Cálculo del Volumen en Dimensión Alta

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El Gran Plan

1. Intro to high dimension and convexity
   - Volume distribution, logconcavity
   - Ellipsoids
   - Lower bounds
2. Algorithms
   - Rounding
   - Volume/Integration
   - Optimization
   - Sampling
3. Probability
   - Markov chains
   - Conductance
   - Mixing of ball walk
   - Mixing of hit-and-run
4. Geometry
   - Isoperimetry
   - Concentration
   - Localization and applications

- Open problems

Progreso en el Cálculo del Volumen

		Power	New aspects
Dyer-Frieze-Kannan 89		23	everything
Lovász-Simonovits 90		16	localization
Applegate-K 90		10	logconcave integration
L 90		10	ball walk
DF 91		8	error analysis
LS 93		7	multiple improvements
KLS 97	$n \times n \times n^3 = n^5$	5	speedy walk, isotropy
LV 03,04		4	annealing, wt. isoper.
LV 06		4	integration, local analysis
Cousins-V. 13, 15		3	Gaussian cooling

Recocido Simulado [LV03, Kalai-V.04]

To estimate $\int f$ consider a sequence $f_0, f_1, f_2, \ldots, f = f_m$ with $\int f_0$ being easy, e.g., constant function over ball.

Then, $\int f = \int f_0 . \frac{\int f_1}{\int f_0} . \frac{\int f_2}{\int f_1} \ldots \frac{\int f_m}{\int f_{m-1}}.$

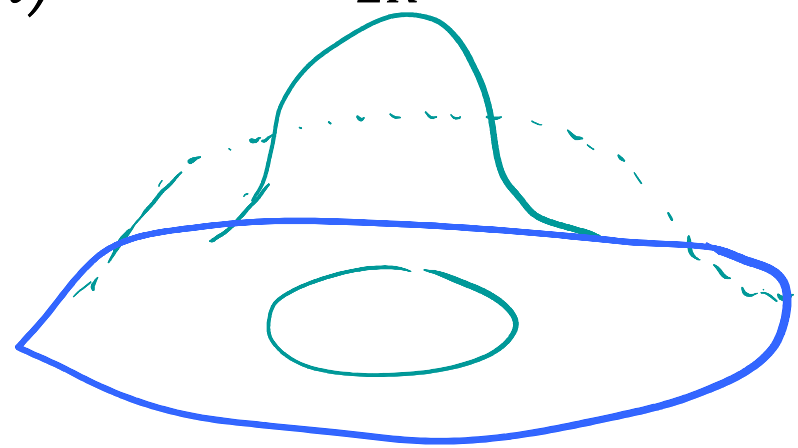
Each ratio can be estimated by sampling:

1. Sample X with density proportional to f_i
2. Compute $Y = \frac{f_{i+1}(X)}{f_i(X)}$

Then, $E(Y) = \int \frac{f_{i+1}(X)}{f_i(X)} . \frac{f_i(X)}{\int f_i(X)} dX = \frac{\int f_{i+1}}{\int f_i}.$

Recocido Simulado [LV06]

- Define: $f_i(X) = e^{-a_i||X||}$ for X $\in K$ y zero afuera.
- $a_0 = 2R, \quad a_{i+1} = a_i/\left(1 + \frac{1}{\sqrt{n}}\right), \quad a_m = \frac{\epsilon}{2R}$
- $m \sim \sqrt{n}\log(2R/\epsilon)$ phases
- $\int f_0 \cdot \frac{\int f_1}{\int f_0} \cdot \frac{\int f_2}{\int f_1} \cdots \frac{\int f_m}{\int f_{m-1}}$.



- The final estimate could be $n^{\Omega(n)}$, so each ratio could be $n^{\sqrt{n}}$ or higher. How can we estimate it with a few samples?!

Recocido [LV03, 06]

- $f_i(X) = e^{-a_i||X||}$
- $a_0 = 2R, \quad a_{i+1} = a_i / \left(1 + \frac{1}{\sqrt{n}}\right), \quad a_m = \frac{\epsilon}{2R}$

Lemma. $\mathrm{Var}\left(Y = \frac{f_{i+1}(X)}{f_i(X)}\right) < 4\, E(Y)^2.$

- Although expectation of Y can be large (exponential even), we need only a few samples to estimate it!

Lo-Ve algorithm: $\sqrt{n} \times \sqrt{n} \times n^3 = n^4$

Varianza del estimador de la relación

Let $f(a_i, X) = e^{-a_i||X||}$, $Y = \frac{f(a_{i+1}, X)}{f(a_i, X)}$

Then, $E(Y) = \int \frac{f(a_{i+1}, X)}{f(a_i, X)} \cdot \frac{f(a_i, X)}{\int f(a_i, X)} dX = \frac{\int f(a_{i+1}, X)}{\int f(a_i, X)}$

$$\frac{E(Y^2)}{E(Y)^2} = \int \frac{f(a_{i+1}, X)^2}{f(a_i, X)^2} \frac{f(a_i, X)}{\int f(a_i, \ X)} dX \cdot \left(\frac{\int f(a_i, X)}{\int f(a_{i+1}, \ X)} \right)^2$$

$$= \frac{\int f(2a_{i+1} - a_i, X) \int f(a_i, X)}{\left(\int f(a_{i+1}, X) \right)^2}$$

Set $\alpha = \frac{1}{\sqrt{n}}$, Recuerda que $a_{i+1} = a_i/(1 + \frac{1}{\sqrt{n}})$. Pues,

$$= \frac{F\big(a_{i+1}(1-\alpha)\big) F\big(a_{i+1}(1+\alpha)\big)}{F(a_{i+1})^2}$$

(would be at most 1, if F were logconcave…)

Propiedades de funciones logconcavos

- Product, minimum of logconcave functions is logconcave
- Convolution is logconcave

$$h(x) = \int_{R^n} f(y)g(x-y)dy$$

- E.g., any marginal:

$$h(x_1, x_2, \dots, x_k) = \int_{R^{n-k}} f(x)dx_{k+1}dx_{k+2} \dots dx_n$$

This follows from Prekopa-Leindler.

Varianza del estimador de relación

Lemma. For any logconcave f and a >0, the function $Z(a) = a^n \int f(a, X)dX$ is also logconcave.

So $a^n F(a)$ is logconcave and

$$(a^n F(a))^2 \geq \big(a(1-\alpha)\big)^n F(a(1-\alpha))\big(a(1+\alpha)\big)^n F(a(1+\alpha))$$

Therefore, for $\alpha \leq \frac{1}{\sqrt{n}}$

$$\frac{F\big(a(1-\alpha)\big)F\big(a(1+\alpha)\big)}{F(a)^2} \leq \left(\frac{1}{(1-\alpha)(1+\alpha)}\right)^n \leq \left(\frac{1}{1-\alpha^2}\right)^n \leq 4.$$

La Planificación de Recocido

- $a_{i+1} = \frac{a_i}{1+\frac{1}{\sqrt{n}}}$: $O^*(\sqrt{n})$ phases and $O^*(\sqrt{n})$ samples/phase

Exercise: How to set a_{i+1} so that we get $O^*(n)$ phases and $O^*(1)$ samples/phase?

Optimización por Recocido

- Maximizar $f(.)$ logconcava ($\approx$ minimizer f convexa)

For $i = 1, \dots, m$:

- $f_i(X) = f(X)^{a_i}$
- $a_0 = \frac{\epsilon}{2R}, \; a_m = \frac{2n}{\epsilon}$
- $a_{i+1} = a_i \left(1 + \frac{1}{\sqrt{n}}\right)$
- Sample with density prop. to $f_i(X)$.

Output X with max $f(X)$.

Recocido: Un Algoritmo Unificado

Integration: $\int f(X)$

- $f_i(X) = f(X)^{a_i}, X \in K$
- $a_1 = 2R, a_m = \frac{\epsilon}{2R},$
- $a_{i+1} = a_i / \left(1 + \frac{1}{\sqrt{n}}\right)$
- Sample from $f_i(X)$.
- Estimate $W_i \sim \int f_{i+1}(X) / \int f_i(X)$
- Output $W = W_1 W_2 \ldots W_m$.

Optimization: $\max f(X)$

- $f_i(X) = f(X)^{a_i}, X \in K$
- $a_0 = \frac{\epsilon}{2R},\ a_m = \frac{2n}{\epsilon}$
- $a_{i+1} = a_i \left(1 + \frac{1}{\sqrt{n}}\right)$
- Sample from $f_i(X)$.
- Output X with max f(X).

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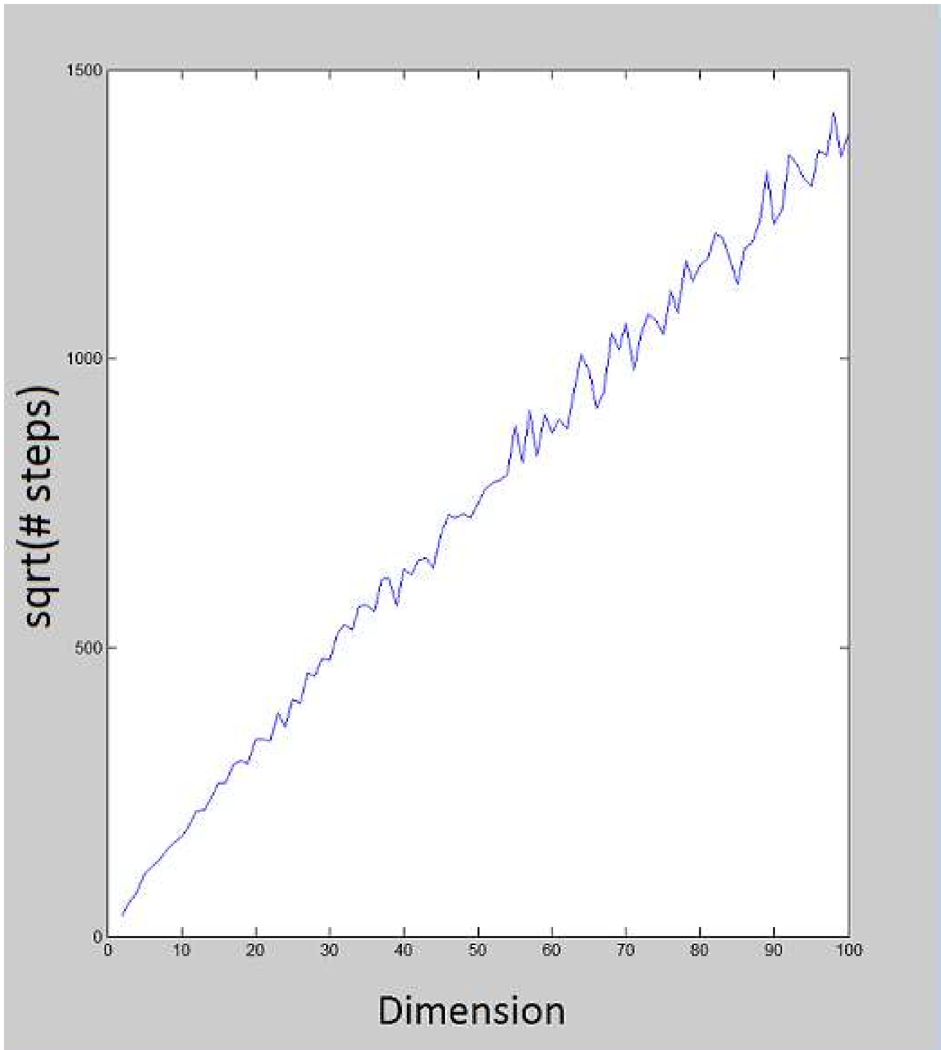
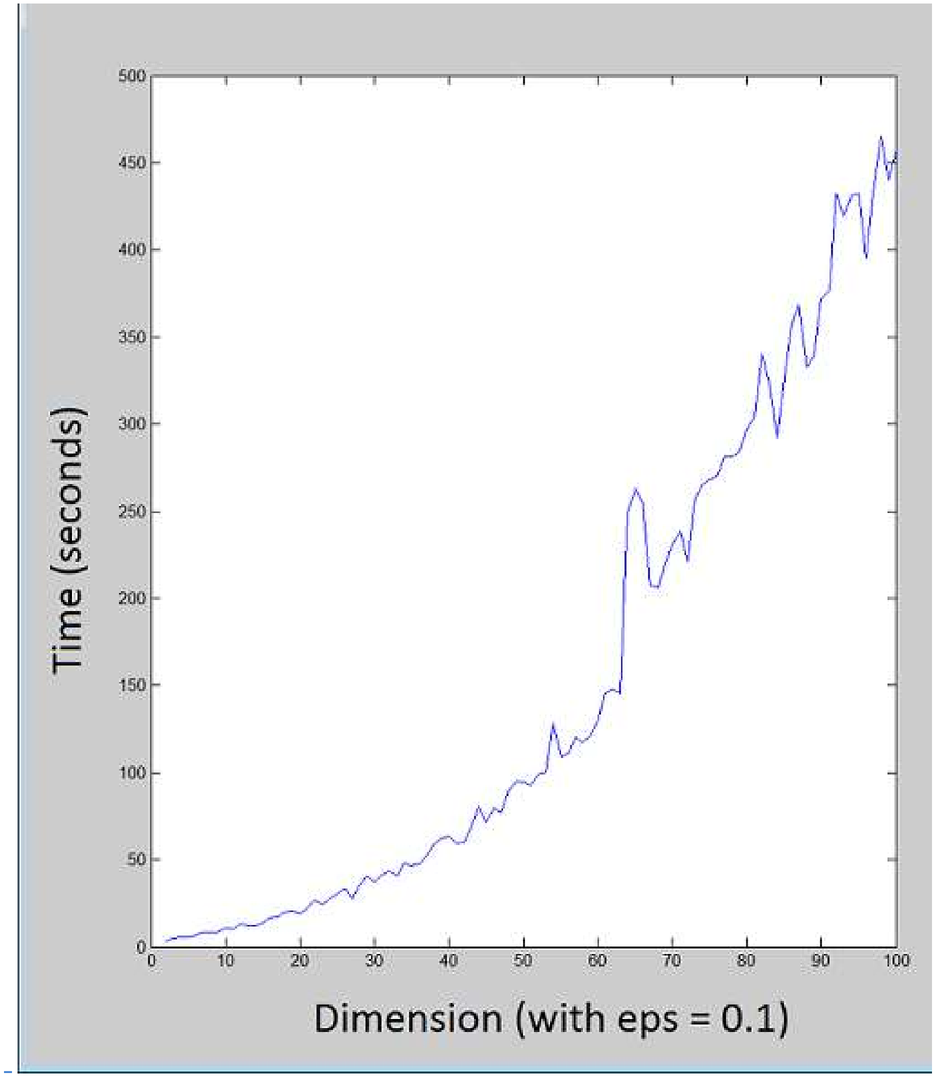
Un Programa para calcular el volumen

- [Cousins-V.13] Matlab implementation of sampling and integration
  - google "volume computation matlab exchange"
  - http://www.cc.gatech.edu/~bcousins/volume.html

- Please download from MATLAB Exchange to try it.

Cubos girados (al azar)





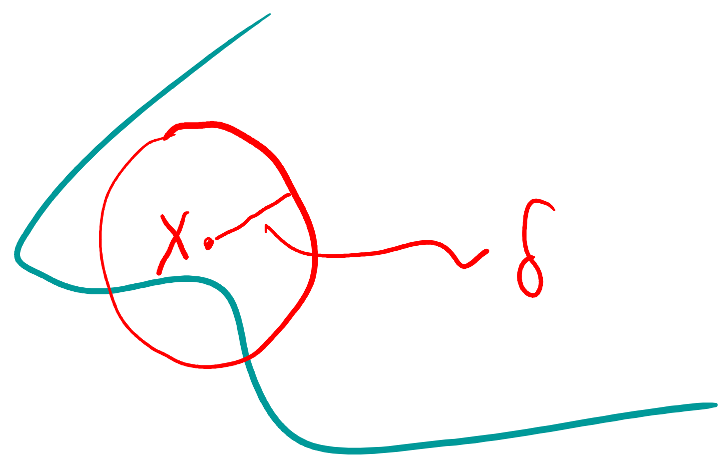
Como Muestrear?

El Paseo de Bola:

At x,

pick random y from $x + \delta B_n$

if y is in K, go to y



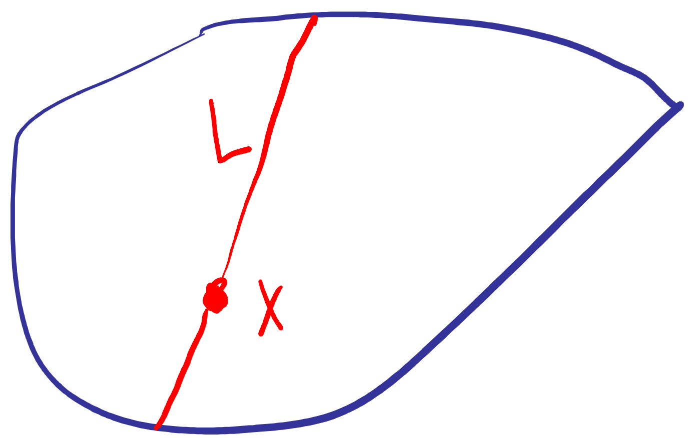
Pega-y-Corre

[Boneh, Smith]

At x,

-pick a random chord L through x

-go to a uniform random point y on L



Parte 3: Probabilidad

- Markov chains
- Convergence
- Conductance
- 1-step coupling

Cadenas de Markov

- State space K
- set of measurable subsets that form a σ-algebra, i.e., closed under finite unions and intersections
- A next step distribution $P_u(.)$ associated with each point u in the state space.
- A starting point.

- $w_0, w_1, \dots, w_k, \dots$ s.t.

$$P(w_k \in A \mid w_0, w_1, \dots, w_{k-1}) = P(w_k \in A \mid w_{k-1})$$

Convergencia

Stationary distribution Q, ergodic "flow" defined as

$$\Phi(A) = \int_A P_u(K\backslash \mathrm{A})\, dQ(u)$$

For a stationary distribution Q, we have

$$\Phi(A) = \Phi(K\backslash \mathrm{A})$$

Exercise: Demostrar que la distribution estacionaria de Pega-y-Corre es uniforme en el conjunto convexo.

Conductancia

Ergodic "flow":

$$\Phi(A) = \int_A P_u(K\backslash A)\,dQ(u)$$

Conductance:

$$\phi(A) = \frac{\Phi(A)}{\min Q(A),\ Q(K\backslash A)}$$

$$\phi = \inf\ \phi(A)$$

Conductancia y convergencia

Mixing rate cannot be faster than $1/\phi$

- it takes $1/\phi$ to escape from some subsets.

Does ϕ give an upper bound? Yes.

For discrete state-space Markov chains:

Thm. [Jerrum-Sinclair] $\frac{\phi^2}{2} \leq 1 - \lambda \leq 2\phi$

Where λ is the second eigenvalue of the transition matrix.

Thus, mixing rate = $\frac{1}{1-\lambda} \leq \frac{2}{\phi^2}$.

La Tasa de Convergencia

- Q : target distribution
- Q_0: starting distribution
- Q_t: distribution after t steps

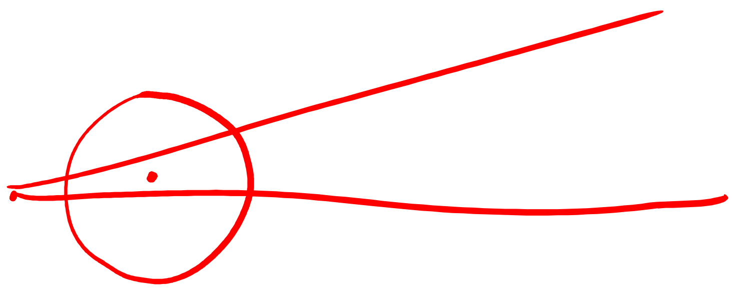
Thm. [LS93]

$$M = \sup_{A \subset K} \frac{Q_0(A)}{Q(A)}: \quad d_{TV}(Q_t, Q) \leq \sqrt{M}\left(1 - \frac{\phi^2}{2}\right)^t$$

$$M = E_{Q_0}\left(\frac{Q_0(x)}{Q(x)}\right): d_{TV}(Q_t, Q) \leq \epsilon + \sqrt{\frac{M}{\epsilon}\left(1 - \frac{\phi^2}{2}\right)^t} \ \forall \epsilon > 0$$

Como limitar la conductancia?

- Local conductance of ball walk can be arbitrarily small



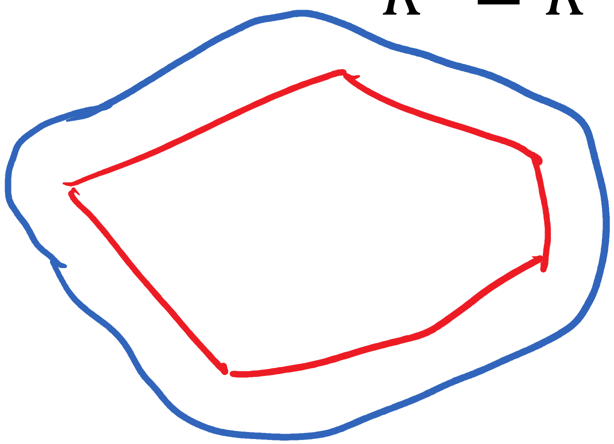
$$\ell(x) = \frac{\mathrm{vol}(x + \delta B_n \cap K)}{\mathrm{vol}(\delta B_n)}$$

- What can we do?

- smooth K
- start with a nearly random point in K.

Suavizar un conjunto convexo

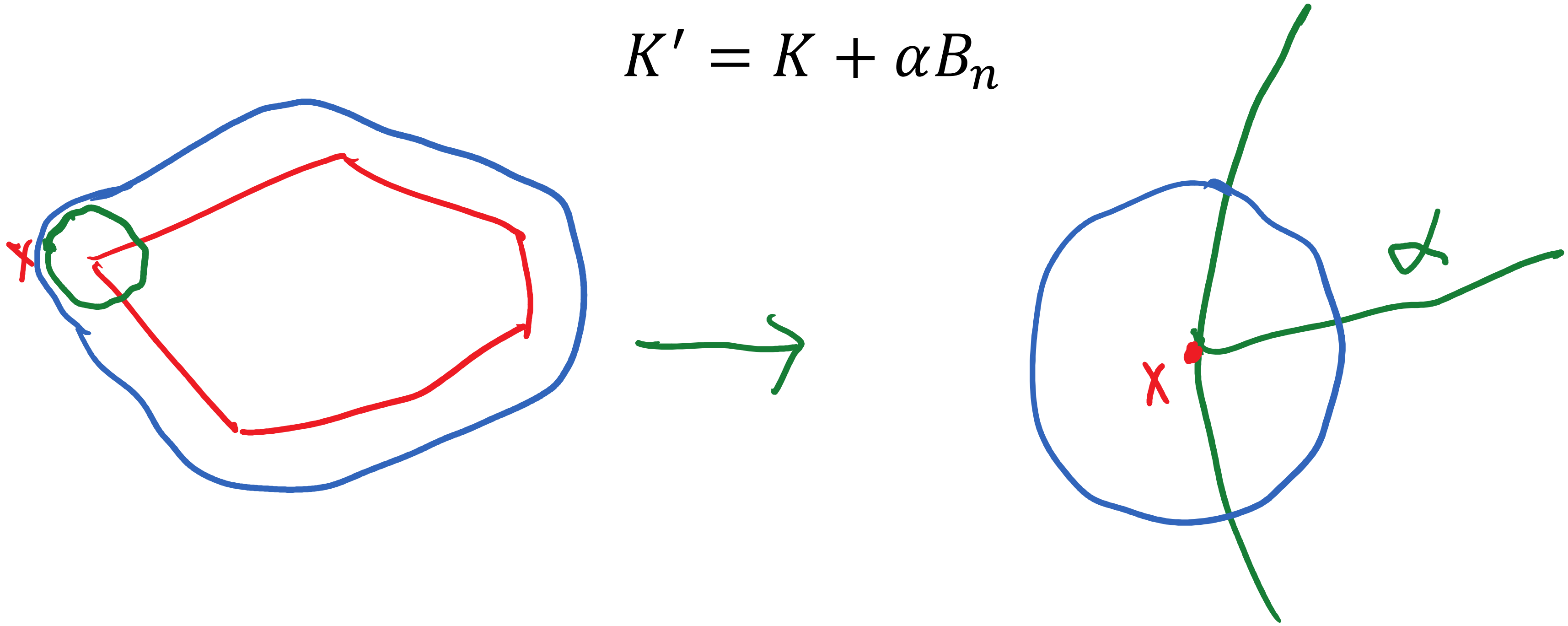
$$K' = K + \alpha B_n$$



- Each point of the original body has a small ball around it.
- What about new points? A ball of radius α containing the point is fully contained in K.

Suavear un conjunto convexo

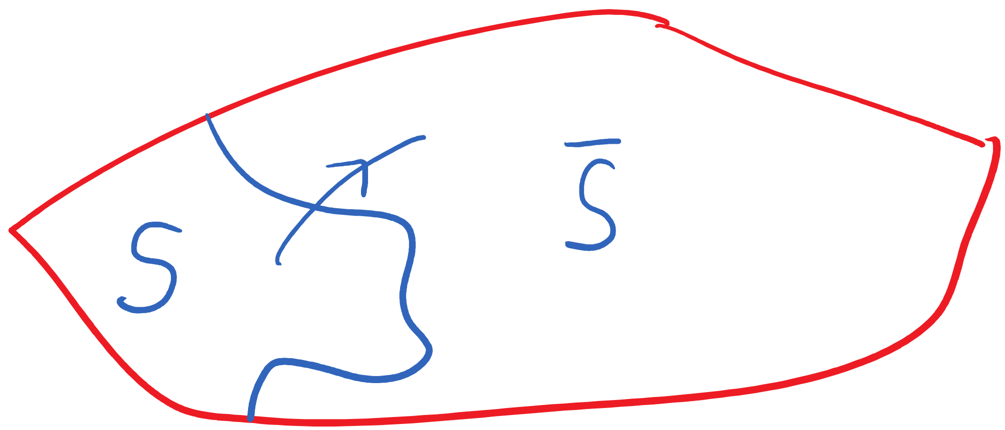
$$K' = K + \alpha B_n$$



- A ball of radius α containing the point is contained in K.
- Choosing $\delta \leq \frac{\alpha}{\sqrt{n}}$ ensures that every point has local conductance at least a constant.

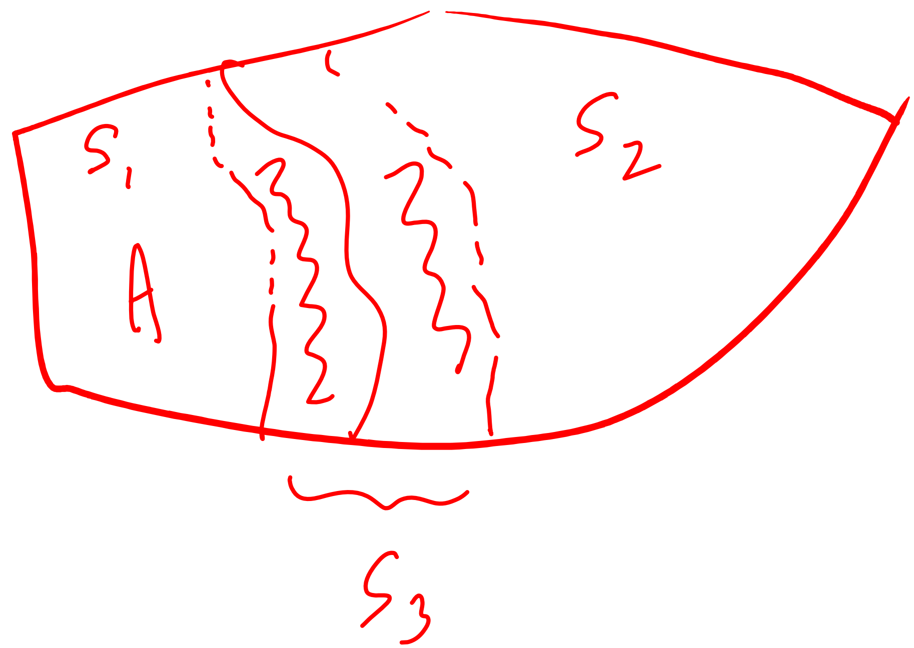
Conductancia

Consider an arbitrary measurable subset S.



We need to show that the escape probability from S is large.

Limitar la conductancia



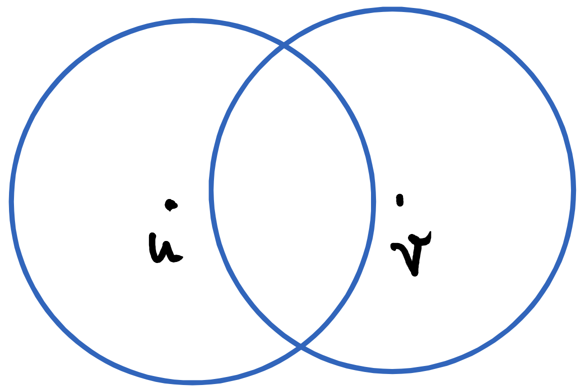
Need:

- Points that do not cross over are far from each other
- If two subsets are far, then the rest of the set is large

Distribuciones de un paso

Idea:

$d(P_u, P_v)$ large



$\Rightarrow$ the balls around u,v have small intersection

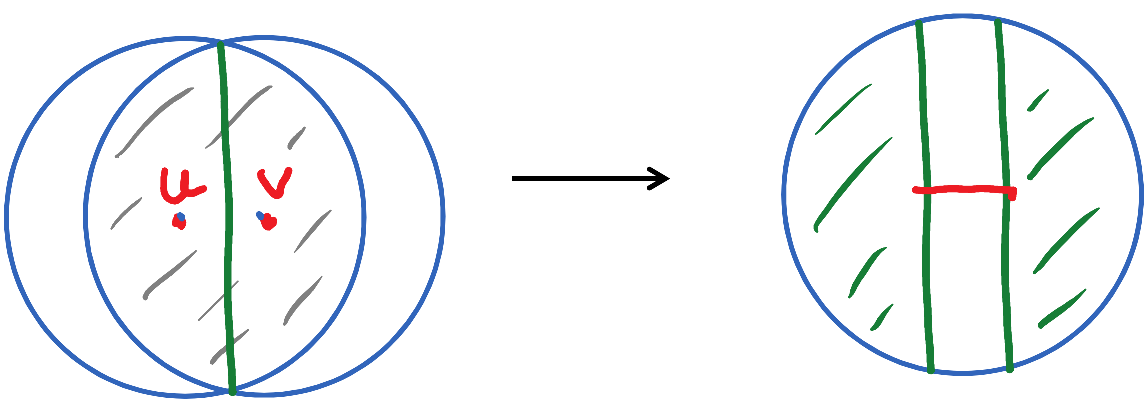
$\Rightarrow$ u,v must be far

Distancia Probabilistica ⇒ Distancia Geometrica

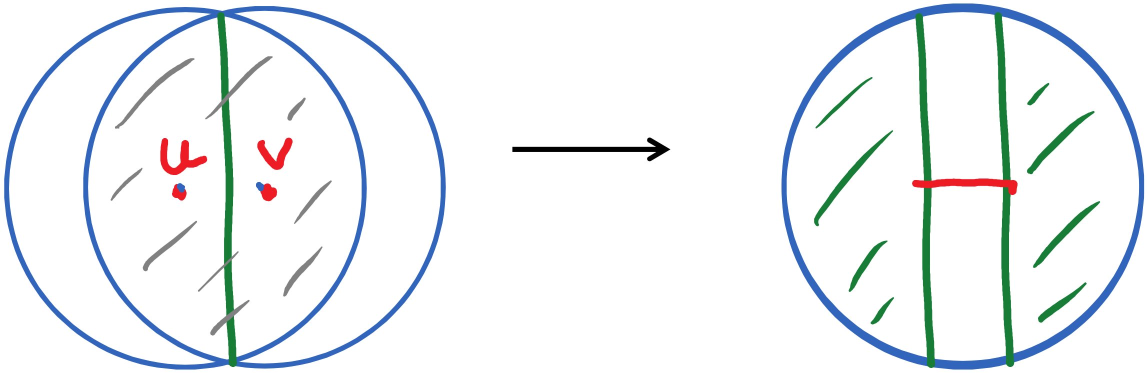
Lemma. $u, v \in K, \ell(u), \ell(v) \geq \ell$ for the ball walk with δ-steps. If

$$d(u,v) \leq \frac{t\delta}{\sqrt{n}},$$

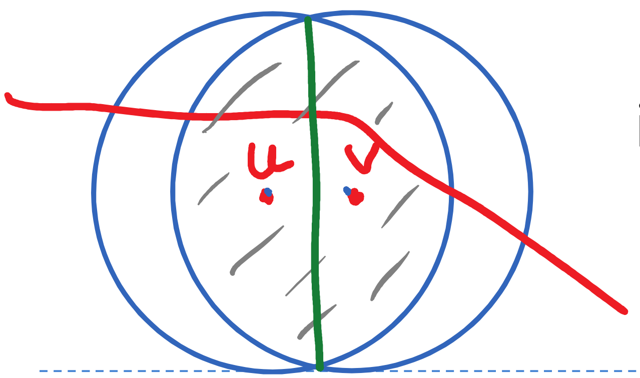
then $d(P_u, P_v) \leq 1 + t - \ell$.



Acoplamiento de las distribuciones de un paso

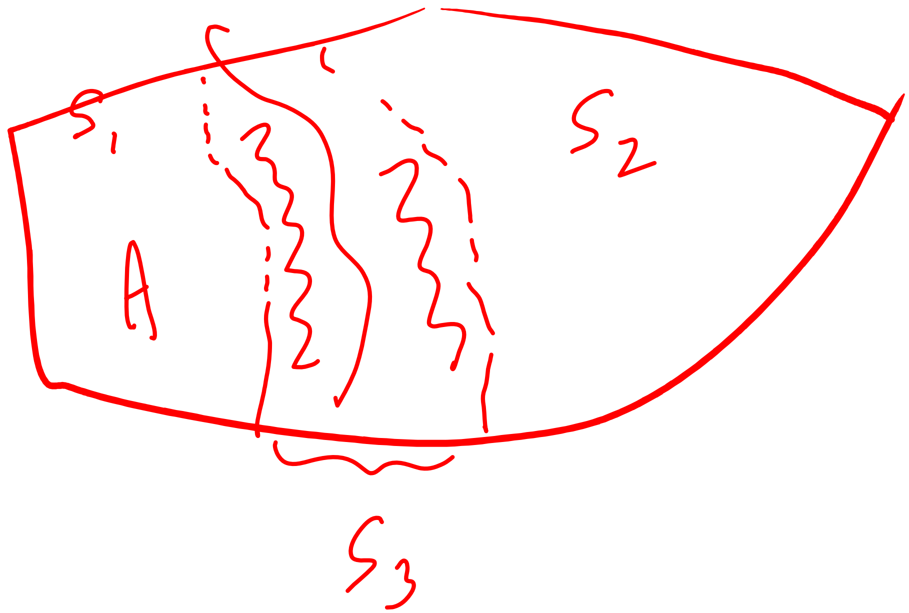


$$d(P_u, P_v) = \mathrm{vol}(\square) \leq t$$



if $\ell < 1,\ d(P_u, P_v) \leq t + 1 - \ell$

Conductancia



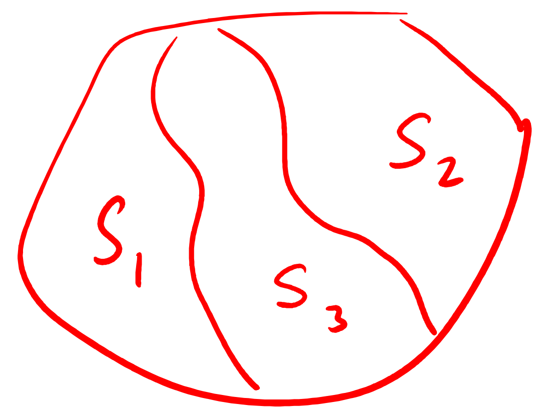
Need:

- Nearby points have overlapping one-step distributions
- Large subsets have large boundaries [isoperimetry]

Isoperimetria

Thm. [LS90,DF91]

$$vol(S_3) \geq \frac{2d(S_1,S_2)}{D} \min vol(S_1), vol(S_2)$$

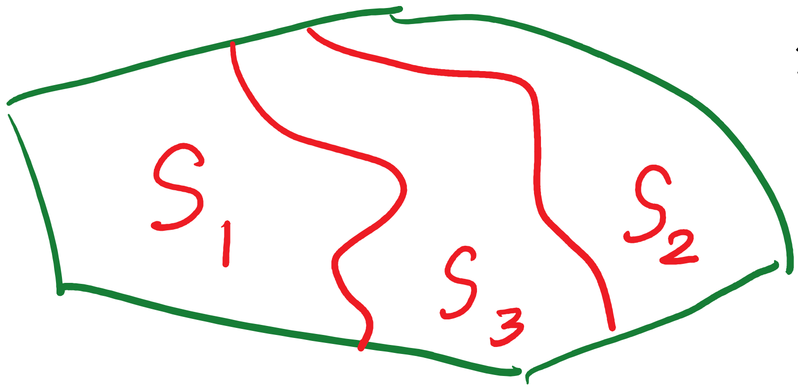


Extends to logconcave densities:

$$\pi_f(S_3) \geq \frac{2d(S_1,S_2)}{D} \min \pi_f(S_1), \pi_f(S_2)$$

D is the diameter of the support.

Isoperimetria (Mañana!)



$$\pi(S_3) \geq \frac{c}{D} d(S_1, S_2) \min \pi(S_1), \pi(S_2)$$

$$R^2 = E_\pi\left(\|x - \bar{x}\|^2\right)$$

A $= E((x - \overline{x})(x - \bar{x})^T)$: covariance matrix of π

$$R^2 = E_\pi\left(\|x - \bar{x}\|^2\right) = Tr(A) = \sum_i \lambda_i(A)$$

Thm. [KLS95]. $\pi(S_3) \geq \frac{c}{R} d(S_1, S_2) \min \pi(S_1), \pi(S_2)$

Conductancia

Thm. Conductance of ball walk is at least $\dfrac{\ell^2\delta}{16\sqrt{n}D}$

Using:

$$\ell = \frac{1}{n}, \qquad \delta = \frac{\ell}{\sqrt{n}} = \frac{1}{n\sqrt{\mathrm{n}}}$$

$$\phi \geq \frac{C}{n^2 D}, \qquad \text{mixing rate} = O(n^4 D^2)$$

Conductancia

Thm. Conductance of ball walk is at least $\dfrac{\ell^2\delta}{16\sqrt{n}D}$

Pf.

$$S_1 = \left\{x \in S : \; P_x(K\backslash S) < \frac{\ell}{4}\right\} \quad S_2 = \left\{x \in K\backslash S : \; P_x(S) < \frac{\ell}{4}\right\}$$

$$S_3 = K\backslash S_1\backslash S_2$$

$$vol(S_1) \geq \frac{vol(S)}{2}, vol(S_2) \geq \frac{vol(K\backslash S)}{2}$$

If not,

$$\int_S P_x(K\backslash S)dx \geq \frac{\ell}{4}.\frac{1}{2}vol(S) \quad \Rightarrow \quad \phi(S) \geq \frac{\ell}{8}.$$

Conductancia: la prueba

Thm. Conductance of ball walk is at least $\frac{\ell^2 \delta}{16\sqrt{n}D}$

Pf.

$$S_1 = \left\{x \in S : \ P_x(K\backslash S) < \frac{\ell}{4}\right\} \quad S_2 = \left\{x \in K\backslash S : \ P_x(S) < \frac{\ell}{4}\right\}$$

For $u \in S_1, v \in S_2$,

$$d(P_u, P_v) \geq 1 - P_u(K\backslash S) - P_v(S) > 1 - \frac{\ell}{2} \ \Rightarrow \ d(u,v) \geq \frac{\ell\delta}{2\sqrt{n}}.$$

$$\begin{aligned} vol(S_3) &\geq \frac{\ell\delta}{\sqrt{n}} \min \ vol(S_1), vol(S_2) \\ &\geq \frac{\ell\delta}{2\sqrt{n}} \min \ vol(S), vol(K\backslash S). \end{aligned}$$

Conductancia: la prueba

Thm. Conductance of ball walk is at least $\frac{\ell^2\delta}{16\sqrt{n}D}$

Pf.

$$\int_S P_x(K\backslash S)dx = \frac{1}{2}\int_S P_x(K\backslash S)dx + \frac{1}{2}\int_{K\backslash S} P_x(S)dx$$

$$\geq \frac{1}{2}.\frac{\ell}{4}.vol(S_3)$$

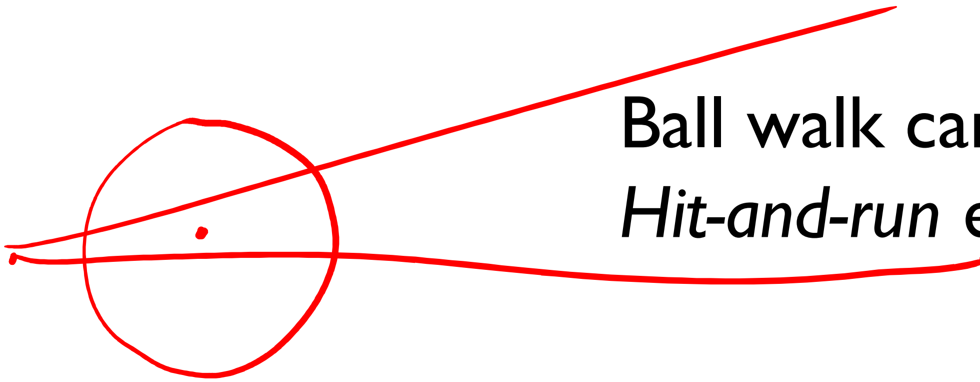
$$\geq \frac{\ell^2\delta}{16\sqrt{n}D}\min\ vol(S), vol(K\backslash S).$$

Convergencia de paseo de bola

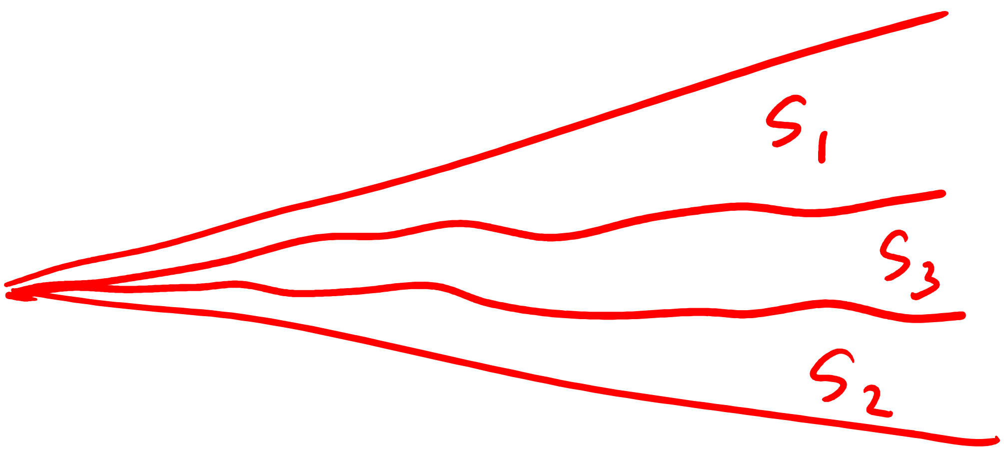
Thm. [LS93, KLS97] If S is convex, then the ball walk with an M-warm start reaches an (independent) nearly random point in poly(n, D, M) steps.

- Strictly speaking, this is not rapid mixing!
- Is there a rapidly-mixing random walk?

Mesclando rapidamente es posible?

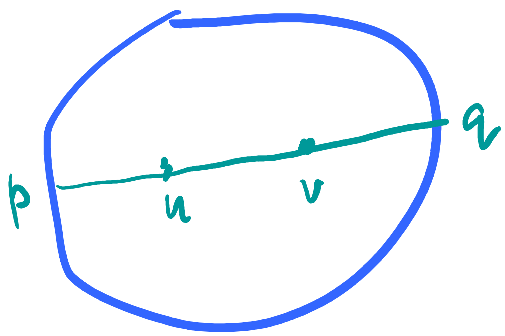


Ball walk can have bad starts, but *Hit-and-run* escapes from corners



Min distance isoperimetry is too coarse

Distancia or pega-y-corre



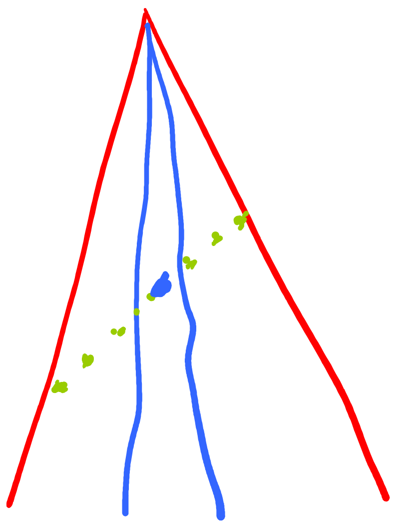
Cross-ratio distance:

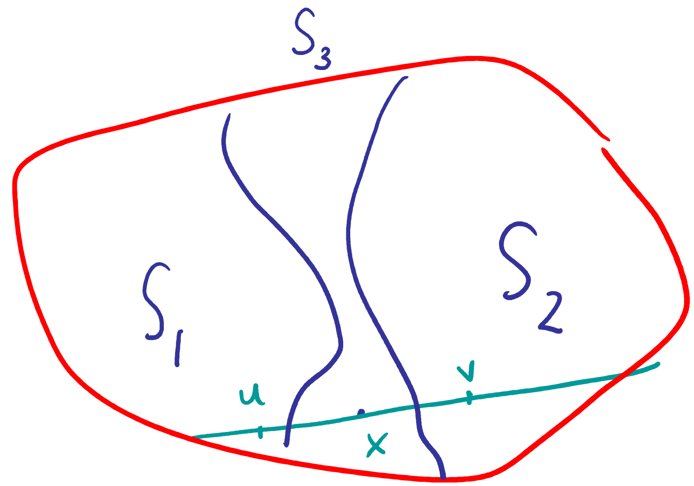
$$d_K(u,v) = \frac{|u-v||p-q|}{|p-u||v-q|}$$

Thm. [L;LV03] $\pi_f(S_3) \geq d_K(S_1,S_2)\pi_f(S_1)\pi_f(S_2)$

Isoperimetria de distancia *promedio*

- How to average distance?
- $h(x) \leq \frac{1}{3} min\ \{d(u,v) :\ u \in S_1, v \in S_2, x \in \ell(x,y)\}$





Thm.[LV04] $\pi(S_3) \geq E\big(h(x)\big)\pi(S_1)\pi(S_2)$

Pega-y-corre mezcla rapidamente

- Thm [LV04]. Hit-and-run mixes in polynomial time from *any* starting point inside a convex body.

- Conductance = $\Omega\left(\frac{1}{nD}\right)$

- Along with isotropic transformation, gives $O^*(n^3)$ sampling algorithm