Algorithms for Network Flows

Lecture 2: Minimum cost flows

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Valparaíso Summer School, 2017

Slides will be available at: `http://nolver.net/home/valparaiso`

Minimum cost flow

Given: Directed graph $G = (V, E)$, edge capacities $u : E \to \mathbb{R}_+$, costs $c : E \to \mathbb{R}$, demands $b : V \to \mathbb{R}$ with $b(V) = 0$.
Goal: Find a b-flow of minimum cost.

- b-flow: function $f : E \to \mathbb{R}_+$ with $\nabla f_i = b_i$ for all $i \in V$, $f(e) \leq u(e)$ for all $e \in E$.
- The cost of f is $\sum_{e \in E} c(e) f(e)$.

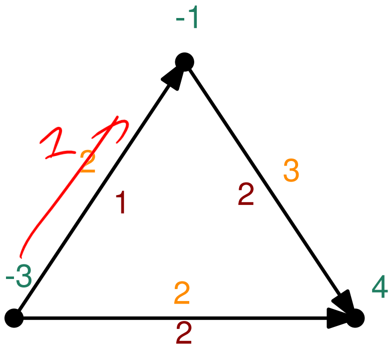
Today's lecture

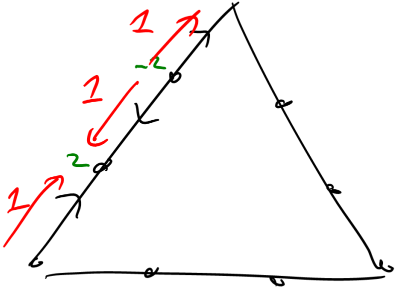
- A strongly polynomial time algorithm for min cost flow due to Goldberg-Tarjan '88.

  (The first strongly polynomial algorithm for the problem was by Tardos '85.)

Transshipment problem

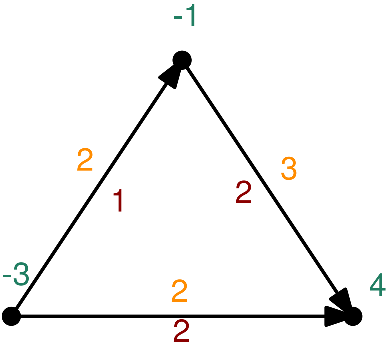
- Same, but with $u(e) = \infty$ for all e.
- Polynomially equivalent

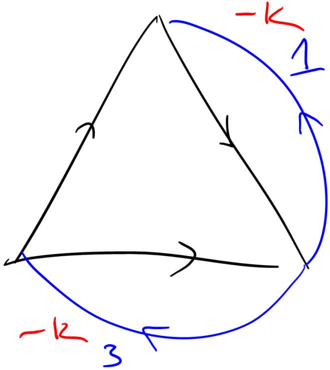




Min cost circulation - also equivalent

Given: Directed graph $G = (V, E)$, edge capacities $u : E \to \mathbb{R}_+$, costs $c : E \to \mathbb{R}$.
Goal: Find a circulation of minimum cost.





A trivial optimality condition

- If $c \geq 0$, then clearly $f = 0$ is the optimal circulation.
- Slightly more: if f is a circulation with $f(e) = 0$ whenever $c(e) > 0$ and $f(e) = u(e)$ whenever $c(e) < 0$, then f is optimal.

For $e \in \overleftrightarrow{E} \setminus E$, define $c(e) := -c(\mathrm{rev}(e))$

Then we can rewrite as simply:

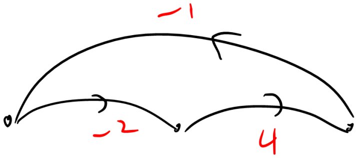
$$c(e) \geqslant 0 \quad \forall e \in E_f .$$

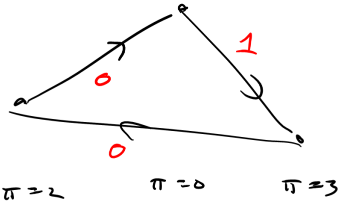
Relabelling

- A labelling (or potential) is any function $\pi : V \to \mathbb{R}$.

Given a labelling π, define the relabelled costs c^π by

$$c^\pi(ij) = c(ij) + \pi_i - \pi_j.$$





$c^\pi(\Gamma) = c(\Gamma)$

$\forall$ cycles Γ

$c^\pi(f) = c(f)$ for any circulation

Relabelling

$c(\mathrm{rev}(e)) = -c(e)$

- A labelling (or potential) is any function $\pi : V \to \mathbb{R}$. e $c(e)$

> Given a labelling π, define the relabelled costs c^π by
>
> $$c^\pi(ij) = c(ij) + \pi_i - \pi_j.$$

Hence if $\exists\, \pi$ s.t. $c^\pi(e) \geq 0 \ \forall e \in E_f$

then f is optimal.

Relabelling

- A labelling (or potential) is any function $\pi : V \to \mathbb{R}$.

Given a labelling π, define the relabelled costs c^π by

$$c^\pi(ij) = c(ij) + \pi_i - \pi_j.$$

Clearly sufficient, is it necessary?

LP, dual and complementary slackness

$$\min \quad \sum_{e \in E} c(e) f(e)$$

$$\text{s.t.} \quad f(e) \le u(e) \quad \forall e \in E \qquad \sigma_e$$

$$f(\delta^+(v)) - f(\delta^-(v)) = 0 \quad \forall v \in V \qquad \pi_i$$

$$f \ge 0$$

Dual:

$$\max \ -\sum u(e)\sigma_e$$

$$\text{s.t.} \ \boxed{-\sigma_{ij} + \pi_j - \pi_i \le c(ij)} \quad \forall ij \in E$$

$$\sigma \ge 0$$

$$\Leftrightarrow \sigma_{ij} \ge -c^\pi(ij)$$

Best choice of σ_e

$$\max \ -\sum_{ij \in E} \max(-c^\pi(ij), 0)\, u(ij)$$

$$\text{s.t.} \quad \pi : V \to \mathbb{R}.$$

Equivalently,

$$\max \sum_{ij \in E} \min(c^\pi(ij), 0)\, u(ij)$$

$$\text{s.t.} \quad \pi : V \to \mathbb{R}$$

Comp. slackness:

$f(e) < u(e) \Rightarrow \alpha_e = 0 \Rightarrow c^{\pi}(ij) \geq 0$

$c^{\pi}(e) > 0 \Rightarrow f(e) = 0.$

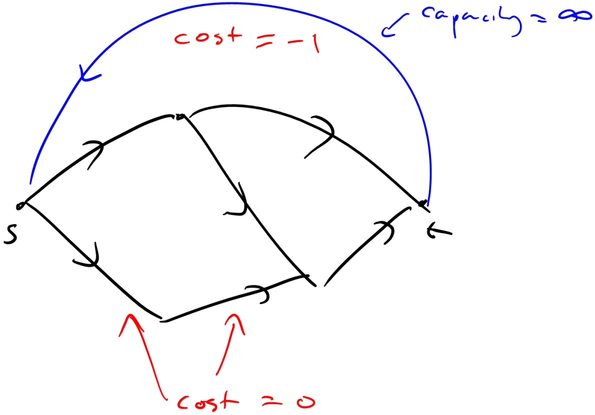
f optimal iff $\nexists$ cycle $\Gamma \subseteq E_f$ with $c(\Gamma) < 0$.

The Goldberg-Tarjan algorithm

$f \leftarrow 0$
while $\exists$ a negative cost cycle **do**
    Find a cycle Γ in G_f of minimum mean cost $c(\Gamma)/|\Gamma|$.
    Push δ := $\min_{e \in C}\{u_f(e)\}$ units of flow backwards around Γ

- Finding a minimum mean cost cycle can be done in time $O(mn)$ by dynamic programming.

Comparing to Edmonds-Karp for max flow



ϵ-optimality

A circulation f is ϵ-optimal if $\exists \pi$ s.t. $c^{\pi}(e) \geq -\epsilon$ for all $\epsilon \in E_f$.

If all costs are integers, and f is ϵ-optimal for some $\epsilon < 1/n$, then f is optimal.

Pf: For any cycle Γ in E_f,

$$c(\Gamma) = c^{\pi}(\Gamma) \geq -\varepsilon |C| > -1$$

≥ 0.

Let

$$\mu(f) := \min_{\Gamma \text{ a cycle in } G_f} \frac{1}{|\Gamma|} \cdot c(\Gamma)$$

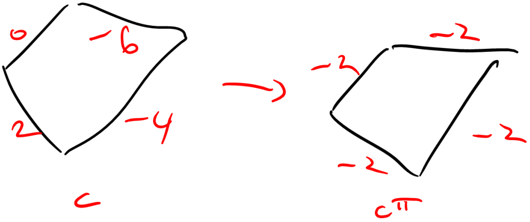
$$\epsilon(f) := \min\{\epsilon : f \text{ is } \epsilon\text{-optimal}\}$$

$c^\pi(e) \geq -\varepsilon(f)$

Lemma

$\mu(f) = -\epsilon(f)$.

$\mu(f) = -2$



Pf: Choose π s.t. $c^{\pi}(e) \geq -\varepsilon(f)$ $\forall e \in E_f$
and from all such π, one minimizing
$|E'|$, $E' = \{ c^{\pi}(e) = -\varepsilon(f) \}$.

$E' \neq \emptyset$

Want: E' contains a cycle Γ,
$$\Rightarrow \mu(f) = \frac{1}{|\Gamma|} c(\Gamma) = -\varepsilon(f)$$

Suppose not. Can choose v s.t. $\delta^{+}_{E'}(v) = \emptyset$

$\delta^{-}_{E'}(v) \neq \emptyset$

Let $\pi'_i = \begin{cases} \pi_i & i \neq v \\ \pi_i + \delta & i = v \end{cases}$

Then for δ small enough

for $e \in \delta^+(v)$, $c^{\pi'}(e) = c^{\pi}(e) - \delta > -\varepsilon(f)$.

Also for $e \in \delta^-(v)$, $c^{\pi'}(e) = c^{\pi}(e) + \delta > -\varepsilon(f)$.

This contradicts our choice of π. $\boxtimes$

Lemma

$\epsilon(f) = -\mu(f)$ is decreasing throughout the algorithm.

Pf: Let f be circulation before augmenting by Γ, f' after.

$$f' = f + \lambda \chi(\Gamma) \qquad \lambda > 0.$$

$$c(\Gamma) < 0.$$

Choose π s.t. $c^{\pi}(e) \geq -\varepsilon(f) \quad \forall e \in E_f$.

A weakly polynomial bound

Lemma

Let f_r be the flow obtained after j iterations of the G-T algorithm. Then $\epsilon(f_{s+m}) \leq (1 - 1/n) \cdot \epsilon(f_s)$ for any s.

Let Γ_r be cycle chosen in iteration r.

So $f_{r+1} = f_r + \lambda_r \cdot \chi(\Gamma_r)$

Let π be s.t. $c^\pi(e) \geq -\varepsilon(f)$ $\forall e \in E_f$.

Let $E^- = \{ e \in \overleftrightarrow{E} : c^\pi(e) < 0 \}$.

Choose $l \geq s$ minimal s.t. $\Gamma_l \not\subseteq E^-$.

- For each $s \le r < \ell$, $\mathrm{rev}(\Gamma_r) \cap E^- = \emptyset$.

$\therefore \; E_{f_{r+1}} \cap E^- \subsetneq E_{f_r} \cap E^-$.

some edge of Γ_r disappears,

So $\ell \le s + m$.

Now
$$\mu(f_\ell) = \frac{1}{|\Gamma_\ell|} \cdot c(\Gamma_\ell)$$
$$= \frac{1}{|\Gamma_\ell|} \cdot c^\pi(\Gamma_\ell)$$
$$\ge \frac{1}{|\Gamma_\ell|} \cdot \Big((|\Gamma_\ell| - 1) \cdot (-\varepsilon(f_s)) + 0 \Big)$$

$$\geq \left(1 - \frac{1}{|\Gamma_e|}\right) \cdot \left(-\mathcal{E}(f_s)\right).$$

After

$O(mn \log(Cn))$ iterations

$\mathcal{E}(f) < \frac{1}{n}.$

A strongly polynomial bound

We call an edge $e \in \overleftrightarrow{E}$ ϵ-frozen if $e \notin E_g$ for any ϵ-optimal circulation g.

Claim

If f is ϵ-optimal w.r.t. π, and $c^\pi(e) \leq -2n\epsilon$, then e is ϵ-frozen.

$c^\pi(e) \geq -\varepsilon \quad \forall e \in E_f$

Pf: Let g be any ε-optimal sol^n.

Suppose $e \in E_g$.

Let $h = f - g$.

h is a circulation.

Claim: $\mathrm{supp}(h) \subseteq E_g$, $\mathrm{rev}(\mathrm{supp}(h)) \subseteq E_f$.

pf: $h(e') > 0 \Rightarrow$ either $f(e') > g(e')$

$\Rightarrow g(e') < u(e')$, $f(e') > 0$

or $g(\mathrm{rev}(e')) > f(\mathrm{rev}(e'))$

$\Rightarrow g(\mathrm{rev}(e')) > 0$, $f(\mathrm{rev}(e')) < u(e)$

Either way, $e' \in E_g$ and $e' \in E_f$. $\boxtimes$

Now: since $e \in E_g \setminus E_f$, $h(e) > 0$.

(Convince yourself!)

$\therefore \exists \Gamma \subseteq \mathrm{supp}(h)$, $e \in \Gamma$.

$\subseteq E_g$.

$\forall e' \in \Gamma$, $\mathrm{rev}(e') \in E_f \therefore c^u(e') \leq \varepsilon$.

$$c(\Gamma) = c^{\pi}(\Gamma) \leq -2n\varepsilon + (|\Gamma|-1)\cdot\varepsilon$$

$$\therefore \frac{1}{|\Gamma|}c(\Gamma) \leq \varepsilon\left(1 - \frac{1}{|\Gamma|} - \frac{2n}{|\Gamma|}\right)$$

$$< -\varepsilon.$$

$\lightning$ $\blacksquare$

The algorithm terminates after $O(m^2 n \log n)$ iterations.

Pf: Let f be circ. at start of some iteration Γ.

Let π be s.t. $c^\pi(e) \geq -\varepsilon(f)$ $\forall e \in E_f$.

Let Γ be chosen cycle.

Then $c^\pi(e) = -\varepsilon(f)$ $\forall e \in \Gamma$.

No edge in Γ is $\varepsilon(f)$-frozen.

Let f' be flow after $mn \ln 2n$ iters.
π' be s.t. $c^{\pi'}(e) \geq -\varepsilon(f')$ $\forall e \in E_{f'}$.

$$\varepsilon(f') \le \left(1 - \frac{1}{n}\right)^{n \ln 2n} \cdot \varepsilon(f)$$

$$\le \frac{1}{2n} \cdot \varepsilon(f)$$

$\exists\, e \in \Gamma$ with $c^{\pi'}(e) \le -\varepsilon(f) \le -2n\,\varepsilon(f')$.

Gives $O(m^2 n \log n)$ bound on #

iterations.

Overview

Max flow	Min cost flow	
Capacity scaling Ahuja-Orlin	**Capacity scaling + contraction** Orlin	
Shortest paths Edmonds-Karp	**Minimum mean cycle** Goldberg-Tarjan	
Push-relabel Goldberg-Tarjan		

State of the art

- **Fastest weakly polynomial algorithm:** $\tilde{O}(m\sqrt{n}\,\mathrm{polylog}\,U)$
  Lee-Sidford '13

- **Fastest strongly polynomial algorithm:**
  $O(m \log n(m + n \log n)) = \tilde{O}(m^2)$ Orlin '93

Exercise

Consider the following variation of the Goldberg-Tarjan algorithm:

$f \leftarrow 0$, $\pi \leftarrow 0$
repeat
    while There exists a cycle $\Gamma \subseteq E_f$ with $c^\pi(e) < 0$ for all $e \in \Gamma$ **do**
        Augment on Γ
    Update π so that $c^\pi(e) \geq -\epsilon(f)$ for all $e \in E_f$.
until $\epsilon(f) = 0$

Show that this runs in time $O(mn^2 \log(Cn))$ (so a factor m faster than what we got for the original Goldberg-Tarjan algorithm).

You may assume that the last step can be done in time $O(mn)$, and that a finding a cycle (if any) in a directed graph can be done in time $O(n)$.

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