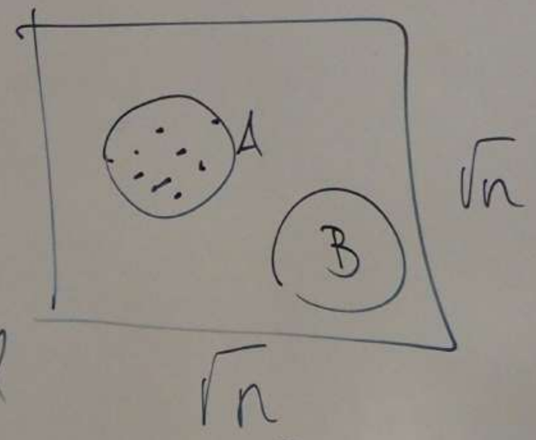
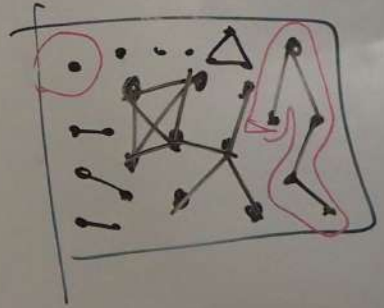
Connectivity of RGGs (Poisson in $[0,\sqrt{n}]^2$)

$G(X_n, r)$, $r = C\sqrt{\log n}$

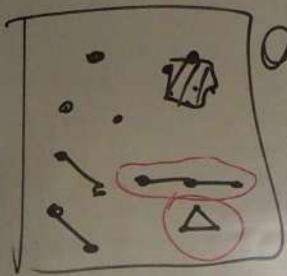


<u>Thm</u>: The threshold for having an isolated vertex and being connected are equal.

<u>Lemma 1</u>: Let $c>0$. Whp, $G(X_n, c\sqrt{\log n})$ does not have 2 components of Euclidean diameter $\geq C\sqrt{\log n}$ for large enough $C = C(c) > 0$.



Lemma 2: Let C as in Lemma 1. Whp the graph $G = G\left(X_n, \sqrt{\frac{\log n - \frac{1}{2}\log\log n}{\pi}}\right)$ does not have a component with >1 point and Euclidean diameter $\leq C\sqrt{\log n}$.



Proof: Suppose H is such a component. Case 1: Euclidean diameter of H is $\leq \frac{(\log\log n)^2}{\sqrt{\log n}} = r_1$. Let $x \in H$.

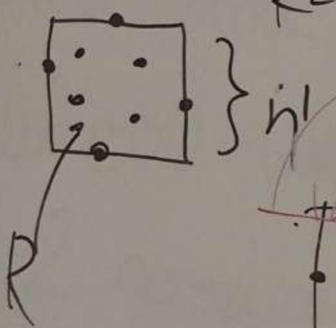
Exercise: fix x. $\mathbb{P}\left(|B(x,r_1) \cap X_n| \geq 1,\ |B(x,r) \setminus B(x,r_1) \cap X_n| = 0 \right)$
$= o(1/n)$

→ Whp, there is no such vertex x.

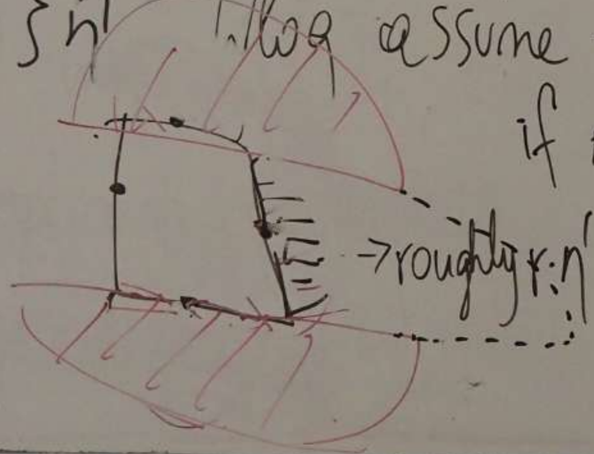
Case 2: Euclidean diameter of $H \geq \eta$.

Consider the following rectangle:

R = smallest axis-parallel rectangle containing H



W.l.o.g. assume the height is at least $\eta' := \frac{\eta}{\sqrt{2}}$



if $\eta' \ll r$: extra area of $2r\eta'$

If $\eta' \geq cr$: extra $\Theta(r^2)$ area empty

Hence, empty area is at least

$$r^2\pi + 2r\eta'(1+o(1)) \geq r^2\pi + \sqrt{2}\, r\eta\,(1+o(1))$$

$$= r^2\pi + (\log\log n)^2$$

$$\mathbb{P}(\text{this area empty}) = \exp\left(-\log n + \tfrac{1}{2}\log\log n - (\log\log n)^2\right) = o\left(\frac{1}{n(\log n)^3}\right).$$

How many ways to choose such areas? At most n choices for starting vertex x.

Other vertices are in rectangle of area $(\log n)$ $\Rightarrow$ at most $(\log n)^3$ choices for other vertices.

$\mathbb{P}(\exists \text{ such area empty}) = o(1)$

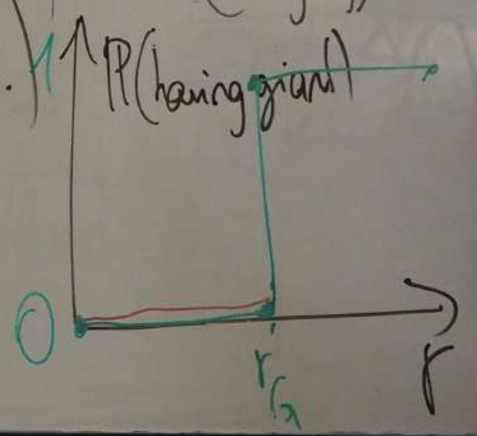
Q: When is there already a giant component in RGGs in $[0,\sqrt{n}]^2$
that is, a component of size $\Theta(n)$?

It will turn out that $c_1 \le r_G \le c_2$ for some constants c_1, c_2.

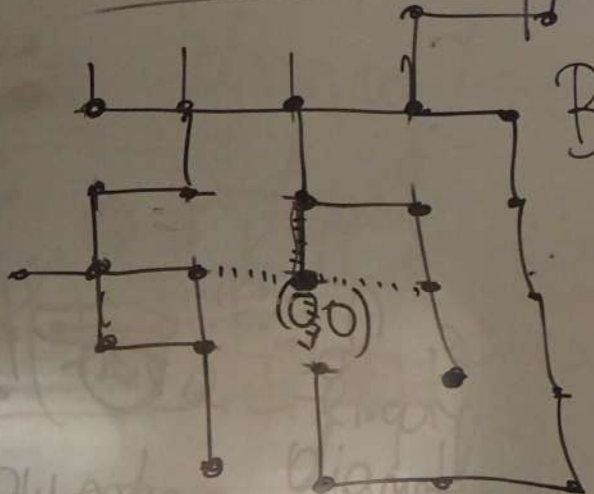
(If $r \le c_1$, then all components have whp at most $O((\log n)^2)$ vertices,
and if $r \ge c_2$, then $\exists!$ component of size $\Theta(n)$.)

Open problem: find value of r_G

Here: show $c_1 \le r_G \le c_2$



Excursion to percolation on $\mathbb{Z}^2$:



Bond percolation: keep each edge independently with probability p, $0 \leq p \leq 1$

Big question: Is there an infinite cluster?

Is there positive probability that $(0,0)$ is in an infinite component?

Define $\theta(p) = \mathbb{P}_p[0 \text{ is in an infinite component}]$

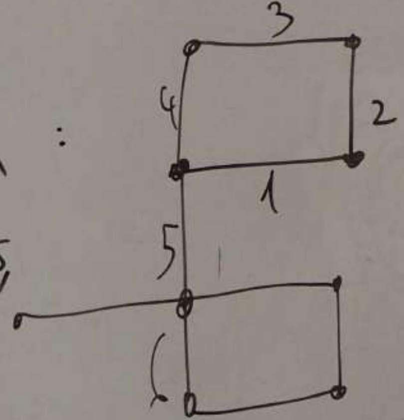
$p_c = \inf\{p : \theta(p) > 0\}$

For $p < p_c$: $\mathbb{P}(x \text{ is } \cancel{\text{in}} \text{ an infinite component}) = 0 \quad \forall x$

$\Longrightarrow$ by union bound, $\mathbb{P}(\exists x \text{ in an infinite comp}) = 0$.

Theorem: $0 < p_c < 1$.

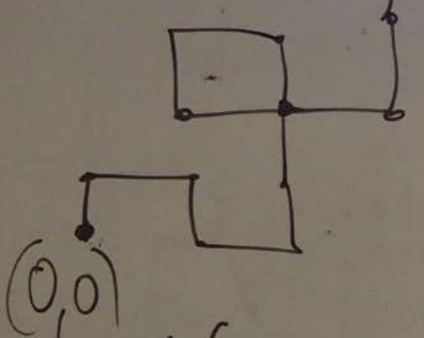
Proof: A self-avoiding path of length n:



path not reusing edges, but can reuse vertices

Define Ω_n to be the set of self-avoiding paths of length n starting at $(0,0)$

Obs: If $(0,0)$ is ~~in~~ an infinite component, then there exists a self-avoiding path of length n, $\forall n \geq 0$ starting from $(0,0)$

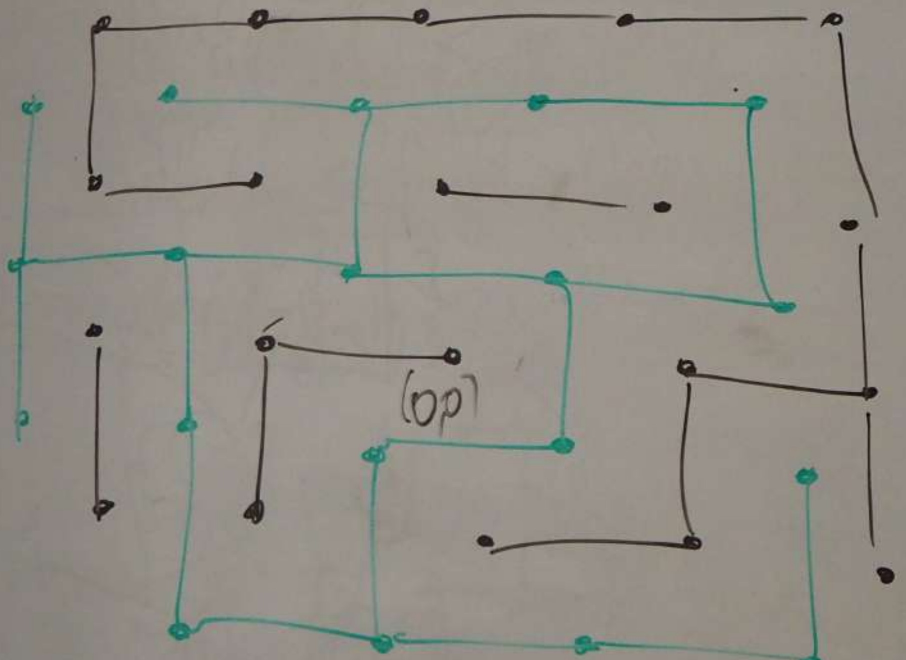


$$\theta(p) \leq \mathbb{P}(\exists \text{ a self-avoiding path of length } n \text{ starting at } (0,0))$$

$$\leq \sum_{\gamma \in \Omega_n} \mathbb{P}(\text{each edge of the path of } \gamma \text{ is still present}) \leq |\Omega_n| \, p^n \leq (4p)^n$$

If $p < 1/4$, then this tends to $0 \Rightarrow p_c \geq 1/4$.

For an upper bound: we introduce the dual graph $(\mathbb{Z}^2)^*$:



$\mathbb{Z}^2$

$(\mathbb{Z}^2)^*$: dual edge is present $\Leftrightarrow$ original associated edge is not present

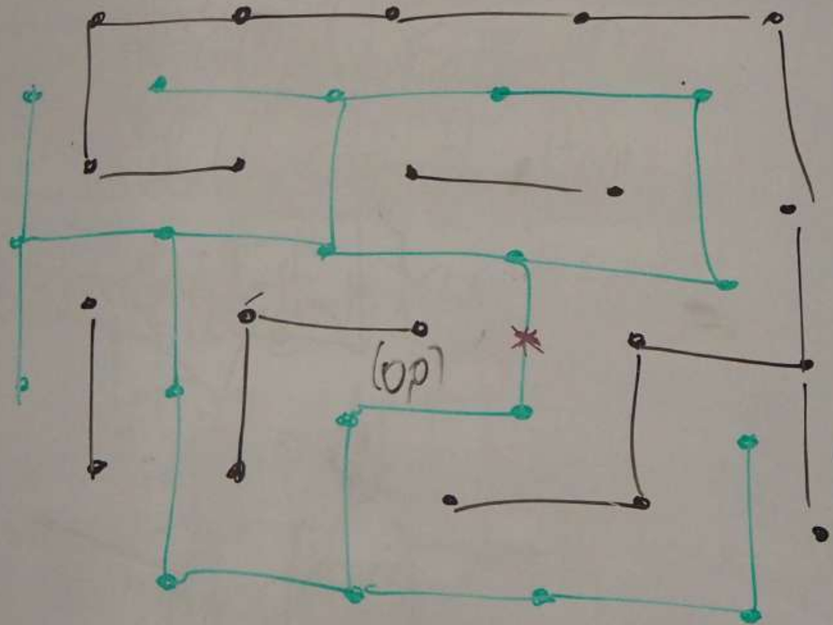
Obs: If $(0,0)$ is not in an infinite component, then there must be a circuit in the dual graph surrounding $(0,0)$.
This circuit must intersect the point $(n+\frac{1}{2}, 0)$ for some $n \in \mathbb{N}$

$$1-\theta(p) \leq \sum_{n \geq 0} \mathbb{P}_p\left[\exists \text{ circuit in } (\mathbb{Z}^2)^* \text{ intersecting } (n+\tfrac{1}{2}, 0), \text{ surrounding the origin } (0,0)\right]$$

$$\leq \sum_{n \geq 0} \mathbb{P}_p\left[\exists \text{ path in } (\mathbb{Z}^2)^* \text{ of length at least } 2n+4 \text{ surrounding the origin } (0,0), \text{ intersecting } (n+\tfrac{1}{2}, 0)\right]$$

$$\leq \sum_{n\geq 0} \left(4(1-p)\right)^{2n+4},$$

for p close to 1 (for ex. $p = 7/8$ still works) this sum is strictly less than 1.



$\Rightarrow 1-\theta(p) < 1$

$\Rightarrow \theta(p) > 0$

$\Rightarrow p_c < 1$

Remark: In fact $p_c = 1/2$