Quantum ergodicity for random matrices

Valparaiso, Disordered models of mathematical physics

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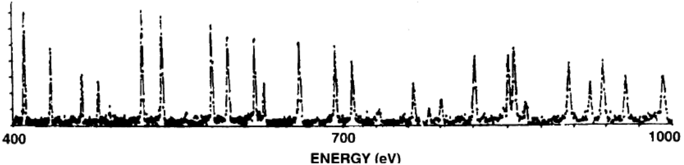
A spacially confined quantum mechanical system can only take certain discrete values of energy. Uranium-238 :



Quantum mechanics postulates that these values are eigenvalues of a certain Hermitian matrix (or operator) H, the Hamiltonian of the system.

The matrix elements H_{ij} represent quantum transition rates between states labelled by i and j.

Wigner's universality idea (1956). *Perhaps I am too courageous when I try to guess the distribution of the distances between successive levels. The situation is quite simple if one attacks the problem in a simpleminded fashion. The question is simply what are the distances of the characteristic values of a symmetric matrix with random coefficients.*

Wigner's model : the Gaussian Orthogonal Ensemble,
(a) Invariance by $H \mapsto U^* H U$, $U \in \mathrm{O}(N)$.
(b) Independence of the $H_{i,j}$'s, $i \leq j$.

The entries are Gaussian. The spectral density is $\frac{1}{Z_N} \prod |\lambda_i - \lambda_j|^\beta e^{-\beta \frac{N}{4} \sum \lambda_i^2}$ with $\beta = 1$ (2, 4 for invariance under unitary or symplectic conjugacy).

(i) Semicircle law as $N \to \infty$, $\rho(E) = \frac{1}{2\pi}\sqrt{(4 - E^2)_+}$.

(ii) Eigenvalues locally converge to a point process parametrized by β (Gaudin, Mehta, Dyson) : for any $E \in (-2, 2)$,

$$\sum_i \delta_{N\rho(E)(\lambda_k - E)} \to \mathrm{Sine}_\beta.$$

(iii) Eigenvectors are Haar-distributed. In particular the, Lévy-Borel law holds : for deterministic $\|\mathbf{q}\|_2 = 1$,

$$\sqrt{N}\langle u_k, \mathbf{q}\rangle \underset{N\to\infty}{\longrightarrow} \mathscr{N}(0,1).$$

Fundamental belief in universality : macroscopic eigenvalues statistics depend on the model, but microscopic statistics only depend on the symmetries.

(i) GOE : Hamiltonians of systems with time reversal invariance

(ii) GUE : no time reversal symmetry (e.g. application of a magnetic field)

(iii) GSE : time reversal but no rotational symmetry

This is not proved for any realistic Hamiltonian.

Easiest models to test Wigner's universal prediction : in the definition of the Gaussian ensembles, either keep

(i) the independence of the entries : Wigner ensembles, $N \times N$ matrix X such that

$$\mathbb{E}(X_{ij}) = 0, \mathbb{E}(X_{ij}^2) = \frac{1}{N}, \text{ higher moments are finite but arbitrary,}$$

(ii) or the conjugacy invariance (Invariant ensembles, universality proved by Deift, Zhou, Bleher, Its, Pastur, Shcherbina...).

This talk is only about Wigner matrices. For eigenvalues, universality is well-established under a variety of assumptions.

Wigner Dyson Mehta conjecture

Correlation functions of symmetric Wigner matrices converge to those of Sine_β ($\beta = 1, 2, 4$ depending on the symmetry class).

Proved :

(i) For Hermitian matrices : Gaussian divisible ensemble (Johansson), smooth distribution of the entries (Erdős, Schlein, Yau), third moment vanishing (Tao-Vu), then arbitrary entries.

(ii) For all symmetry classes : under an extra energy averaging assumption (Erdős, Schlein, Yau, averaging window $N^{-1+\varepsilon}$ by Erdős, Yau, Yin), a four moment matching assumption (Tao, Vu), at fixed energy for arbitrary entries (B, Erdős, Yau, Yin).

What about eigenvectors ?

Random matrix statistics supposedly occur together with strongly delocalized eigenstates

Quantum ergodicity (Shnirelman, Zelditch, Colin de Verdière). Compact manifold $\mathcal{M}$ with ergoic geodesic flow and with volume measure μ Let $(\psi_k)_{k\geq 1}$ denote an orthonormal basis of eigenfunctions of the Laplace-Beltrami operator, associated with increasing eigenvalues. Then for any open set $A \subset \mathcal{M}$ one has

$$\lim_{\lambda\to\infty} \frac{1}{N(\lambda)} \sum_{j:\lambda_j\leq\lambda} \Big| \int_A |\psi_j(x)|^2 \mu(\mathrm{d}x) - \int_A \mu(\mathrm{d}x) \Big|^2 = 0$$

where $N(\lambda) = |\{j : \lambda_j \leq \lambda\}|$.

Quantum unique ergodicity conjecture (Rudnick, Sarnak). If $\mathcal{M}$ is negatively curved,

$$\int_A |\psi_k(x)|^2 \mu(\mathrm{d}x) \underset{k\to\infty}{\longrightarrow} \int_A \mu(\mathrm{d}x).$$

Only known for some arithmetic surfaces (Lindenstraus, Holowinsky-Soundararajan).

Delocalization

For $u_1, \dots, u_N$ (L^2-normalized) eigenvector of a Wigner matrix,

$$\mathbb{P}\left(\sup_k \langle \mathbf{q}, u_k \rangle \leq (\log N)^C N^{-\frac{1}{2}}\right) \geq 1 - N^{-D}$$

for large enough N (Bloemendal, Erdős, Knowles, Schlein, Yau, Yin).

This does not imply quantum unique ergodicity (and reciprocally).

Microscopic scale

For any $\mathbf{q}$ from the canonical basis, for bulk eigenvectors $\sqrt{N}\langle \mathbf{q}, u_k \rangle$ converges to a Gaussian if the first four moments of the entries are 0, 1, 0, 3 (Tao-Vu, Knowles-Yin).

The proof relies on resolvent expansion, moment matching, comparison with GOE.

For generalized Wigner matrices (with Yau)

(1) For any Wigner ensemble, for any $\mathbf{q} \in \mathbb{R}^N$, $\sqrt{N}\langle \mathbf{q}, u_k \rangle$ converges to a Gaussian.

(2) Probabilistic version of QUE, at any scale. For any $A \subset [\![1, N]\!]$ and $\delta > 0$,

$$\mathbb{P}\left(\left|\frac{N}{|A|}\sum_{\alpha\in A}\left(u_k(\alpha)^2 - \frac{1}{N}\right)\right| > \delta\right) \leq C\left(N^{-\varepsilon} + |A|^{-1}\right).$$

Similar results hold for other mean-field models of random matrices (Erdős-Renyi graphs, covariance matrices ...)

Why would you want to study eigenvectors of random matrices ?

(i) Analysis of large data statistics.

(ii) Random graphs are models of a Laplacian in a domain or in a random potential. Paradigm for quantum chaos (Kottos, Smilansky ...).

(iii) QUE is an important tool in the study of eigenvalues statistics of non-mean field models such as band matrices.

The **Dyson Brownian Motion** $\mathrm{d}H_t = \frac{\mathrm{d}B_t}{\sqrt{N}} - \frac{1}{2}H_t\mathrm{d}t$ is an essential interpolation tool in the proof, similarly to the Erdős Schlein Yau approach to eigenvalues universality, which can be summarized as follows :

$$\begin{array}{ccc} & & H_0 \\ & & \updownarrow \\ \widetilde{H}_0 & \overset{\text{(DBM)}}{\longrightarrow} & \widetilde{H}_t \end{array}$$

$\overset{\text{(DBM)}}{\longrightarrow}$: for $t = N^{-1+\varepsilon}$, the eigenvaues of $\widetilde{H}_t$ satisfy universality (optimal time by Erdős, Yau, Yin).

$\updownarrow$: Density argument. For any $t \ll 1$, there exists $\widetilde{H}_0$ such that H_0 and $\widetilde{H}_t$ have the same statistics on the microscopic scale.

For eigenvector universality, no moment matching is required, we will proceed as follows :

$$H_0 \overset{\text{(DBM)}}{\longrightarrow} H_t$$

The coupled eigenvalues/eigenvectors dynamics when the entrie of H are Ornstein-Uhlenbeck motions :

$$\mathrm{d}\lambda_k = \frac{\mathrm{d}B_{kk}}{\sqrt{N}} + \left(\frac{1}{N}\sum_{\ell\neq k}\frac{1}{\lambda_k-\lambda_\ell} - \frac{1}{2}\lambda_k\right)\mathrm{d}t$$

$$\mathrm{d}u_k = \frac{1}{\sqrt{N}}\sum_{\ell\neq k}\frac{\mathrm{d}B_{k\ell}}{\lambda_k-\lambda_\ell}u_\ell - \frac{1}{2N}\sum_{\ell\neq k}\frac{\mathrm{d}t}{(\lambda_k-\lambda_\ell)^2}u_k$$

Let $c_{k\ell} = \frac{1}{N}\frac{1}{(\lambda_k-\lambda_\ell)^2}$. If all $c_{k\ell}$'s were equal, $U = (u_1, \dots, u_N)$ would be the Brownian motion on the unitary group.

Such eigenvector flows first appeared in the 1980s, first in the work of Bru for Wishart matrices, in Norris, Rogers, Williams for the Brownian motion in GL_N, in Anderson, Guionnet, Zeitouni for symmetric matrices.

Relaxation of the Dyson vector flow : first try. Conditionally on the trajectory $(\lambda_i(t), 1 \leq i \leq N)_{t\geq 0}$, the Dyson vector flow has generator

$$L = \sum_{k<\ell} c_{k\ell}(t)(u_k \cdot \partial_{u_\ell} - u_\ell \cdot \partial_{u_k})^2 \text{``} = \Delta\text{''}$$

where Δ is the Laplace-Beltrami for the metric g defined by

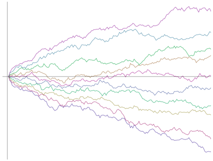
$$\langle E_\alpha, E_\beta \rangle^{(g)} = \frac{2}{c_{k\ell}} 1_{\alpha=\beta},\ \alpha = (k, \ell).$$

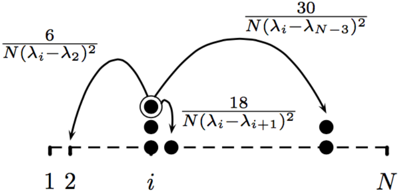
If $\mathrm{Ricci}^{(g)} \geq \tau^{-1} > 0$, the relaxation time is at most τ (Bakry, Émery).
Here, $\frac{\mathrm{Ricci}^{(g)}_{\mathrm{Id}}(E_\alpha, E_\alpha)}{\langle E_\alpha, E_\alpha \rangle^{(g)}}$ is equal to

$$\frac{1}{N} \sum_{i \notin \{k,\ell\}} \frac{1}{(\lambda_k - \lambda_i)(\lambda_i - \lambda_\ell)} = \begin{cases} N^{1-\alpha} \text{ if } |k-\ell| = N^\alpha \text{ and } \lambda_k(t) \equiv \gamma_k, \\ \text{may even be negative otherwise.} \end{cases}$$

There is no general relaxation theory taking into account
(1) time-dependent convexity and (2) initial conditions.

Relaxation of the Dyson vector flow : second try. Configuration $\boldsymbol{\eta}$ of n points on $[\![1, N]\!]$. Number of particles at x : η_x. Configuration obtained by moving a particle from i to j : $\boldsymbol{\eta}^{ij}$.





Dynamics given by $\partial_t f = \mathscr{B}(t) f$ where

$$\mathscr{B}(t) f(\boldsymbol{\eta}) = \sum_{i \neq j} c_{ij}(t) 2\eta_i (1 + 2\eta_j) \left(f(\boldsymbol{\eta}^{i,j}) - f(\boldsymbol{\eta}) \right).$$

The eigenvalues trajectory is a parameter ($c_{ij}(t) = \frac{1}{N} \frac{1}{(\lambda_i(t) - \lambda_j(t))^2}$).

Eigenvectors and random walks. Let $z_k = \sqrt{N}\langle \mathbf{q}, u_k\rangle$, random and time dependent. For a configuration $\boldsymbol{\eta}$ with j_k points at i_k, let

$$f_{t,\boldsymbol{\lambda}}(\boldsymbol{\eta}) = \mathbb{E}\left(\prod_k z_{i_k}^{2j_k} \mid \boldsymbol{\lambda}\right) / \, \mathbb{E}\left(\prod_k \mathscr{N}_{i_k}^{2j_k}\right).$$

Fact 1

$\partial_t f_{t,\boldsymbol{\lambda}}(\boldsymbol{\eta}) = \mathscr{B}(t) f_{t,\boldsymbol{\lambda}}(\boldsymbol{\eta}) = \sum_{i\neq j} c_{ij}(t) 2\eta_i(1+2\eta_j)\left(f(\boldsymbol{\eta}^{i,j}) - f(\boldsymbol{\eta})\right).$

QUE+Normality of the eigenvectors is equivalent to fast relaxation to equilibrium of the eigenvector moment flow.

Relaxation time is N^{-1}. Rigorous proofs by De Giorgi's or Nash's method (non-optimal error terms), or by an ad hoc parabolic maximum principle (optimal error terms).

This PDE analysis is made possible thanks to an explicit reversible measure π for $\mathscr{B}$.

Fact 2

$\pi(\boldsymbol{\eta}) = \prod_{x=1}^N \phi(\eta_x)$ where $\phi(k) = \prod_{i=1}^k \left(1 - \frac{1}{2k}\right)$

Relaxation to equilibrium

Goal : for $t \gg N^{-1}$, $\sup_{\boldsymbol{\eta}\subset \text{bulk}} |f_{\boldsymbol{\lambda},t}(\boldsymbol{\eta}) - 1| \leq N^{-\varepsilon}$.

Key tool : a maximum inequality. If $S_t = \sup_{\boldsymbol{\eta}}(f_t(\boldsymbol{\eta}) - 1)$ is always obtained for a configuration supported in the bulk, then

$$S_t' \leq -N^{1-\varepsilon} S_t + N^{1-\varepsilon}.$$

The bulk condition does not hold, but let's assume it.

Proof of the maximum inequality. For $n = 1$, if $S_t = \sup_k (f_t(k) - 1) = f_t(k_0) - 1$, then for any $\eta > 0$

$$\begin{aligned} S_t' &= \frac{1}{N} \sum_{k \neq k_0} \frac{f_t(k) - f_t(k_0)}{(\lambda_k - \lambda_{k_0})^2} \\ &\leq \sum_{k \neq k_0} \frac{\mathbb{E}\left(\langle u_k(t), \mathbf{q}\rangle^2 \mid \boldsymbol{\lambda}\right) - f_t(k_0)}{(\lambda_k - \lambda_{k_0})^2 + \eta^2} \\ &\leq \frac{1}{\eta}\mathbb{E}(\mathfrak{Im}\langle \mathbf{q}, G(\lambda_{k_0} + \mathrm{i}\eta) q\rangle \mid \boldsymbol{\lambda}) - f_t(k_0) \frac{1}{N\eta} \mathfrak{Im}\mathrm{Tr} G(\lambda_{k_0} + \mathrm{i}\eta) \end{aligned}$$

One concludes by the local semicirle law for $\eta = N^{-1+\varepsilon}$.

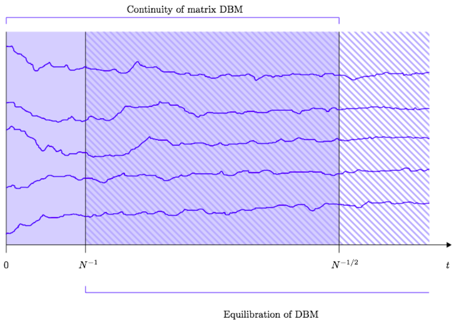
Continuity estimate

We proved QUE for $H + \varepsilon\text{GOE}$ and need to remove this tiny Gaussian convolution. This relies on the *Matrix models* point of view.
Remember

$$\mathrm{d}H_{ij} = \frac{\mathrm{d}B_{ij}}{\sqrt{N}} - \frac{1}{2}H_{ij}\mathrm{d}t.$$

Continuity of the Dyson Brownian motion : if F has uniformly bounded third derivatives and $t \lesssim N^{-1/2}$, we have

$$\mathbb{E}\,F(H_t) - \mathbb{E}\,F(H_0) \to 0.$$



Perturbative analysis in non-perturbative regime.

Let $(M_N)_{N\geq 0}$ be deterministic with eigenvalues satisfying the local semicircle law, eigenvectors $(e_k)_k$.

What do the eigenvectors $(u_k(t))_k$ of $M_N+\sqrt{t}\,\mathrm{GOE}$ look like ?

1. If $t \ll 1/N$, perturbative regime.
2. If $t \sim 1$, Stieltjes-transform and free-probability techniques give the size of all overlaps $\langle u_k(t), e_j\rangle$ (Biane, Knowles-Yin, Allez-Bouchaud).

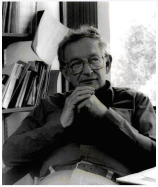
Theorem

In the mesoscopic regime $1/N \ll t \ll 1$, the overlaps are independent Gaussian with variance

$$\mathbb{E}\left(\langle u_k(t), e_j\rangle^2\right) \sim C\ \frac{1}{(Nt)^2+(\gamma_k-\gamma_j)^2}$$

Proof : eigenvector moment flow and homogenization theory .

Anderson's model for metal-insulator transition : $H = -\Delta + \lambda V_\omega$ on $\mathbb{Z}^d \cap [-L, L]^d$.

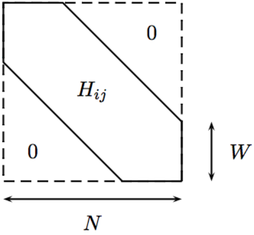


Local eigenvalues statistics :

(i) Supposedly, GOE in the delocalization regime (e.g., small λ for $d = 3$).

(ii) Poisson statistics (Minami) in the localization regime (Frohlich, Spencer, Aizenman, Molchanov), for high disorder.

Band matrices : a toy model for the transition.



(i) Supposedly, if $W \gg \sqrt{N}$, GOE and QUE.

(ii) Supposedly, if $W \ll \sqrt{N}$, Poisson and localization.

Question : QUE for mean-field with optimal error gives QUE and universality for some band matrices ?

Theorem (with Erdős, Yau, Yin, 2015)

Yes when $N/W = O(1)$.

(i) Eigenvectors of Wigner matrices (and non mean-field models) are flat and entries are Gaussian.

(ii) Quantum unique ergodicity is more robust to access than eigenvalues universality : no need for comparison.

(iii) The parabolic PDE for dynamics of moments allows to understand various other problems, like eigenvectors perturbations in a non-perturbative regime.

(iv) QUE is a (the ?) fundamental reason for eigenvalues universality in non-mean fields models.

(v) Basic open problems : (thiner) band matrices, Anderson conjecture in the delocalisation regime, quantum chaos conjecture.