Exercises: Day 3

Exercise 1

Let $\mathcal{G} = (\mathcal{P}_n, r_n)$ be the corresponding random geometric graph in $[0,1]^2$ with $r_n = \sqrt{\log n/((1-\varepsilon)\pi n)}$ for some small $\varepsilon > 0$. Tessellate $[0,1]^2$ into *cells* of side length $r_n/(2k)$ for some large enough constant k. Call a cell *dense* if it has at least 48 vertices, and call cells c_1 and c_2 *close*, if $\sup_{x_1 \in c_1, x_2 \in c_2} d_E(x_1, x_2) \leq r_n$. Let K the number of cells above and to the right of a given cell (at distance at least r_n from any boundary of $[0,1]^2$) that are close to it (note that K can be made large by making k large). A *lattice animal* is a (topologically) diagonally connected collection of cells. Call a lattice animal dense, if it has at least one dense cell (in particular a subsquare is also a lattice animal).

- (**) For K large enough, show that with high probability all lattice animals of size $4K$ are dense, all animals of size $2K$ touching one boundary of $[0,1]^2$ are dense, and all lattice animals of size K touching two sides of $[0,1]^2$ are dense.
- (*) With high probability, for any cell c_1, there exists a cell c_2 that is dense and close to c_1. Hint: Use the first part of this exercise.