Improving Christofides with Randomization & LP

David Shmoys

Cornell University

Traveling Salesman Problem

- (Circuit) Traveling Salesman Problem
  - Given a weighted graph $G = (V, E)$ $(c : E \to \mathbb{R}_+)$, find a minimum Hamiltonian circuit

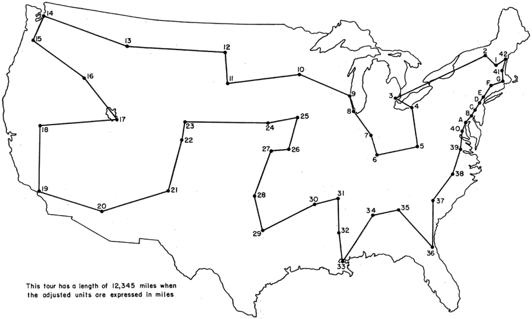

FIG. 16. The optimal tour of 49 cities.

Figure from [Dantzig, Fulkerson, Johnson 1954].

Traveling Salesman Problem

- Metric (circuit) TSP
  - Given a weighted graph $G = (V, E)$ ($c : E \to \mathbb{R}_+$), find a minimum-cost Hamiltonian circuit
  - Triangle inequality holds

    or

    Multiple visits to the same vertex allowed
  - NP-hard
  - Christofides (1976) gave a $3/2$-approximation algorithm

Definition

A ρ-approximation algorithm is a poly-time algorithm that produces a solution of cost within ρ times the optimum

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  - Christofides (1976) gave a $3/2$-approximation algorithm
    - No better performance guarantee known

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A ρ-approximation algorithm is a poly-time algorithm that produces a solution of cost within ρ times the optimum

Traveling Salesman Problem

- Metric *s-t path* TSP
  - Given a weighted graph $G = (V, E)$ $(c : E \to \mathbb{R}_+)$ with *endpoints* $s, t \in V$, find a min-cost s-t Hamiltonian *path*
  - Triangle inequality holds
    or
    Multiple visits to the same vertex allowed
  - NP-hard
  - Hoogeveen (1991) showed that Christofides' algorithm is a $5/3$-approximation algorithm and this bound is tight

Definition

A ρ-approximation algorithm is a poly-time algorithm that produces a solution of cost within ρ times the optimum

Recent Exciting Improvements

- Improvements for asymmetric metric TSP
  - Thin-tree approach of [Asadpour, Goemans, Madry, Oveis Gharan, & Saberi 2010]
  - Augments a randomly selected spanning tree to strongly connected Eulerian subgraph
  - Key role of maximum entropy distribution over spanning trees
- Improvements for unit-weight graphical metric TSP
  - Shortest path metric in an underlying unweighted graph
  - Better approximation than Christofides' [Oveis Gharan, Saberi, Singh 2011], [Mömke, Svensson 2011], [Mucha 2011], [Sebő, Vygen 2012], [Gao 2013]
- Improvements for *s*-*t* path TSP
  - Randomized variant of Christofides' Algorithm [An, Kleinberg, & S 2012]
  - Applies to an *arbitrary* metric
  - Analogues of unit-weight graphical metric results as well
  - Improved analyses of [Sebö 2012] and [Gao 2014]

Even More Recent Exciting Improvements

- Improvements for asymmetric metric TSP
  - Thin-tree approach of [Asadpour, Goemans, Madry, Oveis Gharan, & Saberi 2010]
  - Results of [Marcus, Spielman, & Srivastava 2013] applied in [Anari & Oveis Gharan 2015]
  - Proves stronger integrality gap (but no algorithm)
- Improvements for unit-weight graphical metric TSP
  - Better approximation than Christofides' [Oveis Gharan, Saberi, Singh 2011], [Mömke, Svensson 2011], [Mucha 2011], [Sebő, Vygen 2012], [Gao 2013]
  - Many, many further improvements on a restricted classes of graphs
- Improvements for *s*-*t* path TSP
  - Randomized variant of Christofides' Algorithm [An, Kleinberg, & S 2012]
  - Further algorithmic improvements of [Vygen, 2015, Gottschalk & Vygen, 2015, Sebö & van Zuylen, 2016] $\rightarrow 3/2 + 1/34$

Can Randomization Beat Christofides?

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- Find minimum span. tree $\mathscr{T}_{\text{min}}$
- Augment $\mathscr{T}_{\text{min}}$ into a low-cost Eulerian circuit/path
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  - $O(\log n / \log\log n)$-approx for ATSP
- Oveis Gharan, Saberi, Singh 2011
  - Conjectured $(3/2 - \epsilon)$-approx

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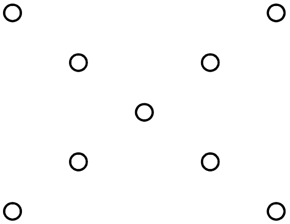
  - Proved for unit-weight graphical metric
- An, Kleinberg, & S 2012 – ϕ-approx for *s*-*t* path TSP
  - Arbitrary metric
  - Simpler random choice

Christofides' Algorithm

- Christofides' algorithm



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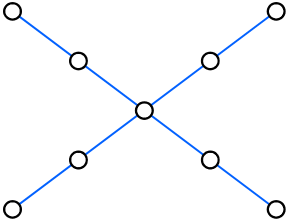

Theorem

Graph G has an Eulerian circuit if and only if G is connected and every vertex of G has even degree

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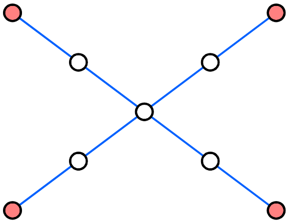

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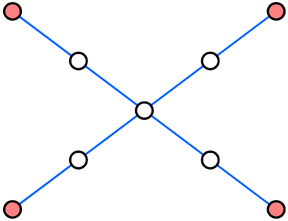

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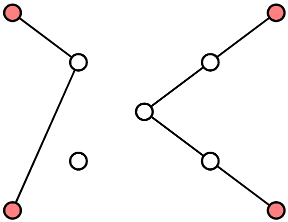

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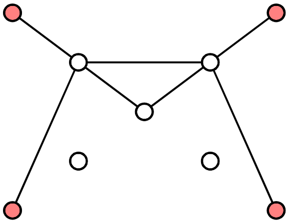

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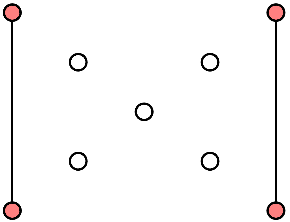

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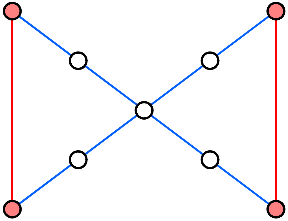

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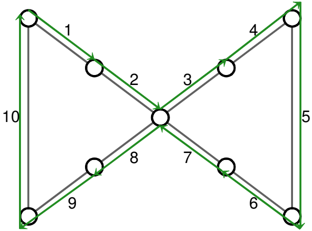
  - Find an Eulerian circuit of $\mathcal{T}_{\text{min}} \cup J$



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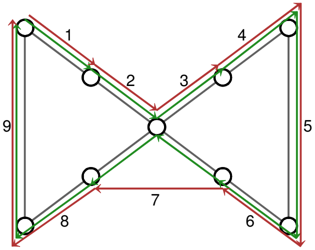
  - Shortcut it into a Hamiltonian circuit H



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Classical Analysis of Christofides' Algorithm

To be done at the board

Christofides' Algorithm, for *s-t* path TSP

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  - Find a minimum spanning tree $\mathscr{T}_{\text{min}}$
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  - Find a minimum T-join J
  - Find an s-t Eulerian *path* of $\mathscr{T}_{\text{min}} \cup J$
  - Shortcut it into an s-t Hamiltonian *path*

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Graph G has an s-t Eulerian path if and only if G is connected and the set of odd-degree vertices is $\{s, t\}$

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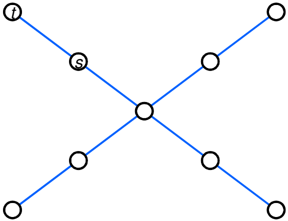

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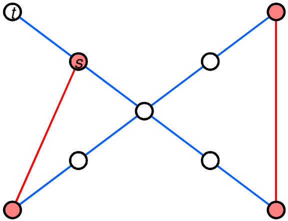

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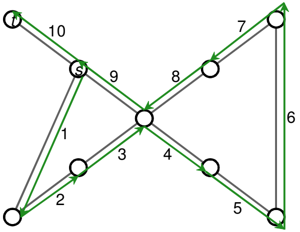

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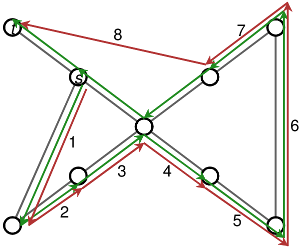

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Path-variant Christofides' algorithm

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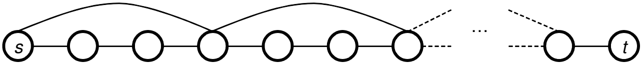
  - $5/3$-approximation algorithm [Hoogeveen 1991]
  - This bound is tight



  - Unit-weight graphical metric:
    distance between two vertices defined as shortest distance on this underlying unit-weight graph

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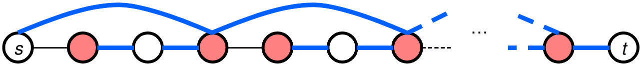

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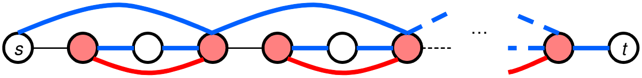

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Hoogeveen's Algorithm for *s*-*t*-path TSP

To be done at the board

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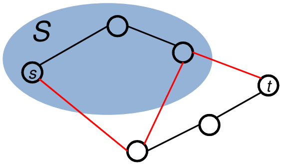
  - $\delta(S)$ for $S \subsetneq V$ denotes the set of edges in cut $(S, \bar{S})$



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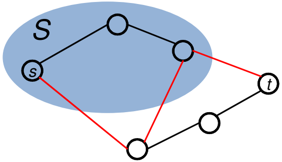

  - For $x \in \mathbb{R}_+^E$ and $F \subset E$,
    - $x(F) := \sum_{f \in F} x_f$
    - Incidence vector of F is $(\chi_F)_e := \begin{cases} 1 & \text{if } e \in F \\ 0 & \text{otherwise} \end{cases}$

Polyhedral Characterizations: Algorithms and Analysis

- Minimum Spanning Tree as Linear Programming Problem
- Each LP has an optimal solution at an extreme point
- Spanning Tree polytope of G $:= \text{conv}\{\chi_{\mathscr{T}} | \mathscr{T}$ is a ST of $G\}$
- How do you write an explicit LP for this geometric object?
- [Edmonds, 1965] Let $\mathcal{P}$ be set of partitions of V
- For partition $\mathcal{S} = (S_1, \ldots, S_k)$, let $\delta(\mathcal{S})$ be set of edges with endpoints in different parts

  For $G = (V, E)$, $\sum_e x_e = n - 1$,

$$\sum_{e \in \delta(\mathcal{S})} x_e \geq k - 1, \quad \forall \mathcal{S} = (S_1, \ldots, S_k) \in \mathcal{P}$$

$$0 \leq x_e \leq 1, \qquad \forall e \in E$$

$$x \in \mathbb{R}^E$$

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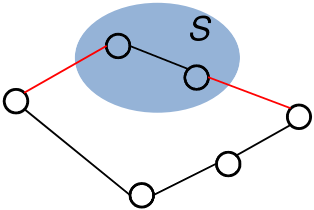
- So can find min spanning tree by solving LP!
- Big LP! Generate just what you “need”
- One important corollary: to prove MST is cheap, exhibit a cheap feasible x: $c(\mathscr{T}_{\min}) \leq c(x)$

Held-Karp TSP Relaxation

- Held-Karp relaxation (for circuit TSP)
  ([Dantzig, Fulkerson, Johnson 1954], [Held, Karp 1970])
  For $G = (V, E)$,

$$\begin{cases} \sum_{e \in \delta(\{v\})} x_e = 2, & \forall v \in V \\ \sum_{e \in \delta(S)} x_e \geq 2, & \forall S \subsetneq V, S \neq \emptyset \\ x_e \in \{0, 1\} & \forall e \in E \end{cases}$$



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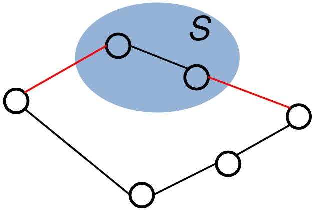

Let x^* be LP optimum;
$c(x^*) \leq c(\mathrm{OPT})$

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  - Any feasible solution to this LP, scaled by $\frac{n-1}{n}$, is in the spanning tree polytope
    - Spanning Tree polytope of G $:=$ conv$\{\chi_{\mathscr{T}} | \mathscr{T}$ is a ST of $G\}$

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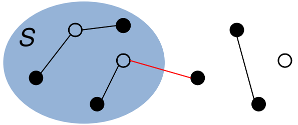
  - $c(\mathscr{T}_{\text{min}}) \leq c(\frac{n-1}{n}x^*) \leq c(x^*) \leq c(OPT)$

Polyhedral Characterization of T-joins

Definition

For $T \subset V$, $J \subset E$ is a T-join if the set of odd-degree vertices in $G' = (V, J)$ is T

- Polyhedral characterization of T-joins [Edmonds, Johnson 1973]



$$\begin{cases} \sum_{e \in \delta(S)} y_e \geq 1, & \forall S \subset V, |S \cap T| \text{ odd} \\ y \in \mathbb{R}_+^E \end{cases}$$

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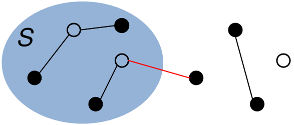
Definition

For $T \subset V$, $J \subset E$ is a T-join if the set of odd-degree vertices in $G' = (V, J)$ is T

- Polyhedral characterization of T-joins [Edmonds, Johnson 1973]



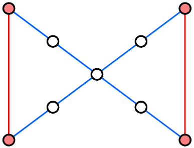
$$\begin{cases} \sum_{e \in \delta(S)} y_e \geq 1, & \forall S \subset V, |S \cap T| \text{ odd} \\ y \in \mathbb{R}_+^E \end{cases}$$

- Call a feasible solution a *fractional T-join*; its cost upper-bounds $c(J)$

LP-based Analysis of Christofides' Algorithm

Theorem (Wolsey 1980)

Christofides' algorithm is a 3/2-approximation algorithm



Proof.

$c(\mathscr{T}_{\min}) \leq c(\frac{n-1}{n}x^*) \leq c(x^*)$

$y^* := \frac{1}{2}x^*$ is a fractional T-join

$$\text{(Held-Karp)} \quad \begin{cases} \sum_{e \in \delta(\{v\})} x_e = 2, & \forall v \in V \\ \sum_{e \in \delta(S)} x_e \geq 2, & \forall S \subsetneq V, S \neq \emptyset \\ 0 \leq x_e \leq 1 & \forall e \in E \end{cases}$$

$$(T\text{-join}) \quad \begin{cases} \sum_{e \in \delta(S)} y_e \geq 1, & \forall S \subset V, |S \cap T| \text{ odd} \\ y \in \mathbb{R}_+^E \end{cases}$$

□

LP-based Analysis of Christofides' Algorithm

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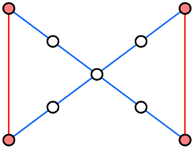
Christofides' algorithm is a $3/2$-approximation algorithm



Proof.

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$y^* := \frac{1}{2} x^*$ is a fractional T-join

$c(J) \leq c(y^*) \leq \frac{1}{2} c(x^*)$

$c(H) \leq c(\mathscr{T}_{\min} \cup J) \leq c(x^*) + c(y^*) \leq \frac{3}{2} c(x^*) \leq \frac{3}{2} c(\mathrm{OPT})$ □

Strength of Held-Karp Relaxation

- Integrality gap
  - Worst-case ratio of the integral optimum to the fractional optimum
  - Natural “limit” of guarantee for algorithm based on LP

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- Path-case
  - $\left[\frac{3}{2}, \frac{5}{3}\right]$; $\frac{3}{2}$?

Path-variant Held-Karp Relaxation

- Path-variant Held-Karp relaxation

  For $G = (V, E)$ and $s, t \in V$,

$$
\begin{cases}
\sum_{e \in \delta(\{s\})} x_e = \sum_{e \in \delta(\{t\})} x_e = 1 & \\
\sum_{e \in \delta(\{v\})} x_e = 2, & \forall v \in V \setminus \{s, t\} \\
\sum_{e \in \delta(S)} x_e \geq 1, & \forall S \subsetneq V, |\{s, t\} \cap S| = 1 \\
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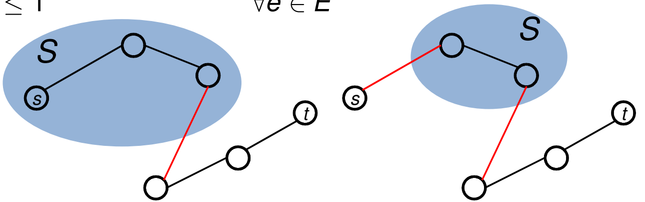
0 \leq x_e \leq 1 & \forall e \in E
\end{cases}
$$

$x \in \mathbb{R}^E$



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Path-variant Held-Karp Relaxation

- Polynomial-time solvable
- Feasible region of this LP is contained in the ST polytope
- Held-Karp solution can be written as a convex combination of (incidence vectors of) spanning trees
- Can find such a decomposition in polynomial time [Grötschel, Lovász, Schrijver 1981]

Algorithm of An, Kleinberg, & S

- Best-of-Many Christofides' Algorithm
  - *Compute an optimal solution x^* to the Held-Karp relaxation*
  - *Rewrite x^* as a convex comb. of spanning trees $\mathscr{T}_1, \ldots, \mathscr{T}_k$*
  - For each $\mathscr{T}_i$:
    - Let T_i be the set of vertices with "wrong" parity of degree: i.e., T_i is the set of even-degree endpoints and other odd-degree vertices in $\mathscr{T}_i$
    - Find a minimum T_i-join J_i
    - Find an s-t Eulerian path of $\mathscr{T}_i \cup J_i$
    - Shortcut it into an s-t Hamiltonian path H_i
  - Output the best Hamiltonian path

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Randomized Algorithm

- Randomized algorithm for simpler analysis
- Sampling Christofides' Algorithm
  - Compute an optimal solution x^* to the Held-Karp relaxation
  - Rewrite x^* as a convex comb. of spanning trees $\mathscr{T}_1, \ldots, \mathscr{T}_k$: $x^* = \sum_{i=1}^k \lambda_i \chi_{\mathscr{T}_i}, \sum_{i=1}^k \lambda_i = 1$
  - *Sample $\mathscr{T}$ by choosing $\mathscr{T}_i$ with probability λ_i*
  - Let T be the set of vertices with "wrong" parity of degree: i.e., T is the set of even-degree endpoints and other odd-degree vertices in $\mathscr{T}$
  - Find a minimum T-join J
  - Find an s-t Eulerian path of $\mathscr{T} \cup J$
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- $\mathrm{E}[c(H)] \leq \rho \cdot \mathrm{OPT} \implies$ Best-of-Many Christofides' Algorithm is ρ-approx. algorithm
- $\Pr[e \in \mathscr{T}] = x^*_e$
  - $\mathrm{E}[c(\mathscr{T})] = \sum_{e \in E} c_e x^*_e = c(x^*)$
  - The rest of the analysis focuses on bounding $c(J)$

Proof of $5/3$-approximation

- Want: a fractional T-join y with $\mathrm{E}[c(y)] \le \frac{2}{3}c(x^*)$
  $x^* :=$ optimal path-variant Held-Karp solution

Proof of 5/3-approximation

- Want: a fractional T-join y with $\mathrm{E}[c(y)] \leq \frac{2}{3}c(x^*)$
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- Circuit case
  - Christofides' algorithm can be interpreted as using half the Held-Karp solution as a fractional T-join
  - Well-known 2-approximation algorithm can be interpreted as using MST as a fractional T-join

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- Is βx^* a fractional T-join for some constant β?

(Held-Karp)
$$\begin{cases} \sum_{e\in\delta(\{s\})} x_e = \sum_{e\in\delta(\{t\})} x_e = 1 & \\ \sum_{e\in\delta(\{v\})} x_e = 2, & \forall v \in V \setminus \{s,t\} \\ \sum_{e\in\delta(S)} x_e \geq 1, & \forall S \subsetneq V, |\{s,t\} \cap S| = 1 \\ \sum_{e\in\delta(S)} x_e \geq 2, & \forall S \subsetneq V, |\{s,t\} \cap S| \neq 1, S \neq \emptyset \\ 0 \leq x_e \leq 1 & \forall e \in E \end{cases}$$

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$$\begin{cases} \sum_{e\in\delta(S)} y_e \geq 1, \quad \forall S \subset V, |S \cap T| \text{ odd} \\ y \in \mathbb{R}_+^E \end{cases}$$

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  - Yes, for $\beta = 1$.
    The present algorithm is a 2-approximation algorithm:
    $\mathrm{E}[c(J)] \leq \mathrm{E}[c(\beta x^*)] = \beta c(x^*)$

	x^*
LB on s-t cut capacities	1
LB on nonseparating cut capacities	2

Proof of $5/3$-approximation

- Want: a fractional T-join y with $\mathrm{E}[c(y)] \leq \frac{2}{3}c(x^*)$.
- Is βx^* a fractional T-join for some β? Yes, for $\beta = 1$.
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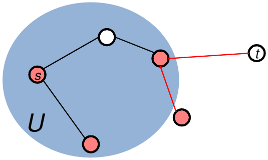
Lemma

An s-t cut $(U, \bar{U})$ that is odd w.r.t. T (i.e., $|U \cap T|$ is odd) has at least two tree edges in it.



	$\chi_{\mathscr{T}}$	x^*
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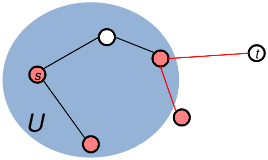

Proof. U contains exactly one of s and t $\Rightarrow$ U has even number of odd-degree vertices

#edges in $\delta(U)$
$= \sum_{v \in U}$ degree of $v - 2 \cdot$ (#edges within U)

□

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Proof of $5/3$-approximation

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Proof of $5/3$-approximation

	$\chi_{\mathscr{T}}$	x^*	y
LB on T-odd s-t cut capacities	2	1	$2\alpha + \beta$
LB on nonseparating cut capacities	1	2	$\alpha + 2\beta$

- $y := \alpha\chi_{\mathscr{T}} + \beta x^*$

Proof of $5/3$-approximation

	$\chi_{\mathscr{T}}$	x^*	y
LB on T-odd s-t cut capacities	2	1	$2\alpha + \beta = 1$
LB on nonseparating cut capacities	1	2	$\alpha + 2\beta = 1$

- $y := \alpha\chi_{\mathscr{T}} + \beta x^*$
  - Choose $\alpha = \beta = \frac{1}{3}$
- $\mathrm{E}[c(y)] = \alpha \mathrm{E}[c(\chi_{\mathscr{T}})] + \beta c(x^*) = (\alpha + \beta)c(x^*)$
- $\mathrm{E}[c(H)] \leq \mathrm{E}[c(\mathscr{T})] + \mathrm{E}[c(J)] \leq (1 + \alpha + \beta)c(x^*)$

Theorem

The given algorithm is a $(1 + \alpha + \beta)$*-approximation algorithm*

Proof of $5/3$-approximation

	$\chi_{\mathcal{T}}$	x^*	y
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Theorem

The given algorithm is a $(1 + \alpha + \beta)$*-approximation algorithm*

- Analysis also works for the original path-variant Christofides' algorithm

First improvement upon $5/3$

	$\chi_{\mathscr{T}}$	x^*	y
LB on T-odd s-t cut capacities	2	1	$2\alpha + \beta$
LB on nonseparating cut capacities	1	2	$\alpha + 2\beta$

- Perturb α and β
  - In particular, decrease α by 2ϵ and increase β by ϵ

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- $E[c(y)] = (\alpha + \beta)c(x^*)$ decreases by $\epsilon c(x^*)$
- $\alpha + 2\beta$ unchanged; nonseparating cuts remain satisfied
- T-odd s-t cuts with small capacity may become violated
  - If violated, by at most $d := 3\epsilon$

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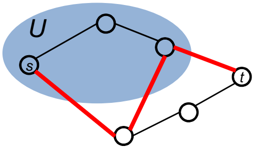
Definition

For $0 < \tau \leq 1$, a τ-narrow cut $(U, \bar{U})$ is an s-t cut with $\sum_{e \in \delta(U)} x_e^* < 1 + \tau$



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Corollary

Each s-t cut $(U, \bar{U})$ with $x^*(\delta(U)) = 1$ is never odd w.r.t. T

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Each s-t cut $(U, \bar{U})$ with $x^*(\delta(U)) = 1$ is never odd w.r.t. T

Proof.

Expected number of tree edges in the cut is equal to $x^*(\delta(U))$:

$$\mathrm{E}[|\delta(U) \cap \mathscr{T}|] = \sum_{e \in \delta(U)} \Pr[e \in \mathscr{T}] = \sum_{e \in \delta(U)} x^*_e = 1$$

So $|\delta(U) \cap \mathscr{T}|$ is identically 1. □

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Proof.

- τ-narrow $\Rightarrow$ Expected # of tree edges in the cut is $< 1 + \tau$
- If $(U, \bar{U})$ is odd w.r.t. T, there must be ≥ 2 tree edges in it $\Rightarrow \Pr[|U \cap T| \text{ odd}] \leq \Pr[|\delta(U) \cap (T)| \geq 2]$
- Expt. # of tree edges in cut is $\geq\ 1 + \Pr[|\delta(U) \cap (T)| \geq 2] \geq 1 + \Pr[|U \cap T| \text{ odd}] \quad \Rightarrow \Pr[|U \cap T| \text{ odd}] < \tau$

□

First improvement upon $5/3$

- Nonseparating cuts & s-t cuts with high capacities are safe
- τ-narrow cuts may be violated when they are T-odd
- This happens with probability smaller than $\tau = O(\epsilon)$
- When this happens, the cut will have deficiency $d = O(\epsilon)$

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- $y := \alpha\chi_{\mathscr{T}} + \beta x^* + r$

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y is a fractional T-join

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$E[c(r)] \leq d\tau c(x^*)$

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- When this happens, the cut will have deficiency $d = O(\epsilon)$
- *Suppose* edge sets of τ-narrow cuts were disjoint
- For each τ-narrow cut $(U, \bar{U})$, define "correction vector" f_U defined as the Held-Karp solution restricted to $\delta(U)$

$$(f_U)_e = \begin{cases} x_e^* & \text{if } e \in \delta(U) \\ 0 & \text{otherwise} \end{cases}$$

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- $y := \alpha\chi_{\mathscr{T}} + \beta x^* + \sum_{U:(U,\bar{U}) \text{ is } \tau-\text{narrow}, |U\cap T| \text{ odd}} d \cdot f_U$

$$
\begin{aligned}
& \mathrm{E}\left[c\left(\sum_{U:(U,\bar{U}) \text{ is } \tau-\text{narrow}, |U\cap T| \text{ odd}} d \cdot f_U\right)\right] \\
\leq\ & c\left(\sum_{U:(U,\bar{U}) \text{ is } \tau-\text{narrow}} \Pr[|U \cap T| \text{ odd}] \cdot d \cdot f_U\right) \\
\leq\ & d\tau c\left(\sum_{U:(U,\bar{U}) \text{ is } \tau-\text{narrow}} f_U\right) \leq d\tau c(x^*)
\end{aligned}
$$

First improvement upon $5/3$

- Nonseparating cuts & *s*-*t* cuts with high capacities are safe
- τ-narrow cuts may be violated when they are T-odd
- This happens with probability smaller than $\tau = O(\epsilon)$
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- The present algorithm is a 1.6572-approximation algorithm *if τ-narrow cuts were disjoint*: $\mathrm{E}[c(y)] \leq (\alpha + \beta + d\tau)c(x^*)$

First improvement upon $5/3$

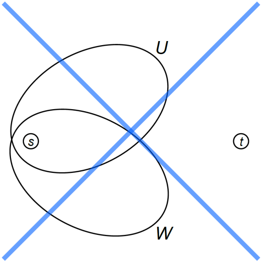
- τ-narrow cuts are not disjoint

First improvement upon $5/3$

- τ-narrow cuts are not disjoint, but "almost" disjoint

Lemma

τ-narrow cuts do not cross: i.e., for τ-narrow cuts $(U, \bar{U})$ and $(W, \bar{W})$ with $s \in U, W$, either $U \subset W$ or $W \subset U$.



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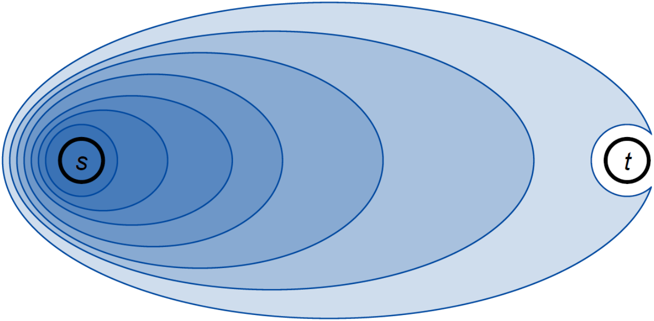

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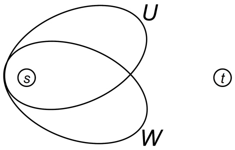

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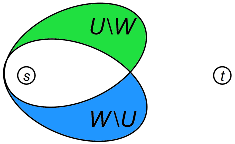
$$x^*(\delta(U)) + x^*(\delta(W)) \geq x^*(\delta(U \setminus W)) + x^*(\delta(W \setminus U)) \geq 2 + 2$$



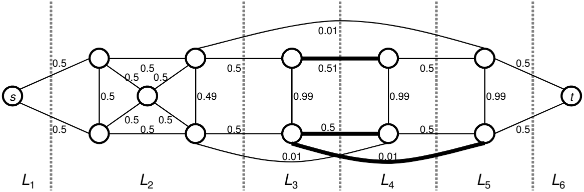
□

First improvement upon $5/3$

Corollary

There exists a partition $L_1, \ldots, L_\ell$ of V such that

- $L_1 = \{s\}$, $L_\ell = \{t\}$, *and*
- $\{U | (U, \bar{U})$ *is* τ*-narrow,* $s \in U\} = \{U_i | 1 \leq i < \ell\}$, *where* $U_i := \cup_{k=1}^{i} L_k$



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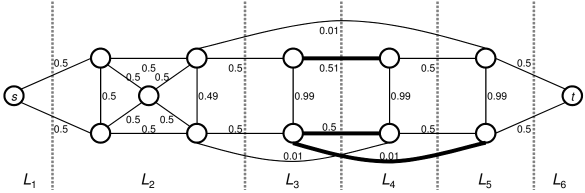

Thick edges show F_3

- We choose "representative edge set" $F_i := E(L_i, L_{\ge i+1})$ for each $\delta(U_i)$. We claim:
  - F_i's are disjoint
  - F_i has large capacity

First improvement upon $5/3$

Lemma

$x^*(F_i) \geq 1 - \frac{\tau}{2}$

Proof.

$A := x^*(E(L_{\leq i-1}, L_i))$, $B := x^*(E(L_i, L_{\geq i+1}))$,
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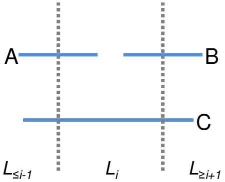

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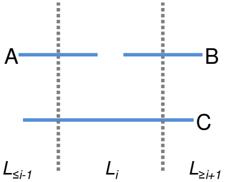

$$B + A \geq 2$$

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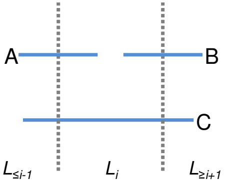

$$\begin{aligned} B + A &\geq 2 \\ B + C &\geq 1 \end{aligned}$$

□

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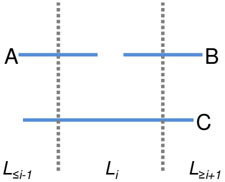

$$
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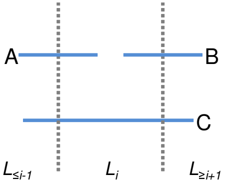

$$\begin{aligned} B + A &\geq 2 \\ B + C &\geq 1 \\ 1 + \tau &> A + C \\ 2B &> 2 - \tau \end{aligned}$$

$$x^*(F_i) = B > 1 - \frac{\tau}{2}$$

□

First improvement upon $5/3$

- τ-narrow cuts are the only cuts that may potentially be violated
- For τ-narrow cuts,
  - deficiency is at most $d = O(\epsilon)$
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- $\mathrm{E}[c(y)] \leq (\alpha + \beta + d\tau)c(x^*)$
  $\mathrm{E}[c(y)] \leq (\alpha + \beta + \frac{d\tau}{1-\frac{\tau}{2}})c(x^*) \leq 0.6577c(x^*)$
- The present algorithm is a 1.6577-approximation algorithm

Intermission.

Traveling Salesman Problem

- Metric (circuit) TSP
  - Given a weighted graph $G = (V, E)$ ($c : E \to \mathbb{R}_+$), find a minimum-cost Hamiltonian circuit
  - Triangle inequality holds
    or
    Multiple visits to the same vertex allowed
  - NP-hard
  - Christofides (1976) gave a $3/2$-approximation algorithm

Definition

A ρ-approximation algorithm is a poly-time algorithm that produces a solution of cost within ρ times the optimum

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    - No better performance guarantee known

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Traveling Salesman Problem

- Metric *s-t path* TSP
  - Given a weighted graph $G = (V, E)$ ($c : E \to \mathbb{R}_+$) with *endpoints* $s, t \in V$, find a min-cost s-t Hamiltonian *path*
  - Triangle inequality holds
    or
    Multiple visits to the same vertex allowed
  - NP-hard
  - Hoogeveen (1991) showed that Christofides' algorithm is a $5/3$-approximation algorithm and this bound is tight

Definition

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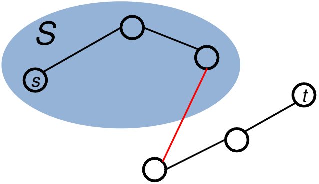
A Quick Recap

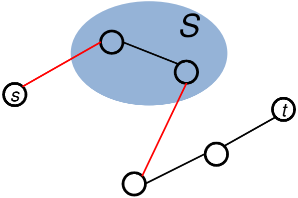
- LP relaxation for s-t path TSP

  For $G = (V, E)$ and $s, t \in V$,

$$
\begin{cases}
\sum_{e\in\delta(\{s\})} x_e = \sum_{e\in\delta(\{t\})} x_e = 1 & \\
\sum_{e\in\delta(\{v\})} x_e = 2, & \forall v \in V \setminus \{s,t\} \\
\sum_{e\in\delta(S)} x_e \geq 1, & \forall S \subsetneq V, |\{s,t\} \cap S| = 1 \\
\sum_{e\in\delta(S)} x_e \geq 2, & \forall S \subsetneq V, |\{s,t\} \cap S| \neq 1, S \neq \emptyset \\
0 \leq x_e \leq 1 & \forall e \in E
\end{cases}
$$

$$x \in \mathbb{R}^E$$





Christofides' Algorithm, for *s*-*t* path TSP

- Christofides' algorithm
  - Find a minimum spanning tree $\mathscr{T}_{\min}$
  - Let T be the set of vertices with "wrong" parity of degree: i.e., *T is the set of even-degree endpoints and other odd-degree vertices in $\mathscr{T}_{\min}$*
  - Find a minimum T-join *J*
  - Find an s-t Eulerian *path* of $\mathscr{T}_{\min} \cup J$
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Graph G has an s-t Eulerian path if and only if G is connected and the set of odd-degree vertices is $\{s, t\}$

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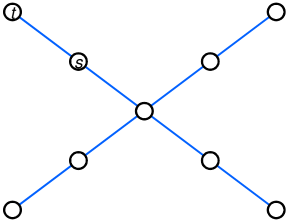

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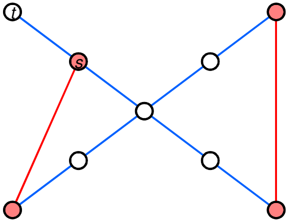

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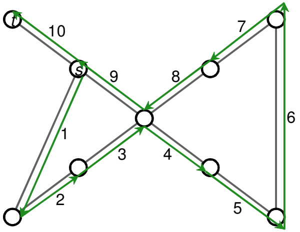

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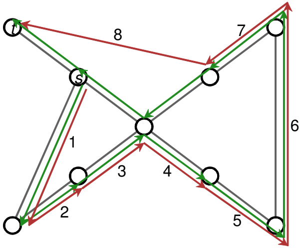

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Algorithm of An, Kleinberg, & S

- Best-of-Many Christofides' Algorithm
  - *Compute an optimal solution x^* to the Held-Karp relaxation*
  - *Rewrite x^* as a convex comb. of spanning trees $\mathscr{T}_1, \ldots, \mathscr{T}_k$*
  - For each $\mathscr{T}_i$:
    - Let T_i be the set of vertices with "wrong" parity of degree: i.e., T_i is the set of even-degree endpoints and other odd-degree vertices in $\mathscr{T}_i$
    - Find a minimum T_i-join J_i
    - Find an s-t Eulerian path of $\mathscr{T}_i \cup J_i$
    - Shortcut it into an s-t Hamiltonian path H_i
  - Output the best Hamiltonian path

Randomized Algorithm

- Randomized algorithm for simpler analysis
- Sampling Christofides' Algorithm
  - Compute an optimal solution x^* to the Held-Karp relaxation
  - Rewrite x^* as a convex comb. of spanning trees $\mathscr{T}_1, \ldots, \mathscr{T}_k$: $x^* = \sum_{i=1}^k \lambda_i \chi_{\mathscr{T}_i}$, $\sum_{i=1}^k \lambda_i = 1$
  - *Sample $\mathscr{T}$ by choosing $\mathscr{T}_i$ with probability λ_i*
  - Let T be the set of vertices with "wrong" parity of degree: i.e., T is the set of even-degree endpoints and other odd-degree vertices in $\mathscr{T}$
  - Find a minimum T-join J
  - Find an s-t Eulerian path of $\mathscr{T} \cup J$
  - Shortcut it into an s-t Hamiltonian path H
- $\Pr[e \in \mathscr{T}] = x^*_e$

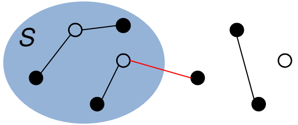
Lemma $\mathrm{E}[c(\mathscr{T})] = \sum_{e \in E} c_e x^*_e = c(x^*)$
Lemma $\mathrm{E}[c(J)] \leq \bigstar \cdot c(x^*)$
Corollary $\mathrm{E}[c(H)] \leq \mathrm{E}[c(\mathscr{T} \cup J)] \leq (1 + \bigstar) c(x^*)$

Polyhedral Characterization of T-joins

Definition

For $T \subset V$, $J \subset E$ is a T-join if the set of odd-degree vertices in $G' = (V, J)$ is T

- Polyhedral characterization of T-joins [Edmonds, Johnson 1973]



$$\begin{cases} \sum_{e \in \delta(S)} y_e \geq 1, & \forall S \subset V, |S \cap T| \text{ odd} \\ y \in \mathbb{R}_+^E \end{cases}$$

- Call a feasible solution a *fractional T-join*; its cost upper-bounds $c(J)$
- Key idea: to bound the cost of the optimal T-join, just exhibit a (cheap) fractional one

Proof of $5/3$-approximation

- Want: a fractional T-join y with $\mathrm{E}[c(y)] \leq \frac{2}{3}c(x^*)$.

Lemma

An s-t cut $(U, \bar{U})$ that is odd w.r.t. T (i.e., $|U \cap T|$ is odd) has at least two tree edges in it.

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	$\chi_{\mathscr{T}}$	x^*	y
LB on T-odd s-t cut capacities	2	1	$2\alpha + \beta = 1$
LB on nonseparating cut capacities	1	2	$\alpha + 2\beta = 1$

- $y := \alpha\chi_{\mathscr{T}} + \beta x^*$
  - Choose $\alpha = \beta = \frac{1}{3}$
- $\mathrm{E}[c(y)] = \alpha \mathrm{E}[c(\chi_{\mathscr{T}})] + \beta c(x^*) = (\alpha + \beta)c(x^*)$
- $\mathrm{E}[c(H)] \leq \mathrm{E}[c(\mathscr{T})] + \mathrm{E}[c(J)] \leq (1 + \alpha + \beta)c(x^*)$

Theorem

The given algorithm is a $(1 + \alpha + \beta)$-approximation algorithm

First improvement upon $5/3$

	$\chi_{\mathscr{T}}$	x^*	y
LB on T-odd s-t cut capacities	2	1	$2\alpha + \beta$
LB on nonseparating cut capacities	1	2	$\alpha + 2\beta$

- Perturb α and β
  - In particular, decrease α by 2ϵ and increase β by ϵ

First improvement upon 5/3

	$\chi_{\mathscr{T}}$	x^*	y
LB on T-odd s-t cut capacities	2	1	$2\alpha + \beta$
LB on nonseparating cut capacities	1	2	$\alpha + 2\beta$

- Perturb α and β
  - In particular, decrease α by 2ϵ and increase β by ϵ
- $E[c(y)] = (\alpha + \beta)c(x^*)$ decreases by $\epsilon c(x^*)$
- $\alpha + 2\beta$ unchanged; nonseparating cuts remain satisfied
- T-odd s-t cuts with small capacity may become violated
  - If violated, by at most $d := 3\epsilon$

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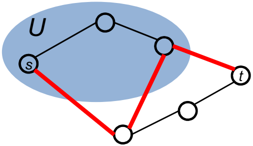
Definition

For $0 < \tau \leq 1$, a τ-narrow cut $(U, \bar{U})$ is an s-t cut with $\sum_{e \in \delta(U)} x_e^* < 1 + \tau$



Feasible Fractional T-join Used Last Time

- Let $\{s\} = U_1 \subset U_2 \subset \cdots \subset U_\ell = V$ be τ-narrow cuts
- There exists a partition $L_1, \ldots, L_\ell$ of V such that
  - $L_1 = \{s\}$, $L_\ell = \{t\}$, and
  - $\{U | (U, \bar{U}) \text{ is } \tau\text{-narrow}, s \in U\} = \{U_i | 1 \leq i < \ell\}$, where $U_i := \cup_{k=1}^{i} L_k$
- We choose "representative edge set" $F_i := E(L_i, L_{\geq i+1})$ for each $\delta(U_i)$
- For each τ-narrow cut $(U_i, \bar{U}_i)$, define "correction vector" f_U as Held-Karp solution restricted to $F_i \subseteq \delta(U_i)$:

  $$(f_U)_e = \begin{cases} x_e^* & \text{if } e \in F_i \\ 0 & \text{otherwise} \end{cases}$$

- $y := \alpha \chi_{\mathscr{T}} + \beta x^* + \sum_{U:(U,\bar{U}) \text{ is } \tau-\text{narrow}, |U \cap T| \text{ odd}} d \cdot \frac{1}{1-\frac{\tau}{2}} \cdot f_U$

Gao's Feasible Fractional T-join

- Let $\{s\} = U_1 \subset U_2 \subset \cdots \subset U_\ell = V$ be τ-narrow cuts
- We choose "representative edge" $e(i)$ for each $\delta(U_i)$ as *cheapest in $\delta(U_i)$*
- For each τ-narrow cut $(U_i, \bar{U}_i)$, define "correction vector" f_U as Held-Karp solution restricted to $F_i \subseteq \delta(U_i)$:

$$(f_U)_e = \begin{cases} 1 & \text{if } e = e(i) \\ 0 & \text{otherwise} \end{cases}$$

- Let T be the wrong parity of degree set for the random spanning tree $\mathscr{T}$:

$$y := \alpha\chi_{\mathscr{T}} + \beta x^* + \sum_{U:(U,\bar{U}) \text{ is } \tau-\text{narrow}, |U \cap T| \text{ odd}} (1 - 2\alpha - \beta x^*(\delta(U)))f_U$$

Gao's Feasible Fractional T-join

- Recall we set $\alpha + 2\beta = 1$; now set τ s.t. $2\alpha + \beta(1+\tau) = 1$

Claim

For each τ-narrow cut U, $1 - 2\alpha - \beta x^*(\delta(U)) \geq 0$.

Proof.

Equivalent to have $2\alpha + \beta x^*(\delta(U)) \leq 1$.
Since $x^*(\delta(U)) < 1 + \tau$, the claim follows. □

Lemma

$$y := \alpha\chi_{\mathscr{T}} + \beta x^* + \sum_{U:(U,\bar{U}) \text{ is } \tau-\text{narrow}, |U\cap T| \text{ odd}} (1 - 2\alpha - \beta x^*(\delta(U))) f_U$$

is a feasible fractional T-join,
where T is the wrong parity of degree set for spanning tree $\mathscr{T}$

Lemma

$y := \alpha\chi_{\mathscr{T}} + \beta x^* +$
$$\sum_{U:(U,\bar{U}) \text{ is } \tau\text{-narrow}, |U \cap T| \text{ odd}} (1 - 2\alpha - \beta x^*(\delta(U))) f_U$$
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Proof.

Consider a set S such that $|S \cap T|$ is odd.

Case 1. If S is not an s-t cut (i.e., $|S \cap \{s,t\}| \neq 1$), then

$$\begin{aligned} y(\delta(S)) &\geq \alpha|\mathscr{T} \cap \delta(S)| + \beta x^*(\delta(S)) \\ &\geq \alpha + 2\beta = 1 \end{aligned}$$

Recall we set $\alpha + 2\beta = 1$

□

Lemma

$y := \alpha\chi_{\mathscr{T}} + \beta x^* +$
$$\sum_{U:(U,\bar{U}) \text{ is } \tau-\text{narrow}, |U\cap T| \text{ odd}} (1 - 2\alpha - \beta x^*(\delta(U)))f_U$$

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Proof.

Consider a set S such that $|S \cap T|$ is odd.

Case 2a. If S is an s-t cut (i.e., $|S \cap \{s, t\}| = 1$) and S is not τ-narrow, then

$$y(\delta(S)) \geq \alpha|\mathscr{T} \cap \delta(S)| + \beta x^*(\delta(S)) \geq 2\alpha + \beta(1 + \tau) = 1$$

Recall we set $\alpha + 2\beta = 1$ and set τ s.t. $2\alpha + \beta(1 + \tau) = 1$

□

Lemma

$y := \alpha\chi_{\mathscr{T}} + \beta x^* +$

$$\sum_{U:(U,\bar{U}) \text{ is } \tau-\text{narrow},|U\cap T| \text{ odd}} (1 - 2\alpha - \beta x^*(\delta(U)))f_U$$

is a feasible fractional T-join,
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Proof.

Consider a set S such that $|S \cap T|$ is odd.

Case 2b. If S is an s-t cut (i.e., $|S \cap \{s,t\}| = 1$) and S is τ-narrow, then

$$\begin{aligned} y(\delta(S)) &\geq \alpha|\mathscr{T} \cap \delta(S)| + \beta x^*(\delta(S)) + \\ &\quad \sum_{U:(U,\bar{U}) \text{ is } \tau-\text{narrow},|U\cap T| \text{ odd}} (1-2\alpha-\beta x^*(\delta(U)))f_U(\delta(S)) \\ &\geq 2\alpha + \beta x^*(\delta(S)) + (1 - 2\alpha - \beta x^*(\delta(S))) \\ &= 1 \end{aligned}$$

□

Lemma

An s-t cut $(U, \bar{U})$ that is odd w.r.t. T (i.e., $|U \cap T|$ is odd) has at least two tree edges in it

Corollary

Each s-t cut $(U, \bar{U})$ with $x^*(\delta(U)) = 1$ is never odd w.r.t. T

Corollary

For any τ-narrow cut $(U, \bar{U})$, $\Pr[|U \cap T| \text{ odd}] < \tau$

Proof.

- τ-narrow $\Rightarrow$ Expected # of tree edges in the cut is $< 1 + \tau$
- If $(U, \bar{U})$ is odd w.r.t. T, there must be ≥ 2 tree edges in it $\Rightarrow \Pr[|U \cap T| \text{ odd}] \leq \Pr[|\delta(U) \cap (T)| \geq 2]$
- Expt. # of tree edges in cut is $\geq\ 1 + \Pr[|\delta(U) \cap (T)| \geq 2] \geq 1 + \Pr[|U \cap T| \text{ odd}] \quad \Rightarrow \Pr[|U \cap T| \text{ odd}] < \tau$

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Corollary

For any τ-narrow cut $(U, \bar{U})$, $\Pr[|U \cap T| \textit{ odd}] \leq x^*(\delta(U)) - 1$

Proof.

- Expected # of tree edges in the cut is $x^*(\delta(U))$
- If $(U, \bar{U})$ is odd w.r.t. T, there must be ≥ 2 tree edges in it $\Rightarrow \Pr[|U \cap T| \text{ odd}] \leq \Pr[|\delta(U) \cap (T)| \geq 2]$
- Expt. # of tree edges in cut is $\geq\ 1 + \Pr[|\delta(U) \cap (T)| \geq 2] \geq 1 + \Pr[|U \cap T| \text{ odd}] \quad \Rightarrow \Pr[|U \cap T| \text{ odd}] \leq x^*(\delta(U)) - 1$

□

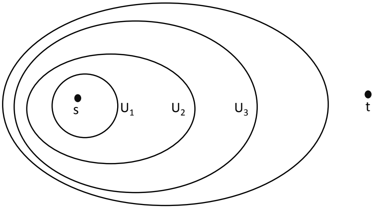
Bounding the total cost of cheapest cut edges

Lemma

Let $\{s\} = U_1 \subset U_2 \subset \cdots \subset U_\ell = V$ *be* τ*-narrow cuts.*
$\sum_{i=1}^{\ell-1} c_{e(i)} \le c(\mathscr{T}_{\min}) \le \sum_e c_e x_e^*$

Proof.

Compute an injective map from U_i to edges of $\mathscr{T}_{\min}$



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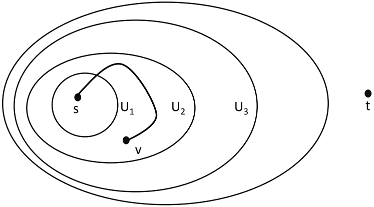
Pick $v \in U_2 - U_1$ and consider s-v path in $\mathscr{T}_{\min}$



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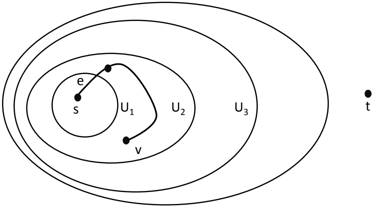

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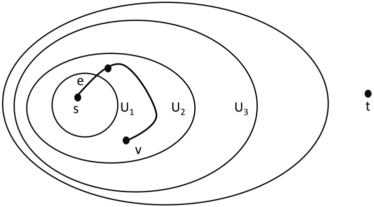
Remove e from $\mathscr{T}_{\min}$, remove U_1, contract s and v, and iterate



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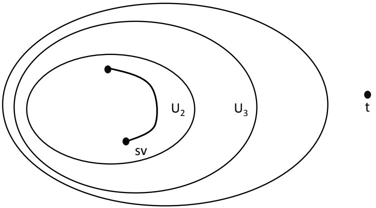

An-Kleinberg-S à la Gao

Theorem

Best-of-many Christofides' algorithm is a deterministic ϕ-approximation algorithm for the s-t path TSP for the general metric, where $\phi = \frac{1+\sqrt{5}}{2} < 1.6181$ is the golden ratio

Proof.

- $y := \alpha\chi_{\mathscr{T}} + \beta x^* +$
  $$\sum_{U:(U,\bar{U}) \text{ is } \tau-\text{narrow}, |U\cap T| \text{ odd}} (1 - 2\alpha - \beta x^*(\delta(U)))f_U$$
- Let $\{s\} = U_1 \subset U_2 \subset \cdots \subset U_\ell = V$ be τ-narrow cuts
- Need to bound $E[c(y)] = \alpha E[c(\mathscr{T})] + \beta c(x^*) +$
  $$\sum_{i=1}^{\ell-1} Pr[|U_i \cap T| \text{ odd}](1 - 2\alpha - \beta x^*(\delta(U_i)))c_{e(i)}$$
  $$\leq \alpha c(x^*) + \beta c(x^*) +$$
  $$\sum_{i=1}^{\ell-1}(x^*(\delta(U_i)) - 1)(1 - 2\alpha - \beta x^*(\delta(U_i)))c_{e(i)}$$

□

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  $\leq (\alpha + \beta)c(x^*) + \max_{z:0<z<\tau} z[1 - 2\alpha - \beta(1 + z)] \sum_{i=1}^{\ell-1} c_{e(i)}$

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  $\leq (\alpha + \beta)c(x^*) + \max_{z:0<z<\tau} z[1 - 2\alpha - \beta(1+z)] \sum_{i=1}^{\ell-1} c_{e(i)}$

  $\leq (\alpha + \beta + \max_{z:0<z<\tau} z[1 - 2\alpha - \beta(1+z)])c(x^*)$

  and set $\alpha = 1 - \frac{2}{\sqrt{5}}$ and $\beta = \frac{1}{\sqrt{5}}$ with $\tau = 3 - \sqrt{5}$

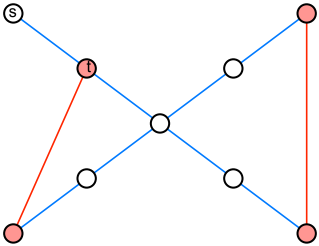
$\square$

Sebö à la Gao

Theorem

Best-of-many Christofides' algorithm is a deterministic 1.6-approximation algorithm for metric s-t path TSP.

- Let $\mathscr{T}_i^{\,st}$ denote the path between s and t in $\mathscr{T}_i$
- *New Idea:* [Guttmann-Beck, Hassin, Khuller, and Raghavachari, 2000] $\mathscr{T}_i - \mathscr{T}_i^{\,st}$ is a T_i-join



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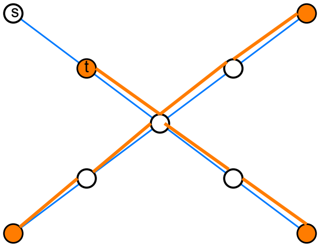

Sebö à la Gao

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Best-of-many Christofides' algorithm is a deterministic 1.6*-approximation algorithm for metric s-t path TSP.*

Corollary

For any τ*-narrow cut* $(U, \bar{U})$, $\Pr[|U \cap T| \text{ odd}] \leq x^*(\delta(U)) - 1$

Corollary

For any τ*-narrow cut* $(U, \bar{U})$, $Pr[|\delta(U) \cap \mathscr{T}| = 1] \geq 2 - x^*(\delta(U))$

Proof.

$$x^*(\delta(U)) = E[|\delta(U) \cap \mathscr{T}] \\ \geq Pr[|\delta(U) \cap \mathscr{T}| = 1] + 2 \cdot Pr[|\delta(U) \cap \mathscr{T}| \geq 2]$$

But

$$Pr[|\delta(U) \cap \mathscr{T}| = 1] + Pr[|\delta(U) \cap \mathscr{T}| \geq 2] = 1 \quad \Rightarrow$$

$$Pr[|\delta(U) \cap \mathscr{T}| = 1] \geq 2 - x^*(\delta(U))$$

□

Sebö à la Gao

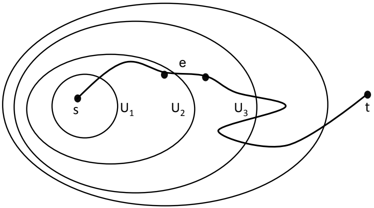
Lemma

Let $\{s\} = U_1 \subset U_2 \subset \cdots \subset U_\ell = V$ *be* τ*-narrow cuts.*
$E[c(\mathscr{T}^{st})] \geq \sum_{i=1}^{\ell-1}(2 - x^*(\delta(U_i)))c_{e(i)}$

Proof.

Consider each realization $\mathscr{T}_j$ of $\mathscr{T}$.

Consider each U_i such that $|\delta(U_i) \cap \mathscr{T}_j| = 1$
Call this edge $\bar{e}_i \quad \Rightarrow \quad \bar{e}_i$ on s-t path



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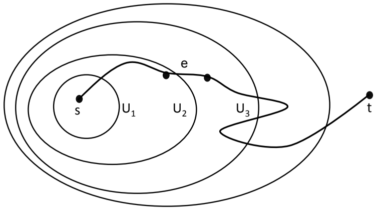
Component of $\mathscr{T}_j - \{\bar{e}_i\}$ with s is graph induced by U_i



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Component of $\mathscr{T}_j - \{\bar{e}_i\}$ with s is graph induced by U_i
$\Rightarrow$ Distinct τ-narrow cuts have distinct unique edges
$c(\mathscr{T}_j^{st}) \geq \sum_{U_i : |\delta(U_i) \cap \mathscr{T}_j| = 1} c(\bar{e}_i) \geq \sum_{U_i : |\delta(U_i) \cap \mathscr{T}_j| = 1} c(e_i)$
Lemma follows by taking expectations and using bound on $Pr[|\delta(U_i) \cap \mathscr{T}| = 1]$. □

Sebö à la Gao

Theorem

Best-of-many Christofides' algorithm is a deterministic 1.6-approximation algorithm for metric s-t path TSP.

Proof.

- Let $\{s\} = U_1 \subset U_2 \subset \cdots \subset U_\ell = V$ be τ-narrow cuts
- $E[c(y)] \leq (\alpha + \beta)c(x^*) + \sum_{i=1}^{\ell-1}(x^*(\delta(U_i)) - 1)(1 - 2\alpha - \beta x^*(\delta(U_i)))c_{e(i)}$

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$$\max_{z:0<z<\tau} \frac{z(1-2\alpha-\beta(1+z)}{1-z} E[c(\mathscr{T}^{st})]$$

- Set $\alpha = 1/9$, $\beta = 4/9$, $\tau = 3/4$ with max at $z = 1/2$
- $E[c(y)] \leq \min\left[E[\mathscr{T} - \mathscr{T}^{st}], \frac{5}{9}c(x^*) + \frac{1}{9}E[c(\mathscr{T}^{st})]\right]$
- Set terms = and use that $E[\mathscr{T} - \mathscr{T}^{st}] + E[c(\mathscr{T}^{st})] = c(x^*)$

□

Unit-Weight Graphical Metric for s-t Path TSP

Improved algorithms for the graphical metric case

- [Oveis Gharan, Saberi, Singh 2011], [Mömke, Svensson 2011], [Mucha 2011]

- [An, Kleinberg, S 2012]Algorithmic use of τ-narrow cuts
- Methods above yield 1.5780-approximation algorithm

- [Sebő & Vygen 2012] 1.5-approximation algorithm
- [Gao 2013] simple 1.5-approximation algorithm & analysis

Gao's Algorithm for Unit-Weight Graphical Metric

- Given metric by $G = (V, E)$
- Include 0, 1, or 2 copies of each edge $e \in E$ to yield Eulerian graph
- Write LP based on this variation: $x_e =$ number of copies
- Partition constraints (as in spanning tree characterization)
- Non s-t cuts must have capacity at least 2
- The algorithm (just a new choice of spanning tree):
  - Solve LP and compute all narrow cuts to get layers L_i
  - Compute spanning tree in each layer
  - Compute $\mathcal{T}$ by adding 1 edge between L_i and L_{i+1}
  - Augment by minimum cost T-join
- Must prove support of x^* always has such a spanning tree
- Analysis: $y = x^*/2$ is feasible fractional T-join

Gao's LP Relaxation

- Let $\mathcal{P}$ be set of partitions of V
- For partition $\mathcal{S} = (S_1, \ldots, S_k)$, let $\delta(\mathcal{S})$ be set of edges with endpoints in different parts

For $G = (V, E)$, minimize $\sum_{e \in E} x_e$
subject to

$$\sum_{e \in \delta(\mathcal{S})} x_e \geq k - 1, \qquad \forall \mathcal{S} = (S_1, \ldots, S_k) \in \mathcal{P}$$

$$\sum_{e \in \delta(S)} x_e \geq 2, \qquad \forall S : \emptyset \subset S \subset V, \ |S \cap \{s, t\}| \neq 1$$

$$0 \leq x_e \leq 2, \qquad \forall e \in E$$

The Structure of the 1-Narrow Cuts

- Let $\{s\} = U_1 \subset U_2 \subset \cdots \subset U_\ell = V$ be 1-narrow cuts.
- Let $L_i = U_i - U_{i-1}$, where $U_0 = \emptyset$.

Claim

Consider the support graph of x^*. For each $1 \leq j \leq k \leq \ell$, the graph induced on the layers $L_j, \ldots, L_k$ is connected.

Proof.

Suppose $1 < j$ and $k < \ell$ - other cases are easier.
Suppose claim is false.
Partition these layers into 2 disconnected sets S_1 and S_2
$x^*(\delta(S_i)) \geq 2$, for each $j \in \{1, 2\}$
Let S_0 and S_3 denote union of lower & higher layers, resp.
These are 1-narrow cuts - $x^*(\delta(S_j)) < 2, j \in \{0, 3\}$.
$4 \leq x^*(\delta(S_1)) + x^*(\delta(S_2)) \leq x^*(\delta(S_0)) + x^*(\delta(S_3)) < 4$ □

The Analysis of Gao's Algorithm

Claim

$x^*/2$ is a fractional T-join, where T is the wrong parity of degree set for the spanning tree $\mathscr{T}$

Proof.

Suppose there exists a set S with $x^*(\delta(S)) < 2$ with $|S \cap T|$ odd.
S must be s-t cut. (Otherwise, LP enforces ≥ 2.)
S must be 1-narrow. (Otherwise, LP again ≥ 2.)
But the number of tree edges crossing each cut U_i is 1.
(By construction.)
But Key Lemma $\Rightarrow$ odd cut has ≥ 2 edges crossing it. □

Some Bigger Picture Questions

- Sampling Christofides' Algorithm
  - Analysis required only that $Pr[e \in \mathcal{T}] = x_e^*$
  - [Asadpour et al.] and [Oveis Gharan et al.] focus on *maximum entropy distribution*:
  - for *each* spanning tree $\mathcal{T}_i$, give weight λ_i to achieve marginal distribution x^* so that λ has maximum entropy (among all such weight functions)
  - What extra power does this distribution provide?
  - Are there other more appropriate sampling approaches?
- Started with approximation algorithm + randomization $\Rightarrow$ improved performance
- Are there other paradigms where this is possible? E.g., GRASP [Resende] - one example is [Buchbinder, Feldman, Naor, Schwartz, SODA 14]

Applications & open questions

- Circuit TSP
  - Is there a better than $3/2$-approximation algorithm?
  - In fact, do we already know such an algorithm for the circuit TSP (and just need the proof)?

Thank you.