Cálculo del Volumen en Dimensión Alta

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El Plan

- Part 1. Intro to high dimension and convexity
  - Volume distribution, logconcavity
  - Ellipsoids
  - Lower bounds
- Part 2. Algorithms
  - Rounding
  - Volume/Integration
  - Optimization
  - Sampling
- Part 3. Probability
  - Markov chains
  - Conductance
  - Mixing of ball walk
  - Mixing of hit-and-run
- Part 4. Geometry
  - Isoperimetry
  - Concentration
  - Localization and applications
- Open problems

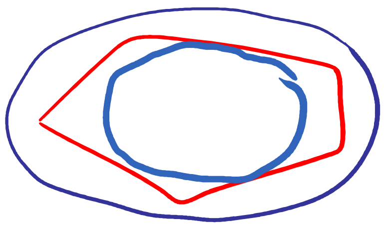
Volumen: Segundo intento: Elipsoides

Ellipsoid #1: John ellipsoid of a convex body K:

E = maximum volume ellipsoid contained in K.

Thm. For any convex body K, the John ellipsoid satisfies

$$E \subseteq K \subseteq nE.$$

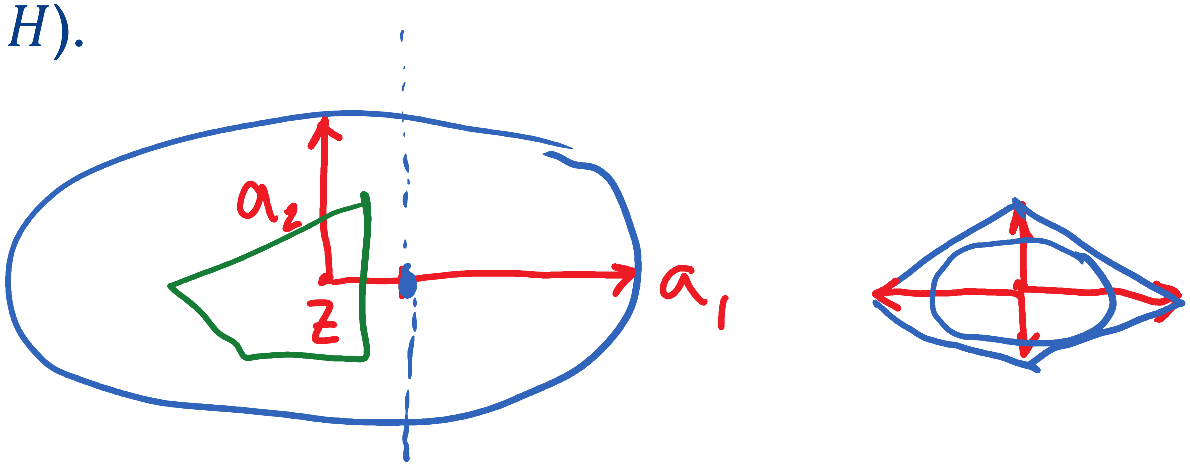


For any centrally-symmetric K, $E \subseteq K \subseteq \sqrt{n}E.$

Exercise 2: Encuentre un ejemplo ajustado.

Elipsoide de John Aproximado

- Variant of the Ellipsoid algorithm:
  - suppose current center is z and axes are $a_1, a_2, \dots, a_n$.
  - Check if $z \pm \frac{a_i}{n} \in K$. If so, output current ellipsoid.
  - If not, then intersect E with halfspace H not containing $z + \frac{a_i}{n}$ and continue algorithm (replace E with min volume ellipsoid containing $E \cap H$).



- Thm. Algorithm outputs E satisfying $E \subseteq K \subseteq n^{1.5} E$.

Volumen por John elipsoide

- Using the Ellipsoid algorithm, in polytime

$$E \subseteq K \subseteq n^{1.5} E$$

- Then

$$vol(E) \leq vol(K) \leq n^{1.5n}\, vol(E)$$

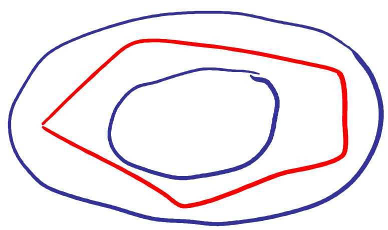
- Polytime, exponential approximation

Elipsoide #2: Elipsoide inertial

- For a convex body K, matrix of inertia:
  - $M = E_K\left((x-\bar{x})(x-\bar{x})^T\right)$
- Inertial ellipsoid: $E = \{y \in R^n: \; y^T M^{-1} y \leq 1\}$

Thm (KLS95). $\sqrt{\frac{n+1}{n}}E \subseteq K \subseteq \sqrt{n(n+1)}E$

- Also a factor n sandwiching, but a *different* ellipsoid.
- Shown earlier up to constants by Milman-Pajor.



Posición Isotropico

- For any distribution with bounded second moments, there is an affine transformation to make it isotropic.
- Applying this to a convex body K:

$$E_K(x) = 0, \qquad E_K\left(xx^T\right) = I_n.$$

- Thus K "looks like a ball" up to second moments.

- How close is it really to a ball?
- K lies between two balls with radii within a factor of n.

Volumen por aproximación elipsoidal

- The Inertial ellipsoid can be approximated to within any constant factor (we'll see how)
- Therefore:

$$E \subseteq K \subseteq 2n\, E \;\; \Rightarrow \; vol(E) \leq vol(K) \leq (2n)^n \, vol(E).$$

- Polytime algorithm, $O(n)^n$ approximation
- Can we do better?

Elipsoide #3: Elipsoide de Milman

- For two compact sets A, B,
- $N(A, B)$ = #translates of A needed to cover B.

Thm (Milman). For any convex body K, there is an ellipsoid E s.t.,

$$N(K, E), N(E, K) \leq 2^{O(n)}$$

- Many important consequences in convex geometry.

Thm (Dadush-V.). Deterministic complexity of computing a Milman ellipsoid is $2^{O(n)}$.

- $2^{O(n)}$ time, $2^{O(n)}$ approximation.
- Can we do better?!

Complejidad de Volumen

Thm [E86, BF87]. For any deterministic algorithm that uses at most n^a membership calls to the oracle for a convex body K and computes two numbers A and B such that $\mathrm{A} \leq \mathrm{vol}(\mathrm{K}) \leq \mathrm{B}$, there is some convex body for which the ratio B/A is at least

$$\left(\frac{cn}{a \log n}\right)^{\frac{\mathrm{n}}{2}}$$

where c is an absolute constant.

Thm [DF88]. Computing the volume of an explicit polytope $Ax \leq b$ is #P-hard, even for a totally unimodular matrix A and rational b.

Estimación determinista

Thm [BF]. For deterministic algorithms:

# oracle calls	approximation factor
n^a	$\left(\frac{cn}{a \log n}\right)^{\frac{n}{2}}$
$\left(\frac{1}{\epsilon}\right)^n$	$(1+\epsilon)^n$

Thm [Dadush-V.13].

Approximation factor of $(1+\epsilon)^n$ in time $\left(\frac{1}{\epsilon}\right)^{O(n)} \mathrm{poly}(n)$.

Un Limite Inferior

- [Elekes]
- Membership oracle answers "YES" for points in unit ball, "No" for points outside.
- After m queries, volume of K is between volume of convex hull and volume of unit ball.
- Lemma. $vol(conv\{x_1, x_2, \ldots, x_m\}) \leq \frac{m}{2^n} . vol(B^n)$
- Need exponentially many queries!

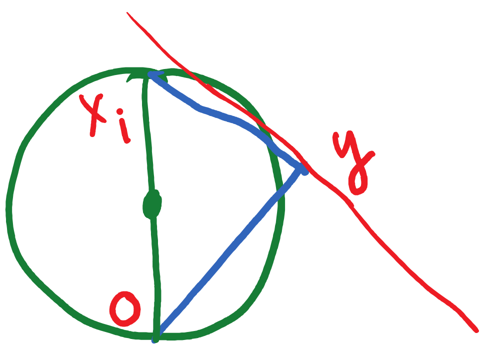
Un Limite Inferior

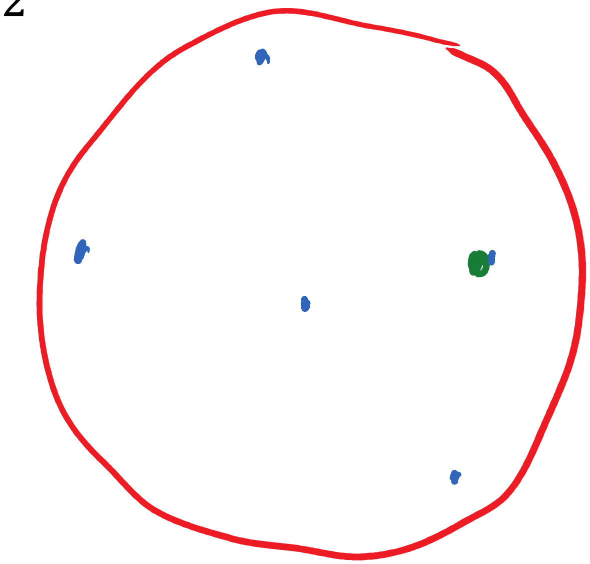
Lemma. $vol(conv\{x_1, x_2, \dots, x_m\}) \leq \frac{m}{2^n} . vol(B^n)$

Proof. Let $B_i =$ ball of radius $\frac{||x_i||}{2}$ around $\frac{x_i}{2}$.

Claim 1. $\sum_i vol(B_i) \leq m . \frac{vol(B^n)}{2^n}$.

Claim 2. $conv\{x_1, x_2, \dots, x_m\} \subset \bigcup_i B_i$.





$y \notin B_i \Rightarrow \angle 0yx_i$ is acute, i.e., 0 and x_i are on the same side of orthogonal hyperplane through y.

Hence, $y \notin conv\{x_1, x_2, \dots, x_n\}$.

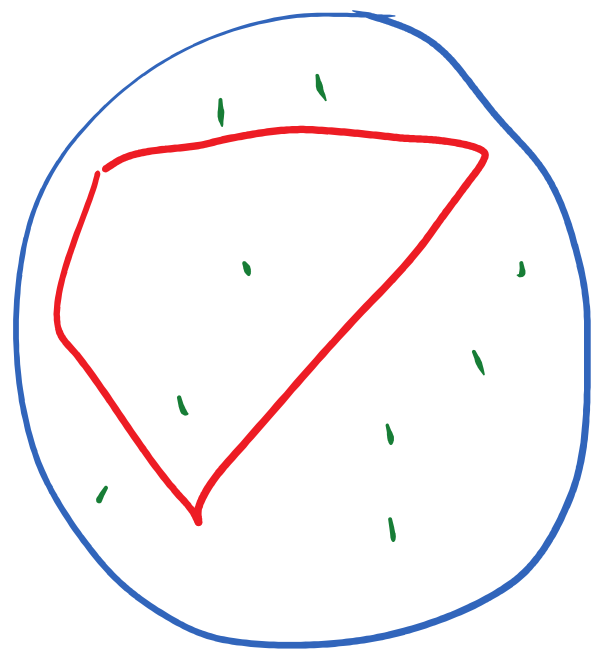
Volumen/Integración *Aleatoria*

[DFK89]. Polytime randomized algorithm that estimates volume to within relative error $(1+\epsilon)$ with probability at least $1-\delta$ in time poly(n, $\frac{1}{\epsilon}$, $\log\left(\frac{1}{\delta}\right)$).

[Applegate-K91]. Polytime randomized algorithm to estimate integral of any (Lipshitz) logconcave function.

Volumen: Tercer intento: Muestreo

- Pick random samples from ball/cube containing K.
- Compute fraction c of sample in K.
- Output c.vol(outer ball).

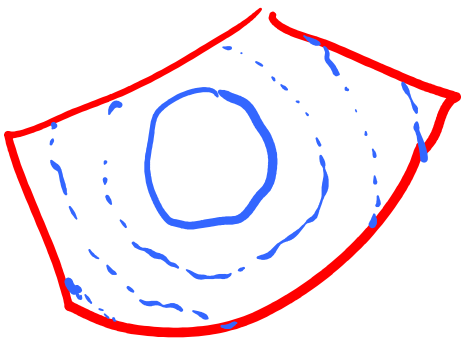


- Need too many samples!

Volumen por Muestreo [DFK89]

$$B \subseteq K \subseteq RB.$$

Let $K_i = K \cap 2^{i/n}B,\ i = 0, 1, \dots, m = n \log R.$



$$\mathrm{vol(K)} = \mathrm{vol(B)}.\frac{\mathrm{vol}(\mathrm{K}_1)}{\mathrm{vol}(\mathrm{K}_0)}\frac{\mathrm{vol}(\mathrm{K}_2)}{\mathrm{vol}(\mathrm{K}_1)}\cdots\frac{\mathrm{vol}(\mathrm{K}_\mathrm{m})}{\mathrm{vol}(\mathrm{K}_{\mathrm{m}-1})}.$$

Estimate each ratio with random samples.
(Markov Chain Monte-Carlo method)

Volumen por Muestreo

$$K_i = K \cap 2^{i/n} B, \qquad i = 0, 1, \dots, m = n \log R.$$

$$\mathrm{vol(K)} = \mathrm{vol(B)}. \frac{\mathrm{vol(K_1)}}{\mathrm{vol(K_0)}} \frac{\mathrm{vol(K_2)}}{\mathrm{vol(K_1)}} \dots \frac{\mathrm{vol(K_m)}}{\mathrm{vol(K_{m-1})}}.$$

Claim. $\mathrm{vol(K_{i+1})} \leq 2.\mathrm{vol(K_i)}$.

$$K_{i+1} \subseteq 2^{1/n}\left(K \cap 2^{i/n} B\right) = 2^{1/n} K_i$$

Varianza de la estimación [DF91]

$$K_i = K \cap 2^{i/n} B, \qquad i = 0, 1, \dots, m = n \log R.$$

$$\text{vol(K)} = \text{vol(B)}. \frac{\text{vol}(\text{K}_1)}{\text{vol}(\text{K}_0)} \frac{\text{vol}(\text{K}_2)}{\text{vol}(\text{K}_1)} \dots \frac{\text{vol}(\text{K}_\text{m})}{\text{vol}(\text{K}_{\text{m}-1})}$$

$$\frac{Var(Y_1 Y_2 \dots Y_m)}{E(Y_1 Y_2 \dots Y_m)^2} = \prod_i \frac{E\left(Y_i^2\right)}{E(Y_i)^2} - 1 = \prod_i \left(1 + \frac{Var(Y_i)}{E(Y_i)^2}\right) - 1$$

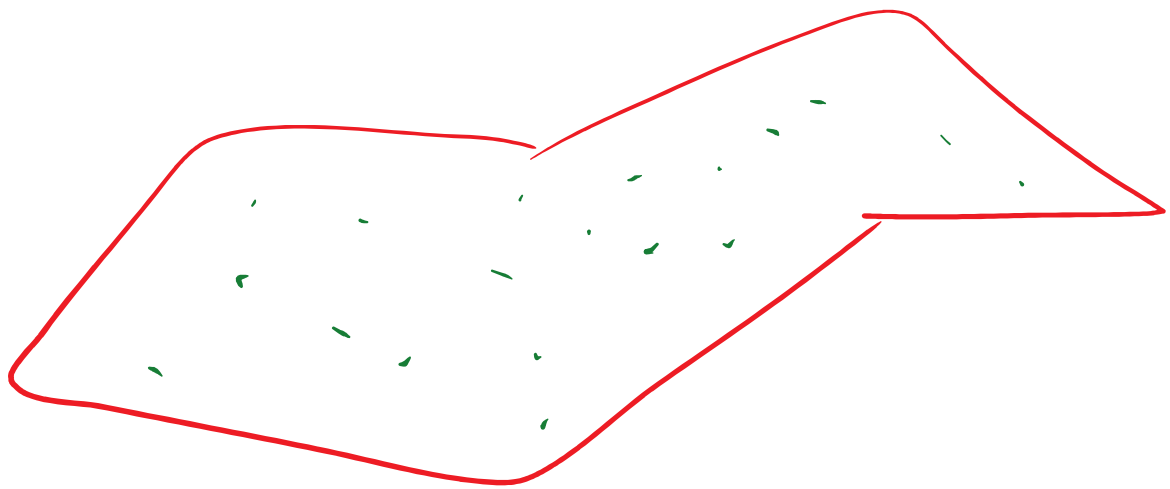
- $\frac{Var(Y_i)}{E(Y_i)^2} \leq \frac{c}{k}$ using k samples in each phase.
- So $k = \frac{m}{\epsilon^2}$ samples in each phase suffice.
- Total number of samples $= m. \frac{m}{\epsilon^2} = O^*\left(n^2\right)$.
- But, how to sample?

Muestreo

Input: function f: $R^n \rightarrow R_+$ specified by an oracle,

point x with f(x)>0, error parameter ε.

Output: A point y from a distribution within distance ε of distribution with density proportional to f.



Aplicaciónes

Given a blackbox for sampling logconcave densities, we get efficient algorithms for:

- Rounding
- Convex Optimization
- Volume Computation/Integration
- some Learning problems

Redondear por Muestreo

1. Sample m random points from K;
2. Compute sample mean and sample covariance matrix
   - $z = E(x) \qquad A = E((x-z)(x-z)^T).$
3. Output $B = A^{-\frac{1}{2}}$.

B(K-z) is nearly isotropic.

Thm. C(ϵ).n random points suffice to get $E\left(||A-I||_2\right) \leq \epsilon.$

[Adamczak et al; improving on Bourgain, Rudelson]

I.e., for any unit vector v, $1-\epsilon \leq E\left(\left(v^T x\right)^2\right) \leq 1+\epsilon.$

Complejidad de Muestreo

Thm. [KLS97] For a convex body, the ball walk with an M-warm start reaches a nearly independent, nearly random point in poly(n, R, M) steps.

$$M = \sup \frac{Q_0(S)}{Q(S)} \quad or \quad M = E_{Q_0}\left(\frac{Q_0(x)}{Q(x)}\right)$$

Thm. [LV03]. Same holds for arbitary logconcave density functions. Complexity is $O^*(M^2 n^2 R^2)$.

- Isotropic transformation makes R=O($\sqrt{n}$); M can be kept at O(1).

KLS'97 volume algorithm: $n \times n \times n^3 = n^5$

Progreso en el Cálculo del Volumen

	Power	New aspects
Dyer-Frieze-Kannan 89	23	everything
Lovász-Simonovits 90	16	localization
Applegate-K 90	10	logconcave integration
L 90	10	ball walk
DF 91	8	error analysis
LS 93	7	multiple improvements
KLS 97	5	speedy walk, isotropy
LV 03,04	4	annealing, wt. isoper.
LV 06	4	integration, local analysis
Cousins-V. 13, 15	3	Gaussian cooling