Improving Christofides with Randomization & LP

David Shmoys

Cornell University

Traveling Salesman Problem

- (Circuit) Traveling Salesman Problem
  - Given a weighted graph $G = (V, E)$ $(c : E \rightarrow \mathbb{R}_+)$, find a minimum Hamiltonian circuit

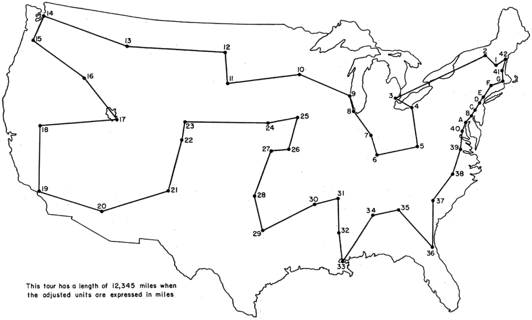

Fig. 16. The optimal tour of 49 cities.

Figure from [Dantzig, Fulkerson, Johnson 1954].

Christofides' Algorithm, for *s*-*t* path TSP

- Christofides' algorithm
  - Find a minimum spanning tree $\mathscr{T}_{\text{min}}$
  - Let T be the set of vertices with "wrong" parity of degree: i.e., *T is the set of even-degree endpoints and other odd-degree vertices in $\mathscr{T}_{\text{min}}$*
  - Find a minimum T-join J
  - Find an s-t Eulerian *path* of $\mathscr{T}_{\text{min}} \cup J$
  - Shortcut it into an s-t Hamiltonian *path*

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Graph G has an s-t Eulerian path if and only if G is connected and the set of odd-degree vertices is $\{s, t\}$

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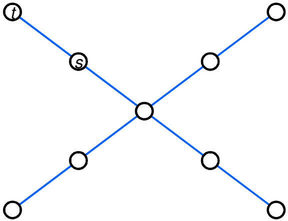

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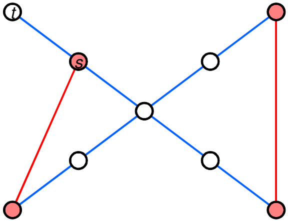

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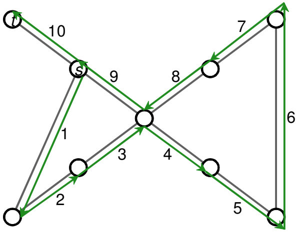

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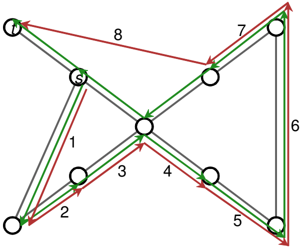

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Path-variant Christofides' algorithm

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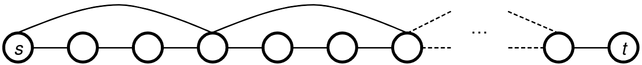
  - $5/3$-approximation algorithm [Hoogeveen 1991]
  - This bound is tight



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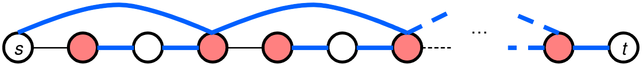

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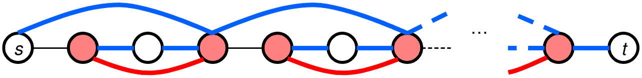

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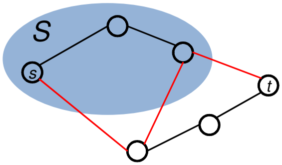

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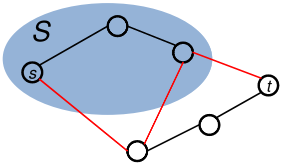

  - For $x \in \mathbb{R}_+^E$ and $F \subset E$,
  - $x(F) := \sum_{f \in F} x_f$
  - Incidence vector of F is $(\chi_F)_e := \begin{cases} 1 & \text{if } e \in F \\ 0 & \text{otherwise} \end{cases}$

Polyhedral Characterizations: Algorithms and Analysis

- Minimum Spanning Tree as Linear Programming Problem
- Each LP has an optimal solution at an extreme point
- Spanning Tree polytope of G $:= \text{conv}\{\chi_{\mathscr{T}} | \mathscr{T}$ is a ST of $G\}$
- How do you write an explicit LP for this geometric object?
- [Edmonds, 1965] Let $\mathcal{P}$ be set of partitions of V
- For partition $\mathcal{S} = (S_1, \ldots, S_k)$, let $\delta(\mathcal{S})$ be set of edges with endpoints in different parts

  For $G = (V, E)$, $\sum_e x_e = n - 1$,

$$\sum_{e \in \delta(\mathcal{S})} x_e \geq k - 1, \quad \forall \mathcal{S} = (S_1, \ldots, S_k) \in \mathcal{P}$$

$$0 \leq x_e \leq 1, \qquad \forall e \in E$$

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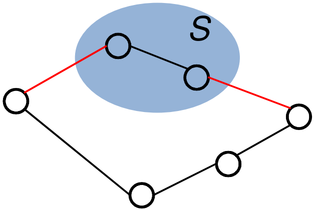
- So can find min spanning tree by solving LP!
- Big LP! Generate just what you "need"
- One important corollary: to prove MST is cheap, exhibit a cheap feasible x: $c(\mathscr{T}_{\min}) \leq c(x)$

Held-Karp TSP Relaxation

- Held-Karp relaxation (for circuit TSP)
  ([Dantzig, Fulkerson, Johnson 1954], [Held, Karp 1970])
  For $G = (V, E)$,

$$\begin{cases} \sum_{e \in \delta(\{v\})} x_e = 2, & \forall v \in V \\ \sum_{e \in \delta(S)} x_e \geq 2, & \forall S \subsetneq V, S \neq \emptyset \\ x_e \in \{0, 1\} & \forall e \in E \end{cases}$$



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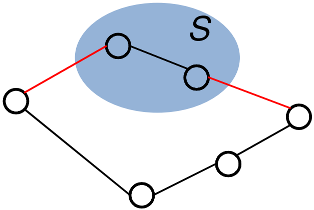

Let x^* be LP optimum;
$c(x^*) \leq c(\mathrm{OPT})$

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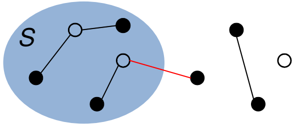
  - $c(\mathscr{T}_{\min}) \leq c(\frac{n-1}{n}x^*) \leq c(x^*) \leq c(OPT)$

Polyhedral Characterization of T-joins

Definition

For $T \subset V$, $J \subset E$ is a T-join if the set of odd-degree vertices in $G' = (V, J)$ is T

- Polyhedral characterization of T-joins [Edmonds, Johnson 1973]



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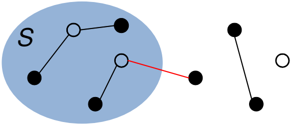


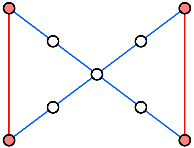
$$\begin{cases} \sum_{e \in \delta(S)} y_e \geq 1, & \forall S \subset V, |S \cap T| \text{ odd} \\ y \in \mathbb{R}_+^E \end{cases}$$

- Call a feasible solution a *fractional T-join*; its cost upper-bounds $c(J)$

LP-based Analysis of Christofides' Algorithm

Theorem (Wolsey 1980)

Christofides' algorithm is a 3/2-approximation algorithm



Proof.

$c(\mathscr{T}_{\min}) \leq c(\frac{n-1}{n}x^*) \leq c(x^*)$

$y^* := \frac{1}{2}x^*$ is a fractional T-join

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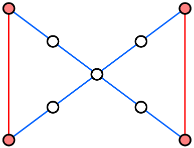

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$c(J) \leq c(y^*) \leq \frac{1}{2}c(x^*)$

$c(H) \leq c(\mathscr{T}_{\text{min}} \cup J) \leq c(x^*) + c(y^*) \leq \frac{3}{2}c(x^*) \leq \frac{3}{2}c(\text{OPT})$ $\square$

Strength of Held-Karp Relaxation

- Integrality gap
  - Worst-case ratio of the integral optimum to the fractional optimum
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- Path-case
  - $\left[\frac{3}{2}, \frac{5}{3}\right]$; $\frac{3}{2}$?

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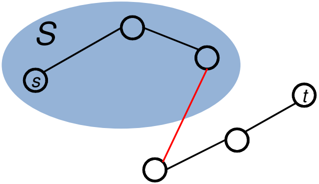

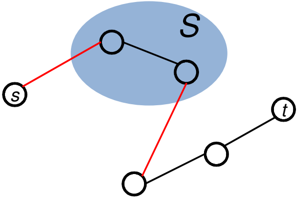

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- Polynomial-time solvable
- Feasible region of this LP is contained in the ST polytope
- Held-Karp solution can be written as a convex combination of (incidence vectors of) spanning trees
- Can find such a decomposition in polynomial time [Grötschel, Lovász, Schrijver 1981]

Algorithm of An, Kleinberg, & S

- Best-of-Many Christofides' Algorithm
  - *Compute an optimal solution x^* to the Held-Karp relaxation*
  - *Rewrite x^* as a convex comb. of spanning trees $\mathcal{T}_1, \ldots, \mathcal{T}_k$*
  - For each $\mathcal{T}_i$:
    - Let T_i be the set of vertices with "wrong" parity of degree: i.e., T_i is the set of even-degree endpoints and other odd-degree vertices in $\mathcal{T}_i$
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Randomized Algorithm

- Randomized algorithm for simpler analysis
- Sampling Christofides' Algorithm
  - Compute an optimal solution x^* to the Held-Karp relaxation
  - Rewrite x^* as a convex comb. of spanning trees $\mathscr{T}_1, \ldots, \mathscr{T}_k$: $x^* = \sum_{i=1}^k \lambda_i \chi_{\mathscr{T}_i}, \sum_{i=1}^k \lambda_i = 1$
  - *Sample $\mathscr{T}$ by choosing $\mathscr{T}_i$ with probability λ_i*
  - Let T be the set of vertices with "wrong" parity of degree: i.e., T is the set of even-degree endpoints and other odd-degree vertices in $\mathscr{T}$
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- $\Pr[e \in \mathscr{T}] = x_e^*$
  - $\mathrm{E}[c(\mathscr{T})] = \sum_{e \in E} c_e x_e^* = c(x^*)$
  - The rest of the analysis focuses on bounding $c(J)$

Proof of $5/3$-approximation

- Want: a fractional T-join y with $\mathrm{E}[c(y)] \leq \frac{2}{3}c(x^*)$
  $x^* :=$ optimal path-variant Held-Karp solution

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- Circuit case
  - Christofides' algorithm can be interpreted as using half the Held-Karp solution as a fractional T-join
  - Well-known 2-approximation algorithm can be interpreted as using MST as a fractional T-join

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- Is βx^* a fractional T-join for some constant β?

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$$(T\text{-join}) \quad \begin{cases} \sum_{e \in \delta(S)} y_e \geq 1, \quad \forall S \subset V, |S \cap T| \text{ odd} \\ y \in \mathbb{R}_+^E \end{cases}$$

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- Is βx^* a fractional T-join for some constant β?
  - Yes, for $\beta = 1$.
    The present algorithm is a 2-approximation algorithm:
    $\mathrm{E}[c(J)] \leq \mathrm{E}[c(\beta x^*)] = \beta c(x^*)$

	x^*
LB on *s*-*t* cut capacities	1
LB on nonseparating cut capacities	2

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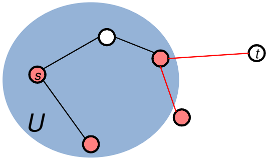
- How about $\alpha\chi_{\mathscr{T}}$?
  - s-t cuts do have some slack in this case

Lemma

An s-t cut $(U, \bar{U})$ that is odd w.r.t. T (i.e., $|U \cap T|$ is odd) has at least two tree edges in it.



	$\chi_{\mathscr{T}}$	x^*
LB on *T-odd* s-t cut capacities	2	1
LB on nonseparating cut capacities	1	2

Proof of $5/3$-approximation

- Want: a fractional T-join y with $\mathrm{E}[c(y)] \leq \frac{2}{3}c(x^*)$.
- Is βx^* a fractional T-join for some β? Yes, for $\beta = 1$.
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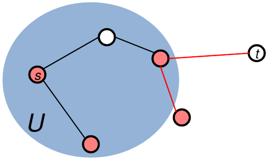

Proof. U contains exactly one of s and t $\Rightarrow$ U has even number of odd-degree vertices

#edges in $\delta(U)$
$= \sum_{v \in U}$ degree of $v - 2 \cdot$ (#edges within U)

□

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- Yes, for $\alpha = 1$.
  The present algorithm is a 2-approximation algorithm:
  $E[c(J)] \leq E[c(\alpha\chi_{\mathscr{T}})] = \alpha c(x^*)$

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Proof of $5/3$-approximation

	$\chi_{\mathscr{T}}$	x^*	y
LB on T-odd s-t cut capacities	2	1	$2\alpha + \beta$
LB on nonseparating cut capacities	1	2	$\alpha + 2\beta$

- $y := \alpha\chi_{\mathscr{T}} + \beta x^*$

Proof of $5/3$-approximation

	$\chi_{\mathscr{T}}$	x^*	y
LB on T-odd s-t cut capacities	2	1	$2\alpha + \beta = 1$
LB on nonseparating cut capacities	1	2	$\alpha + 2\beta = 1$

- $y := \alpha\chi_{\mathscr{T}} + \beta x^*$
  - Choose $\alpha = \beta = \frac{1}{3}$
- $\mathrm{E}[c(y)] = \alpha \mathrm{E}[c(\chi_{\mathscr{T}})] + \beta c(x^*) = (\alpha + \beta)c(x^*)$
- $\mathrm{E}[c(H)] \leq \mathrm{E}[c(\mathscr{T})] + \mathrm{E}[c(J)] \leq (1 + \alpha + \beta)c(x^*)$

Theorem

The given algorithm is a $(1 + \alpha + \beta)$*-approximation algorithm*

Proof of 5/3-approximation

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- Analysis also works for the original path-variant Christofides' algorithm

First improvement upon $5/3$

	$\chi_{\mathscr{T}}$	x^*	y
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- Perturb α and β
  - In particular, decrease α by 2ϵ and increase β by ϵ

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- $E[c(y)] = (\alpha + \beta)c(x^*)$ decreases by $\epsilon c(x^*)$
- $\alpha + 2\beta$ unchanged; nonseparating cuts remain satisfied
- T-odd s-t cuts with small capacity may become violated
  - If violated, by at most $d := 3\epsilon$

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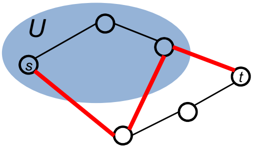
Definition

For $0 < \tau \leq 1$, a τ-narrow cut $(U, \bar{U})$ is an s-t cut with $\sum_{e \in \delta(U)} x^*_e < 1 + \tau$



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Corollary

Each s-t cut $(U, \bar{U})$ with $x^*(\delta(U)) = 1$ is never odd w.r.t. T

$$(T\text{-join}) \quad \begin{cases} \sum_{e \in \delta(S)} y_e \geq 1, & \forall S \subset V, |S \cap T| \text{ odd} \\ y \in \mathbb{R}_+^E \end{cases}$$

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Proof.

Expected number of tree edges in the cut is equal to $x^*(\delta(U))$:

$$\mathrm{E}[|\delta(U) \cap \mathscr{T}|] = \sum_{e \in \delta(U)} \Pr[e \in \mathscr{T}] = \sum_{e \in \delta(U)} x_e^* = 1$$

So $|\delta(U) \cap \mathscr{T}|$ is identically 1. □

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Proof.

- τ-narrow $\Rightarrow$ Expected # of tree edges in the cut is $< 1 + \tau$
- If $(U, \bar{U})$ is odd w.r.t. T, there must be ≥ 2 tree edges in it $\Rightarrow \Pr[|U \cap T| \text{ odd}] \leq \Pr[|\delta(U) \cap (T)| \geq 2]$
- Expt. # of tree edges in cut is $\geq\ 1 + \Pr[|\delta(U) \cap (T)| \geq 2] \geq 1 + \Pr[|U \cap T| \text{ odd}] \qquad \Rightarrow \Pr[|U \cap T| \text{ odd}] < \tau$

□

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- Nonseparating cuts & s-t cuts with high capacities are safe
- τ-narrow cuts may be violated when they are T-odd
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$E[c(r)] \leq d\tau c(x^*)$