a polynomial number of samples

Two-Stage Recourse Model

Given : Probability distribution over inputs.

Stage I : Make some advance decisions – plan ahead or hedge against uncertainty.

Observe the actual input scenario.

Stage II : Take recourse. Can augment earlier solution paying a recourse cost.

Choose stage I decisions to minimize

(stage I cost) + (expected stage II recourse cost).

2-Stage Steiner Tree Problem

Given a set of points N (with root) in a metric space, integer inflation factor λ, and distribution over 2^N

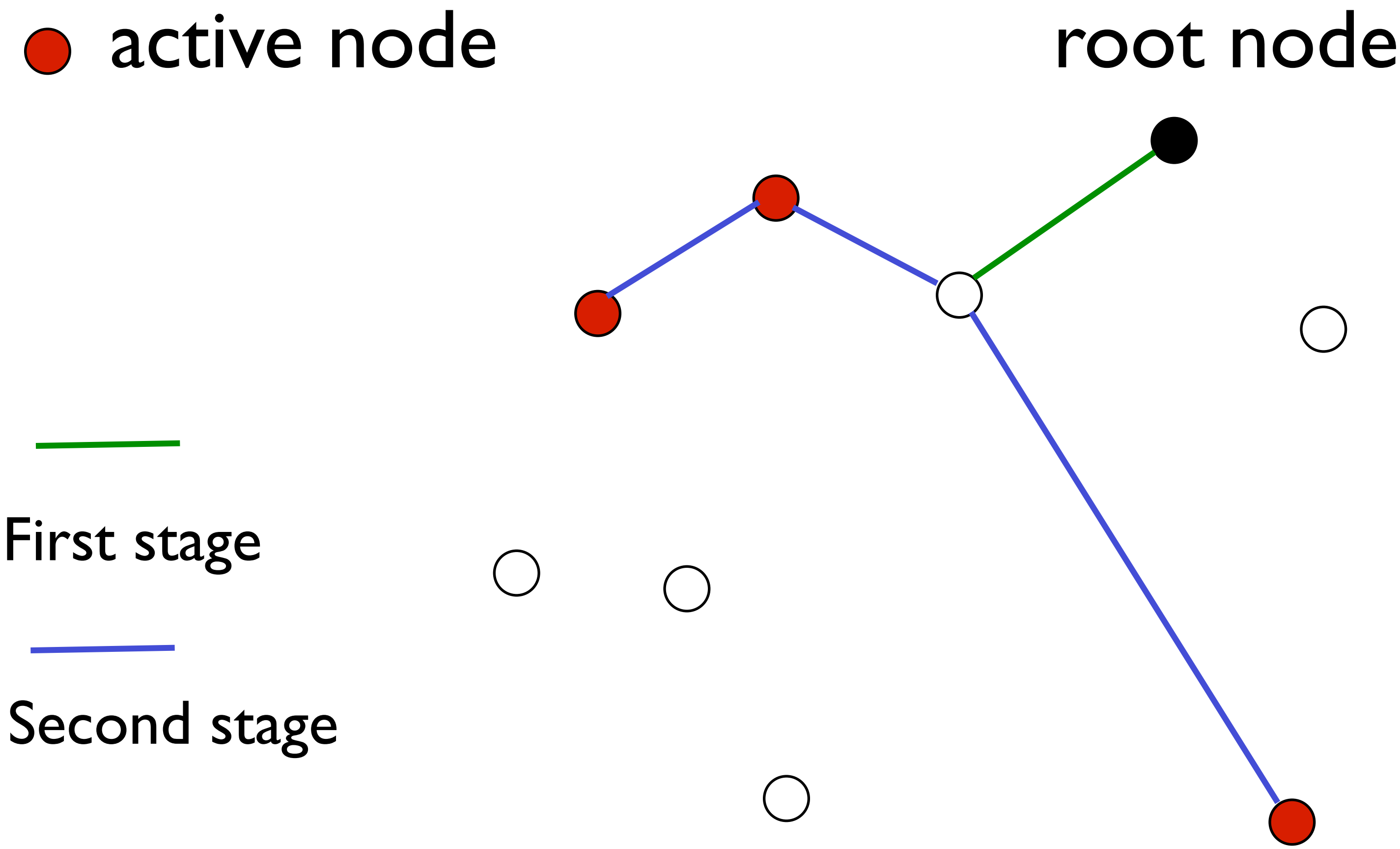
Stage I: install edges A_I – cost of e is c_e

Set of active terminals $T \subseteq N$ is selected (including root)

Stage II: install edges A_{II} s.t. $A_I \cup A_{II}$ is Steiner tree on T - cost of edge e is λc_e

Goal: Minimize
(cost of edges installed in stage I) +
$\lambda\, \mathbf{E}_{T \subseteq N}$ [cost of edges installed for scenario T].

An Example



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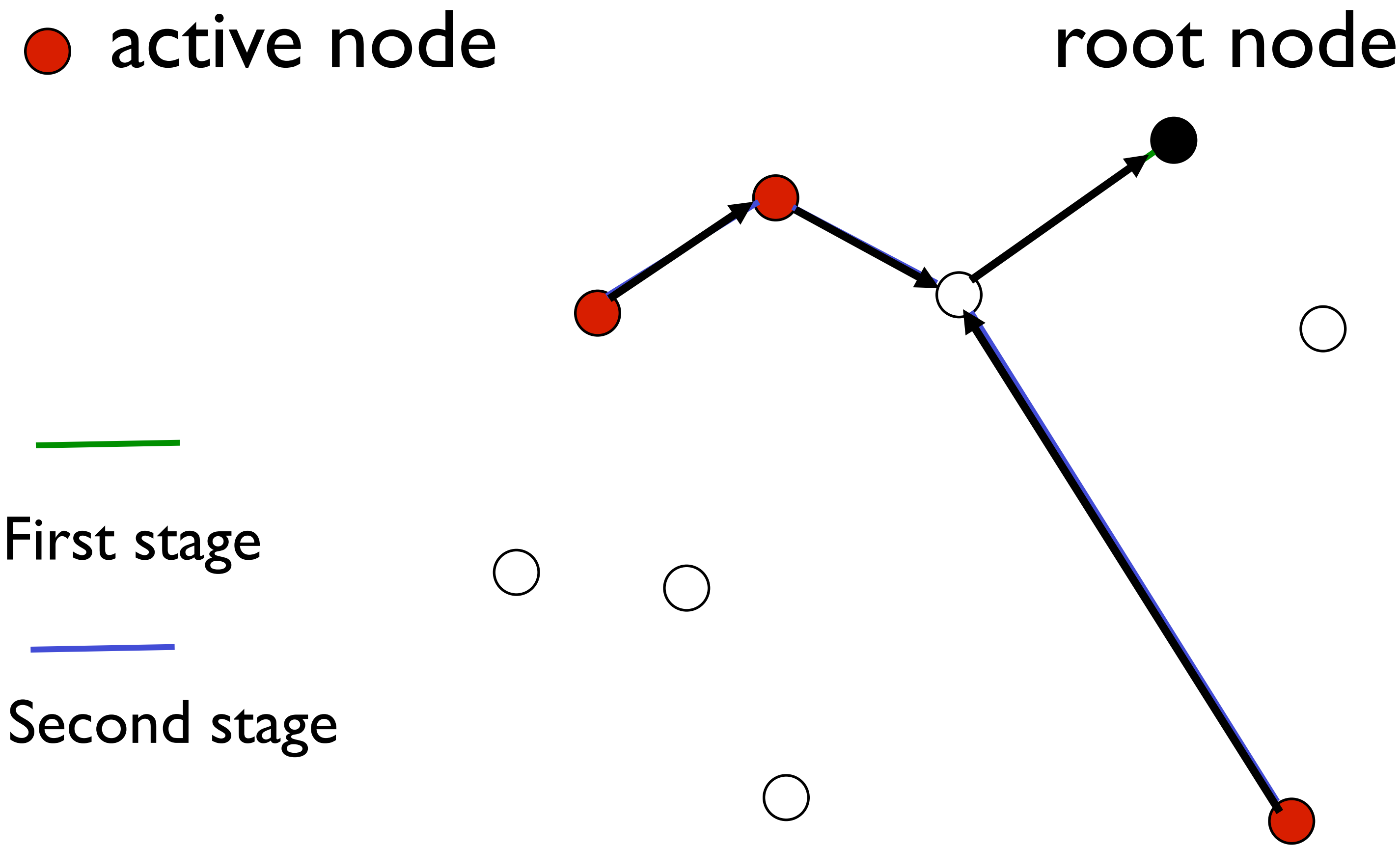
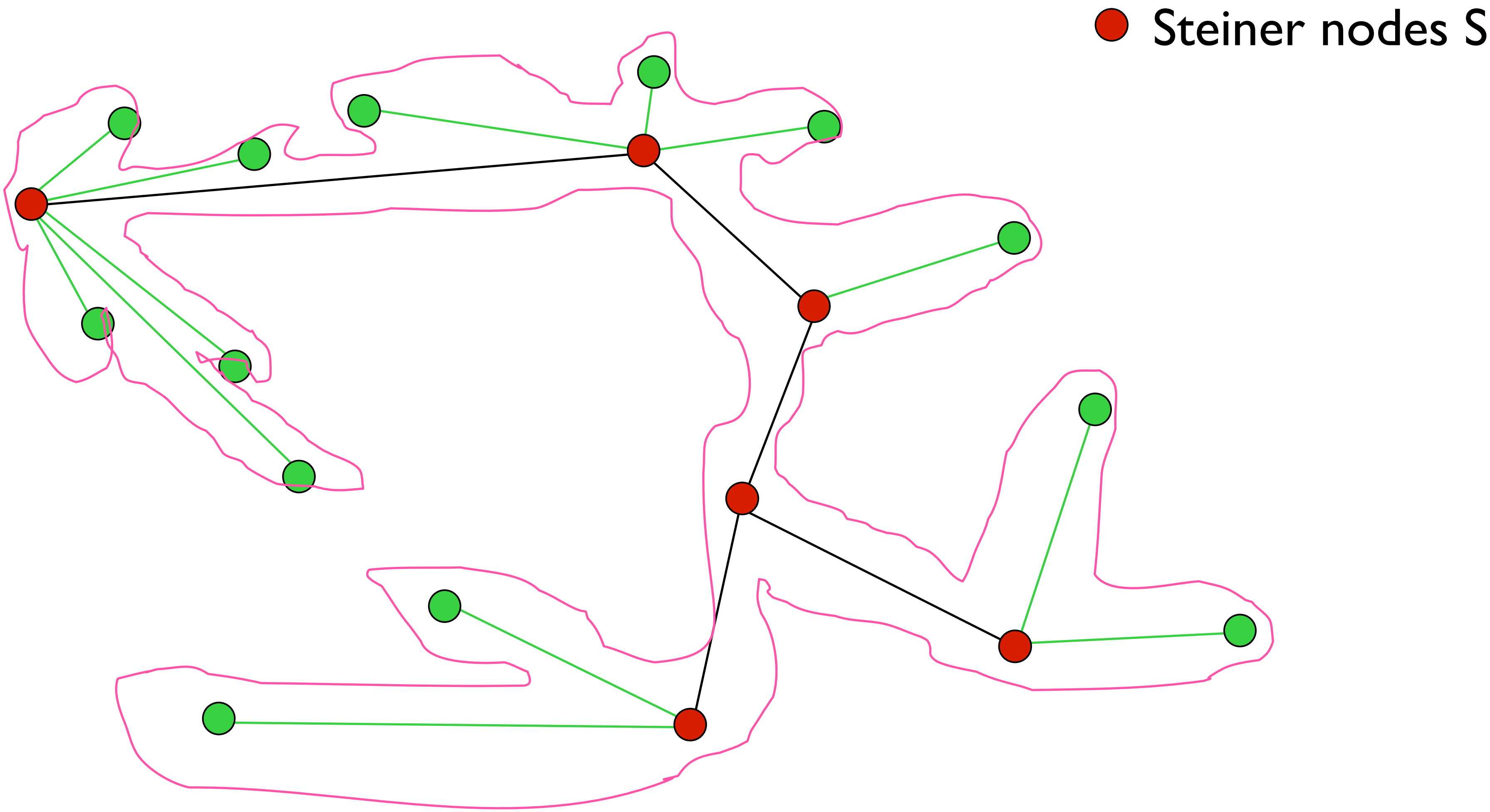


Deterministic Steiner Tree



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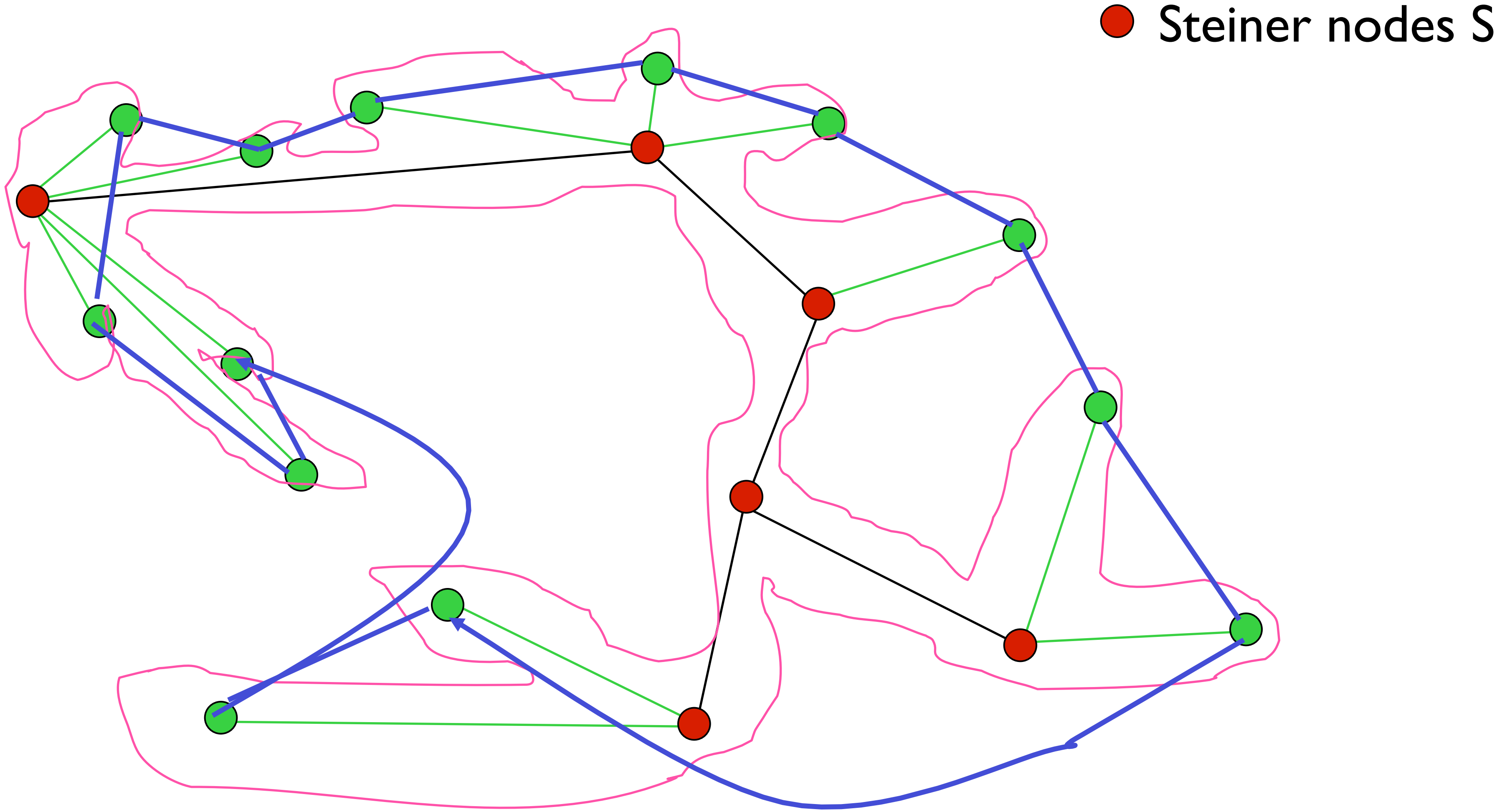

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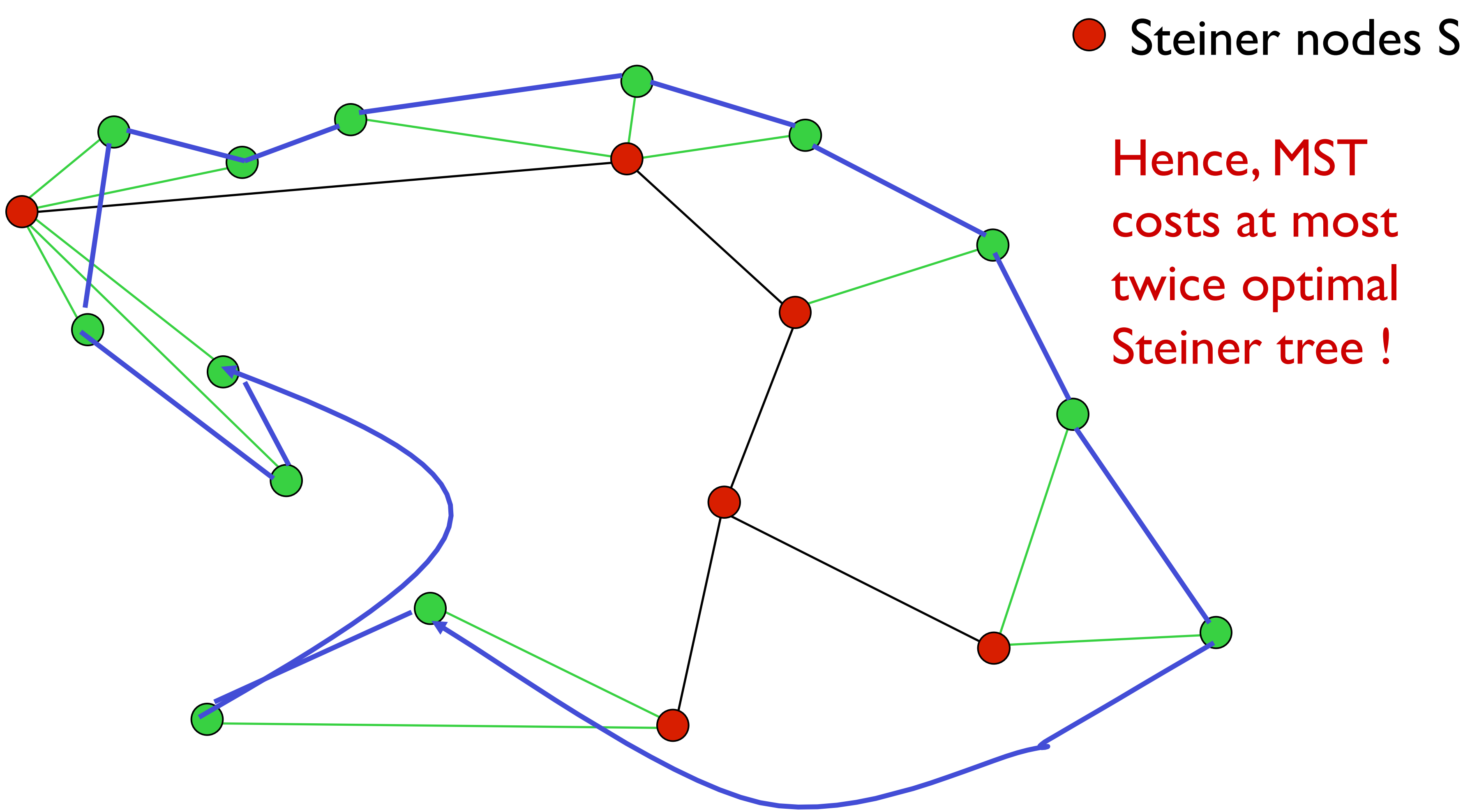

Hence, MST costs at most twice optimal Steiner tree !

Walking “around” optimal Steiner tree gives connected graph, so can view as connected graph on just terminal nodes (by shortcuts)

Boosted Sampling Algorithm
(Gupta, Pál, Ravi, Sinha)

- Draw λ independent samples $S_1, S_2, \ldots, S_\lambda \rightarrow S$
- First stage decision: compute minimum spanning tree (MST) for S (including root), and install those edges $\rightarrow$ Alg_I
- Observe scenario T (independently drawn from same dist)
- Compute (rooted) minimum spanning tree on $S \cup T$,
  (but make cost of edges Alg_I all 0)
  and let e[j] be edge from j to its parent
- Let $Alg_{II} \leftarrow \{ e[j] : j \in T \}$

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Theorem Boosted Sampling is a 4-Approximation Algorithm

First Stage Cost

Optimal cost $Z^* = c(Opt_I) + \lambda\, E_{T \subseteq N} [\, c(Opt_{II}(T))\,]$

We compute MST on $S \leftarrow S_1 \cup \ldots \cup S_\lambda$ for Stage I

(this is 2-approximation for S)

How expensive is it to connect S? Could use

$$Opt_I \cup Opt_{II}(S_1) \cup \ldots \cup Opt_{II}(S_\lambda)$$

Each S_i is identical random T so its expected cost is

$$c(Opt_I) + \lambda\, E_{T \subseteq N} [\, c(Opt_{II}(T))] \rightarrow Z^*$$

Since MST is 2-approximation algorithm $\Rightarrow$

expected Stage I cost is at most $2Z^*$

Cost sharing role of parental edge

- Build a MST on a set $S \cup T$ (plus root)
- Focus on parental edge $e[j]$ for each $j \in S \cup T$
- Total edge cost is $\sum_{j \in S \cup T} c_{e[j]}$
- But this is ≤ twice cost of optimal Steiner tree on $S \cup T$
- Attribute share $c_{e[j]}/2$ of optimal cost to j
- Total share cost is ≤ optimal Steiner tree cost

Second Stage Cost

- Algorithm computes Steiner tree for $S_I \cup \ldots \cup S_\lambda \cup T$
- Consider $T \leftarrow Opt_I \cup Opt_{II}(S_I) \cup \ldots \cup Opt_{II}(S_\lambda) \cup Opt_{II}(T)$
- Role of $\lambda+1$ sets, $S_I, \ldots, S_\lambda, T$ is symmetric
- $E[c(T)] \le c(Opt_I) + (\lambda+1)\, E[c(Opt_{II}(S_i))] \le (\lambda+1)/\lambda\ Z^*$
- Form $D_1, \ldots, D_\lambda$ by deleting nodes in multiple sets
- $\sum_{j \in T\text{-}S} c_{e[j]} + \sum_i \sum_{j \in D_i} c_{e[j]} \le 2c(T)$
- By symmetry, $E[\sum_{j \in T\text{-}S} c_{e[j]}] \le 2c(T)/(\lambda+1)$
- Hence, $E[\sum_{j \in T\text{-}S} c_{e[j]}] \le 2Z^* / \lambda \Rightarrow$ Stage II cost $\le 2Z^*$!

$\Rightarrow$ Boosted Sampling is 4-approximation algorithm

Discrete Stochastic Optimization and Approximation Algorithms

- Area of emerging importance
- Rich source of algorithmic questions
- Can one prove a strong result for approximate stochastic dynamic programming? [Levi Roundy & S] [Halman, Klabjan, Mostagir, Orlin & Simchi-Levi]
- When is sampling information good enough to derive near-optimal solutions?
- Reconsider some well-studied problems but now in "black box" model, not just specific distributions
- Expectation is not enough

Thank You.