XV Summer School in Discrete Mathematics, Valparaiso, Chile

Exercises on the Traveling Salesman Problem

January 6-10, 2020

Lecture 1:

1. (*) Prove that in any undirected graph there is always an even number of odd-degree vertices.

2. Given a complete, undirected graph $G = (V, E)$ with edge costs $c(i, j) \geq 0$ that are symmetric ($c(i, j) = c(j, i)$ for all $i, j \in V$) and obey the triangle inequality ($c(i, j) \leq c(i, k) + c(k, j)$ for all $i, j, k \in V$), this question asks you to explore the problem of finding a minimum-cost *path* that visits all vertices.

   (a) (**) Suppose the path may start in any vertex and end in any vertex, visiting all other vertices in between. Show that such a path, whose length is at most (3/2) times the optimal path length, can be found using a Christofides-Serdyukov-style Tree+Matching algorithm. (*Hint:* It may be necessary to add points to the instance so the desired matching exists.)

   (b) (**) Suppose we are given $s, t \in V$, and the goal is to find the minimum-cost path that starts at s and ends at t, visiting all other vertices in between. Give the natural analog of the Christofides-Serdyukov Tree+Matching algorithm for this problem, and find an example that show that this algorithm is *not* a $\frac{3}{2}$-approximation algorithm.

   (c) (***) Prove that the algorithm from (b) is a $\frac{5}{3}$-approximation algorithm.

Lecture 2:

3. (*) Let $G = (V, E)$ is 2-edge-connected and subcubic, i.e., every vertex has degree at most three.

   (a) Argue that every vertex in G has degree two or three, so that $|V| = |V_2| + |V_3|$, where V_2 is the set of degree two vertices, and V_3 is the set of degree three vertices.

   (b) Express $|E|$ in terms of $|V_2|, |V_3|$.

   (c) Suppose we use the same procedure as in lecture to find a removable pairing (R, P) (i.e., we take a DFS tree and pair each backedge with a tree-edge as described in lecture; for the root we ignore one backedge (so that if the root has degree two, its incident edges do not form a pair in P)).
   Show that this indeed gives a removable pairing (R, P), and that $\frac{4}{3}|E| - \frac{2}{3}|R| \leq \frac{4}{3}|V|$.

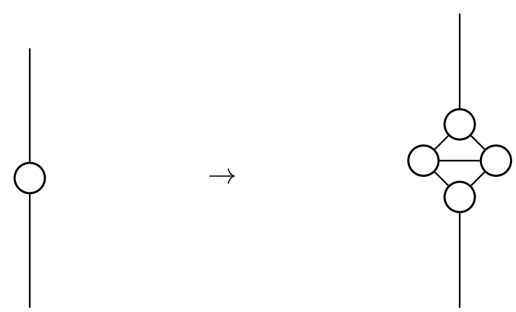
   (d) Conclude by using Theorem 1 (see the next question) that there is a $\frac{4}{3}$-approximation algorithm for the Graph-TSP problem on any 2-edge-connected, subcubic graph G.

4. (*) Extend the theorem on slide 29 of Lecture 2 to subcubic 2-edge-connected graphs, i.e.,

   Theorem 1. *Given a 2-edge-connected* subcubic *graph G and a removable pairing (R, P), there is an Eulerian multigraph with at most $\frac{4}{3}|E| - \frac{2}{3}|R|$ edges.*

   Hint: The proof in lecture showed this by giving an algorithm that gives the desired Eulerian multigraph: set $c(e) = 1$ for $e \in E \setminus R$, $c(e) = -1$ for $e \in R$; compute a minimum-cost perfect matching M; let F be obtained from E by removing the edges in $M \cap R$ and adding a copy of the edges in $M \setminus R$. We proved that (V, F) is an Eulerian multigraph with at most $\frac{4}{3}|E| - \frac{2}{3}|R|$ edges.

   Since we are now considering a subcubic graph G, we should not use the same algorithm (because it would make the degree of the degree-2 vertices become odd). To modify the algorithm, consider replacing each vertex of degree two by the following "gadget".

   Let E_g be the edges inside the gadgets, and let $G' = (V', E \cup E_g)$ be the resulting graph. Explain how to use a minimum-cost perfect matching in this graph (where you should explain how to set the costs of the edges) to prove Theorem 1.

5. (**) Consider a solution x to the subtour LP relaxation for the TSP that is a *fractional 2-matching*: $x(e) \in \{0, 1/2, 1\}$ for all $e \in E$, and all the edges with $x(e) = 1/2$ form vertex-disjoint cycles with odd numbers of edges. A *2-matching* is a set of edges such that each vertex has degree two, and each connected component has at least 3 vertices in it (and is thus a collection of cycles, each of which has at least 3 vertices in it). A *graphical 2-matching* is a multiset of edges such that each vertex has even degree, and each connected component has at least 3 vertices in it. In this exercise, we will show that we can find a 2-matching of cost at most 4/3 times the value of the LP solution.

   To do this, consider a graph G' obtained from the LP solution by any path of edges e with $x(e) = 1$ and replacing the path with a single edge, and including any edge e with $x(e) = 1/2$. The resulting graph is cubic (each vertex has degree exactly 3) and is 2-edge-connected (argue to yourself that this must be true). Let the cost $c'(e)$ of any edge e in the graph G' be the cost of the path (if e corresponds to a path in the original graph) or the negative of the cost of the edge e (if the edge e had $x(e) = 1/2$) in the original graph. Compute a minimum-cost perfect matching M. We now construct a set of edges F. If $e \in M$ corresponded to a path in the original graph, then add two copies of each edge in the path to F. If $e \notin M$ and e corresponded to a path in the original graph, then include one copy of each edge in the path in F. If for edge e we had $x(e) = 1/2$ and $e \notin M$, then add this edge to F (if $e \in M$ then we do not add this edge to F).

(a) Argue that the resulting set of edges F must be a graphical 2-matching.

(b) Prove that the cost of the edges in F must be at most $\frac{4}{3}\sum_{e \in E} c(e)x(e)$.

(c) Prove that there is a 2-matching of cost at most the cost of the edges in F.

Lecture 3:

5. (*) Recall the following standard linear programming relaxation of the traveling salesman problem:

$$\text{Min} \quad \sum_{e \in E} c(e)x(e)$$

subject to:

$$\begin{aligned} & x(\delta(i)) = 2, && \forall i \in V, \\ & x(\delta(S)) \geq 2, && \forall S \subset V, S \neq \emptyset, \\ & 0 \leq x_e \leq 1, && \forall e \in E. \end{aligned}$$

Let x^* be an optimal solution to the LP, and let $n = |V|$. Show that $z(e) = \frac{n-1}{n}x^*(e)$ for every $e \in E$ gives a feasible solution for the spanning tree LP:

$$\text{Min} \quad \sum_{e \in E} c(e)z(e)$$

subject to:

$$\begin{aligned} & z(E) = |V| - 1, && \\ & z(E(S)) \leq |S| - 1, && \forall S \subset V, S \neq V, \\ & 0 \leq z(e) \leq 1, && \forall e \in E. \end{aligned}$$

6. (*) Let T be a spanning tree, $\mathrm{W}_T = \mathrm{Odd}_T \triangle \{s, t\}$ the set of vertices whose degree parity needs to be fixed if we want to obtain an s-t traveling salesman path.

Let $S \subset V$. In lecture, we claimed that if $|\mathrm{W}_T \cap S|$ is odd, then

$$|T \cap \delta(S)| = \begin{cases} \text{even} & \text{if } \delta(S) \text{ is an } s\text{-}t \text{ cut}, \\ \text{odd} & \text{if } \delta(S) \text{ is a non } s\text{-}t \text{ cut}. \end{cases}$$

Prove that this is indeed correct.

Hint: consider $\sum_{i \in S} |T \cap \delta(i)|$.

7. (*) Let T be a spanning tree, and let M be a minimum-cost matching of $\mathrm{W}_T = \mathrm{Odd}_T \triangle \{s, t\}$.

Prove that $c(M) \leq \frac{1}{3}(c(T) + OPT_{LP})$ by showing that, if x^* is a feasible solution to the s-t path TSP LP, then setting $z(e) = \frac{1}{3}x^*(e) + \frac{1}{3}\chi_T(e)$ for all $e \in E$ gives a feasible solution to the W_T-matching LP.[1]

Lecture 4:

7. (*) Recall the LP relaxation for the s-t TSP path problem:

$$\text{Min} \quad \sum_{e \in E} c(e)x(e)$$

subject to:

$$x(\delta(i)) = \begin{cases} 1, & \forall i = s, t, \\ 2, & \forall i \neq s, t, \end{cases}$$

$$x(\delta(S)) \geq \begin{cases} 1, & \forall\ s\text{-}t \text{ cuts } \delta(S), \\ 2, & \forall \text{ non } s\text{-}t \text{ cuts } \delta(S), \end{cases}$$

$$0 \leq x(e) \leq 1, \qquad \forall e \in E.$$

Let x^* be an optimal solution to the LP, and let OPT_{LP} be the optimal value of the LP.

In this problem, we will show that we can find a s-t traveling salesman path of cost at most $\frac{3}{2}OPT_{LP}$ if x^* is half-integral, i.e., if $x^*(e) \in \{0, \frac{1}{2}, 1\}$ for all $e \in E$.

(a) Prove that $x^*(\delta(S))$ is integral for any set $S \subset V$ if x^* is half-integral.

(b) Prove that any solution generated by An, Kleinberg, Shmoys's Best-of-Many algorithm has cost at most $\frac{3}{2}OPT_{LP}$ if x^* is half-integral.

8. (**) Let $G = (V, E)$ be a 2-edge-connected graph, and suppose G has two special vertices, $s, t \in V$, where in addition to being 2-edge-connected, G satisfies that $|\delta(S)| \geq 3$ for all non s-t cuts $\delta(S)$. In this question, you will show that there exist a multisubset E' of E with $|E'| \leq |E|$ such that (V, E') has an s-t Eulerian path.

(a) Show that $x(e) = \frac{2}{3}$ for all $e \in E$ is a feasible solution to the following linear program.

$$\text{Min} \quad \sum_{e \in E} x(e)$$

subject to:

$$x(\delta(S)) \geq \begin{cases} 1, & \forall\ s\text{-}t \text{ cuts } \delta(S), \\ 2, & \forall \text{ non } s\text{-}t \text{ cuts } \delta(S), \end{cases}$$

$$0 \leq x_e \leq 1, \qquad \forall e \in E.$$

(b) Call $\delta(S)$ a narrow cut if $x(\delta(S)) < 2$ for x from (a), i.e. if $|\delta(S)| < 3$. Show that the narrow cuts are disjoint, i.e., $\delta(S) \cap \delta(S') = \emptyset$ for distinct narrow cuts $\delta(S), \delta(S')$.

[1] χ_T is the characteristic vector of T, and has $\chi_T(e) = 1$ if $e \in T$, $\chi_T(e) = 0$ if $e \notin T$.

(c) Show that there exists a spanning tree T in G such that $|T \cap \delta(S)| = 1$ for every narrow cut $\delta(S)$.

(d) Let F be the forest obtained by removing the edges in the narrow cuts from the tree T from (c).

Let $\mathrm{W}_F = \mathrm{Odd}_F \triangle \{s, t\}$. Let M be a minimum-cost W_F-matching, where the cost $c(e)$ of edge $e = (i, j)$ is the shortest path distance between i and j in G. Show that $c(M) \leq \frac{1}{3}|E| + |L|$, where $L = T \setminus F$.

(e) Show that $(V, F \sqcup M)$ is connected, and that s, t are the only two vertices that have odd degree (i.e., $(V, F \sqcup M)$ has an s-t Eulerian path).

(f) Conclude that replacing each edge in M (which may not be an edge in G) by the shortest path between i and j gives a multisubset E' of E with $|E'| \leq |E|$ such that (V, E') has an s-t Eulerian path.