Algorithms for Network Flows

Lecture 3: Generalized flows I

Neil Olver

VU University Amsterdam

CWI Centrum Wiskunde & Informatica

Valparaíso Summer School, 2017

Slides will be available at: `http://nolver.net/home/valparaiso`

Today's lecture

- A second path to a strongly polynomial algorithm for min cost flow
- The generalized flow model
- Building up our toolbox and intuition

Another route to a strongly polynomial algorithm

- The key to the strongly polynomial analysis was that

$$c^{\pi}(e) \geq 2n\epsilon(f) \quad \Rightarrow \quad e \notin E_{f^*} \text{ for any optimal } f^*.$$

$$c^{\pi}(e) \geq -\epsilon(f) \quad \forall e \in E_f$$

Another route to a strongly polynomial algorithm

- The key to the strongly polynomial analysis was that

$$c^{\pi}(e) \geq 2n\epsilon(f) \quad \Rightarrow \quad e \notin E_{f^*} \text{ for any optimal } f^*.$$

- There is a "dual" version of this. **We switch to the transshipment setting.**

Call $e \in E$ contractible if $c^{\pi^*}(e) = 0$ for any optimal dual solution π^*.

- If we can prove that an edge e is contractible, we will be able to reduce the problem to a smaller instance.

Edge contraction

$$\min \sum_{e \in E} c(e) f(e)$$
$$\text{s.t.} \quad \nabla f_i = b_i \quad \forall i \in V$$
$$f \geq 0$$

$$\max \sum_{i \in V} b_i \pi_i$$
$$\text{s.t.} \quad \pi_i - \pi_j \leq c(ij) \quad \forall ij \in E$$

$c^{\pi^*}(uv) = 0 \qquad \pi_u^* - \pi_v^* = c(uv)$

So replace π_u by $\pi_v + c(uv)$

$$\max \sum_{i \in V \setminus \{u,v\}} b_i \pi_i + (b_u + b_v) \pi_v + b_u c(uv)$$

How can we show that an edge is contractible?

How can we show that an edge is contractible?

Lemma

Let $f : E \to \mathbb{R}_+$ and $\pi : V \to \mathbb{R}$ be such that $c^\pi(e) \geq 0$ for all $e \in E_f$. Let

$$\mathrm{Ex}(f) := \sum_{i \in V} \max\{\nabla f_i - b_i, 0\}.$$

If $f(\hat{e}) > \mathrm{Ex}(f)$, then $\hat{e}$ is contractible.

Pf: Let f^* be an opt. flow, chosen s.t. $\|f - f^*\|_1$.

Let $h = f^* - f = \sum \lambda_i \chi(P_i)$

Claim: Each P_i is a path from

$$V^- = \{ i : \nabla f_i < b_i \}$$

to $$V^+ = \{ i : \nabla f_i > b_i \}$$

Pf: Suppose P_i is a cycle.

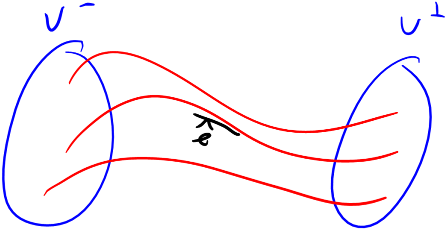
$$\text{supp}(h) \subseteq E_f, \quad \text{rev}(\text{supp}(h)) \subseteq E_{f^*}$$

π^* optimal dual sol^n.

$$c^{\pi}(P_i) \geq 0$$

$$\therefore c^{\pi^*}(P_i) \geq 0.$$

$\therefore \hat{f} = f^* - \delta \cdot P_i$ is "better" ↯. ☒.



$$h(\hat{e}) \le Ex(f)$$

$$\therefore f^{n}(\hat{e}) > 0.$$

Scheme of a strongly polymomial algorithm *Orlin '93*

FIND-OPTIMAL-DUAL(G):

1: Adjust f, π maintaining $c^\pi(e) \geq 0$ for all $e \in E_f$, to produce an edge e' with $f(e') > \mathrm{Ex}(f)$.
2: $\pi' \leftarrow$ FIND-OPTIMAL-DUAL($G/\{e'\}$)
3: "Uncontract" π' to get π^*

Scheme of a strongly polymomial algorithm *Orlin '93*

FIND-OPTIMAL-DUAL(G):

1: Adjust f, π maintaining $c^\pi(e) \geq 0$ for all $e \in E_f$, to produce an edge e' with $f(e') > \text{Ex}(f)$.
2: $\pi' \leftarrow$ FIND-OPTIMAL-DUAL($G/\{e'\}$)
3: "Uncontract" π' to get π^*

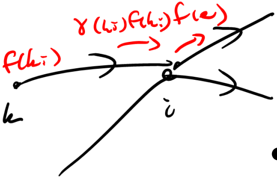
- Once an optimal dual π^* has been found, it's easy to find an optimal flow f^* by complementary slackness.

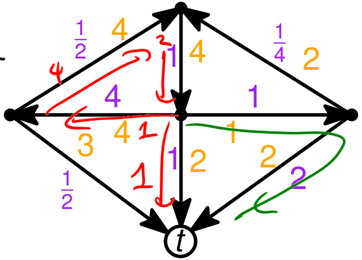
Generalized flow maximization

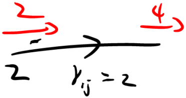
Given: Directed graph $G = (V, E)$, edge capacities $u : E \to \mathbb{R}_+$, gains $\gamma : E \to \mathbb{R}_{++}$, sink $t \in V$.
Goal: Find a generalized flow maximizing the net flow into t

- A generalized flow is a function $f : E \to \mathbb{R}_+$ with $f(e) \leq u(e)$ for all $e \in E$, and $\nabla f_i = 0$ for all $i \in V \setminus \{t\}$, where

$$\nabla f_i := \sum_{e \in \delta^-(i)} \gamma(e) f(e) - \sum_{e \in \delta^+(i)} f(e).$$



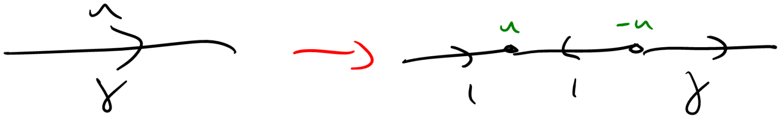




An equivalent formulation

- We can replace edge capacities by node demands $b : V \to \mathbb{R}$.

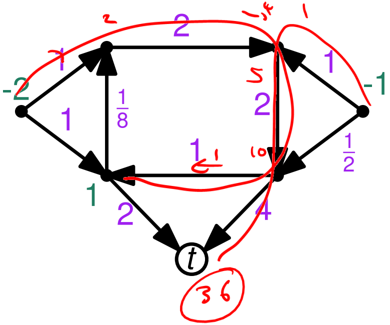
$$
\begin{aligned}
\max \quad & \nabla f_t \\
\text{s.t.} \quad & \nabla f_i = b_i \qquad \forall i \neq t \\
& f \geq 0
\end{aligned}
$$



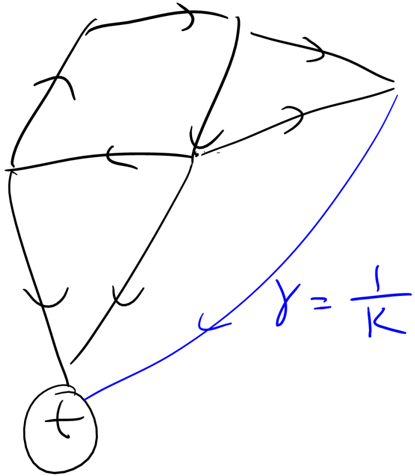
An equivalent formulation

- We can replace edge capacities by node demands $b : V \to \mathbb{R}$.

$$\begin{aligned} \max \quad & \nabla f_t \\ \text{s.t.} \quad & f_i = b_i \qquad \forall i \neq t \\ & f \geq 0 \end{aligned}$$



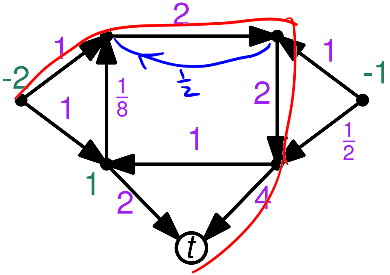
- **Extra assumption:** There is a path from i to t in E, for each $i \in V$.



- **Extra assumption:** There is a path from i to t in E, for each $i \in V$.
- We can allow flow to be discarded

$$
\begin{aligned}
\max \quad & \nabla f_t \\
\text{s.t.} \quad & f_i \geq b_i \qquad \forall i \neq t \\
& f \geq 0
\end{aligned}
$$

Residual graph



- For $e \in E$, define $\gamma(\text{rev}(e)) = 1/\gamma(e)$.
- Given $f : E \to \mathbb{R}_+$, the residual capacity of an arc $e \in \overleftrightarrow{E}$ is

$$u_f(e) = \infty \qquad \forall e \in E$$
$$u_f(\text{rev}(e)) = f(e) \cdot \gamma(e) \qquad \forall e \in E$$

Generalized flow and LP

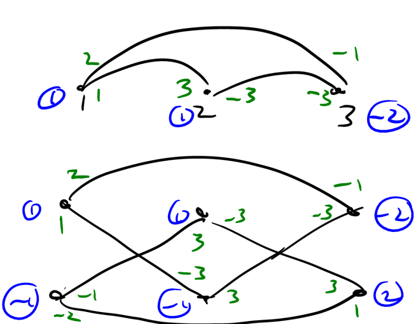
- Consider the feasibility problem

$$Ax = b, x \geq 0,$$

where $A \in \mathbb{R}^{mn}$ and each column of A has at most 2 nonzero entries.

> This is equivalent to the decision version of generalized flow maximization. *Hochbaum*

$$\begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & -3 \\ 0 & -1 & -3 \end{pmatrix} \underline{x} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$



- What about the optimization LP

$$\min c^T x \qquad \text{s.t.} \qquad Ax = b, x \geq 0,$$

same conditions on A?

- This is equivalent to the minimum cost generalized flow problem.

$$\min \sum c(e) f(e)$$

$$\nabla f_i = b_i$$

$$f \geq 0$$

- What about the optimization LP

$$\min c^T x \qquad \text{s.t.} \qquad Ax = b, x \geq 0,$$

  same conditions on A?

- This is equivalent to the minimum cost generalized flow problem.

- We don't know a strongly polynomial algorithm for this problem!

- Primal feasibility $\equiv$ max. generalized flow: Végh '14
- Dual feasibility $A^T y \leq c$: Megiddo '83

Flow-generating cycles

A cycle $C \in \overleftrightarrow{E}$ is called

- a flow-generating cycle if $\gamma(C) := \prod_{e \in C} \gamma(e) > 1$
- a flow-absorbing cycle if $\gamma(C) < 1$
- a unit cycle if $\gamma(C) = 1$

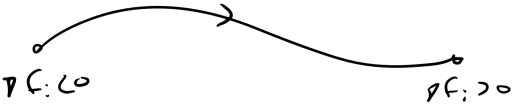
Flow decomposition

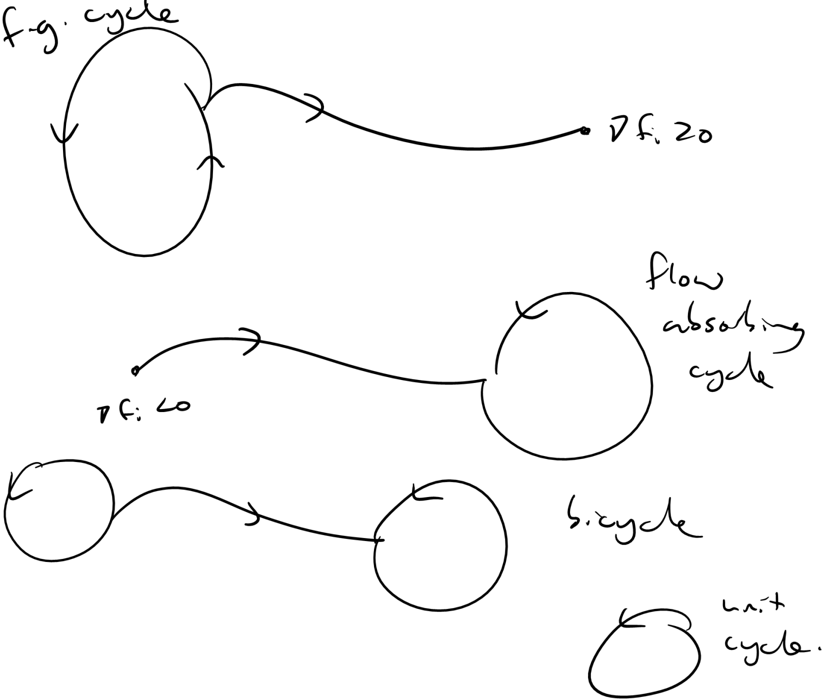
Let $f : E \to \mathbb{R}_+$ satisfy $f \leq u$. We say $g : E \to \mathbb{R}_+$ conforms to f if

$$\mathrm{supp}(g) \subseteq \mathrm{supp}(f)$$
$$\nabla g_i > 0 \Rightarrow \nabla f_i > 0$$
$$\nabla g_i < 0 \Rightarrow \nabla f_i < 0$$

Lemma

Let $f : E \to \mathbb{R}_+$ satisfy $f \leq u$. Then $f = \sum_{r=1}^{k} \lambda_r f^{(r)}$, where $k \leq m$, $\lambda \geq 0$, and each $f^{(r)}$ is an elementary flow conforming to f.

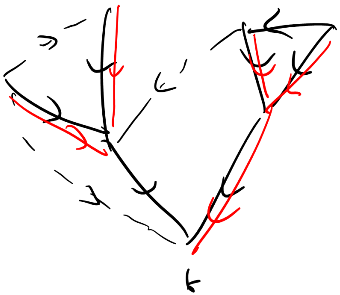



f.g. cycle
$\nabla f_i \geq 0$
flow absorbing cycle
$\nabla f_i \leq 0$
bicycle
unit cycle.

A trivial optimality condition

We call a network lossy if $\gamma(e) \leq 1$ for all $e \in E$.

Suppose the network is lossy, f is feasible with $\nabla f_i = b_i$ for all $i \neq t$, and $\gamma(e) = 1$ for all $e \in \text{supp}(f)$.
Then f is optimal.



Relabelling

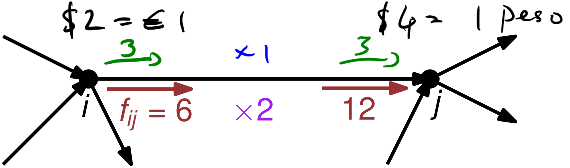
- A labelling is any function $\mu : V \to \mathbb{R}_{++}$.

Given a labelling μ, define the relabelled gains γ^μ and relabelled demands b^μ by

$$\gamma_{ij}^\mu = \frac{\mu_i}{\mu_j} \cdot \gamma_{ij}, \qquad b_i^\mu = \frac{1}{\mu_i} \cdot b_i.$$

Given a flow f on the original instance, define the relabelled flow f^μ by

$$f_{ij}^\mu = \frac{1}{\mu_i} \cdot f_{ij}.$$



Relabelling

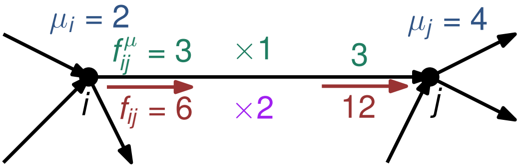
- A labelling is any function $\mu : V \to \mathbb{R}_{++}$.

Given a labelling μ, define the relabelled gains γ^μ and relabelled demands b^μ by

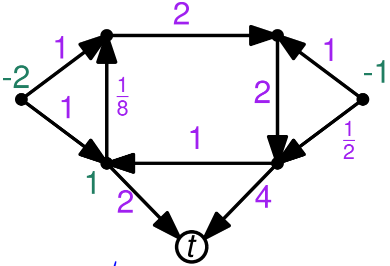
$$\gamma^\mu_{ij} = \frac{\mu_i}{\mu_j} \cdot \gamma_{ij}, \qquad b^\mu_i = \frac{1}{\mu_i} \cdot b_i.$$

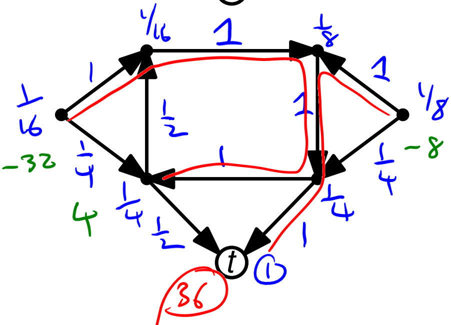
Given a flow f on the original instance, define the relabelled flow f^μ by

$$f^\mu_{ij} = \frac{1}{\mu_i} \cdot f_{ij}.$$



- Relabelled instance is completely equivalent to the original one





If $\nabla f_i = b_i$ forall $i \neq t$, and there exists a labelling μ s.t. G^μ is a lossy network with $\gamma^\mu(e) = 1$ for all $e \in \text{supp}(f^\mu)$, then f is optimal.

- Sufficient, but is it necessary?

If $\nabla f_i = b_i$ forall $i \neq t$, and there exists a labelling μ s.t. G^μ is a lossy network with $\gamma^\mu(e) = 1$ for all $e \in \mathrm{supp}(f^\mu)$, then f is optimal.

- Sufficient, but is it necessary?

$$
\begin{aligned}
\min \quad & -z \\
\text{s.t.} \quad & \nabla f_i \geq b_i \quad \forall i \\
& \nabla f_t \geq z \\
& f \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\max \quad & \sum_{i \neq t} b_i \theta_i \\
\text{s.t.} \quad & \gamma_{ij} \theta_j - \theta_i \leq 0 \quad \forall ij \in E \\
& \theta_t = 1
\end{aligned}
$$

Let $\mu_i = \frac{1}{\theta_i}$

$$
\begin{aligned}
\max \quad & \sum_{i \neq t} b_i / \mu_i \quad (b^\mu_i) \\
\text{s.t.} \quad & \gamma_{ij} \frac{\mu_i}{\mu_j} \leq 1 \quad (\gamma^\mu_{ij}), \quad \mu_t = 1
\end{aligned}
$$

Given $f \in \mathbb{R}_+^E$, $\mu \in \mathbb{R}_{++}^V$, (f, μ) is called a fitting pair if:

- μ is dual feasible
- $f_e > 0$ implies $\gamma_e^\mu = 1$.

$\gamma_e^\mu \le 1 \quad \forall e \in E_f$

- So if (f, μ) is a fitting pair and $\nabla f_i = b_i$ for all $i \neq t$, then f and μ are both optimal.
- Given a feasible f, there does not always exist a μ so that (f, μ) is a fitting pair...

Lemma

If there are no flow-generating cycles in E_f, then we can efficiently find labels μ s.t. $\gamma^{\mu}(e) \leq 1$ for all $e \in E_f$.

Pf: let $c(e) = -\log \gamma(e) \quad \forall e \in E$

$$\gamma(C) > 1 \iff \sum_{e \in C} c(e) < 0.$$

No neg cost cycles $\Rightarrow$ π s.t. $\pi_j \leq \pi_i + c(i,j)$

$$\mu_i = 2^{\pi_i} \quad \forall i$$

Cancelling flow generating cycles

Use a multiplicative version of Goldberg-Tarjan:

while $\exists$ a flow-generating cycle **do** maximum
    Find a cycle C in G_f of ~~minimum~~ mean gain $\gamma(C)^{1/|C|}$
    Augment as much flow as possible around C

Cancelling flow generating cycles

Use a multiplicative version of Goldberg-Tarjan:

1: **while** $\exists$ a flow-generating cycle **do**
2: Find a cycle C in G_f of minimum mean gain $\gamma(C)^{1/|C|}$
3: Augment as much flow as possible around Γ

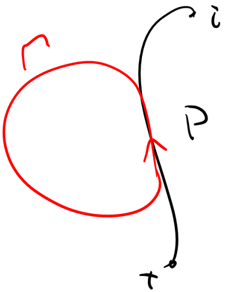
- Weakly polynomial analysis is basically the same
- Strongly polynomial analysis is harder *Radzik '93*

Onaga's algorithm

1: Let f be an initial feasible solution ($\nabla f_i \geq b_i$ for all $i \neq t$)
2: Cancel all flow-generating cycles
3: **while** $\exists$ a node with $\nabla f_i > b_i$ **do**
4: Find a highest gain path P from i to t
5: Augment as much flow as possible via t

Lemma

After step 2, E_f never has any flow-generating cycles.



Lemma

Assuming rational input, Onaga's algorithm terminates with a maximum generalized flow.

Lemma

Assuming rational input, Onaga's algorithm terminates with a maximum generalized flow.

...But unfortunately this is not polynomial.

(Because if all gains are unit, it's exactly Ford-Fulkerson)

A weakly polynomial algorithm

Simpler version of an algorithm of Goldberg-Plotkin-Tardos '91; see Wayne '99, Shigeno '04.

- Assume $b_i \in \mathbb{Z}$, $|b_i| \leq B$, and $\gamma(e) = \frac{p_e}{q_e}$ with $p_e, q_e \leq B$.

> A most improving path is a path in E_f that brings the largest amount of flow to the sink from a node with $\nabla f_i > b_i$.

Choose f satisfying $\nabla f_i = b_i$ for all $i \neq t$
repeat
    Cancel all flow-generating cycles
    Augment flow along a most improving path
until increase in ∇f_t in the iteration is less than B^{-2n}/m
Cancel all flow-generating cycles
Find a μ fitting f
μ will be an optimal dual solution; find f^* by complementary slackness.

Optimal duals to optimal primals

If μ is optimal, can compute an optimal g in strongly polynomial time.

$$\mathrm{supp}(g) \subseteq \{e : \gamma^{\mu}_{e} = 1\}$$

g^{μ} is a sol^n to a regular flow problem.

Exercises

1. Explain how the generalized flow problem can be easily solved in strongly polynomial time if $b_i \leq 0$ for all $i \neq t$.

2. Suppose our generalized network has $b_i \in \mathbb{Z}$, $|b_i| \leq B$, and $\gamma(e) = \frac{p_e}{q_e}$ with $p_e, q_e \leq B$, where B is some integer. Assume $it \in E$ for each $i \neq t$.

   Suppose f is feasible ($\nabla f_i \geq b_i$ for all $i \neq t$), and that (f, μ) is a fitting pair. Prove that if

$$\mathrm{Ex}(f) := \sum_{i \neq t} (\nabla f_i - b_i) < B^{-3n},$$

   then there exists g with (g, μ) a fitting pair and $\nabla g_i = b_i$ for all $i \neq t$. (This implies that g and μ are both optimal.)

   Hint: work in the relabelled network with demands b_i^μ, gains $\gamma^\mu(e)$. Make use of integrality properties of regular (not generalized) flows.

References

A. V. Goldberg, S. A. Plotkin, and É. Tardos.
Combinatorial algorithms for the generalized circulation problem.
Mathematics of Operations Research, 16(2):351, 1991.

M. Shigeno.
A survey of combinatorial maximum flow algorithms on a network with gains.
Journal of the Operations Research Society of Japan, 47(4):244–264, 2004.

K. Wayne.
Generalized maximum flow algorithms, 1999.