Extremal Processes of Gaussian Processes Indexed by Trees

Anton Bovier

with

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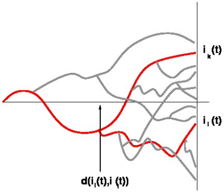

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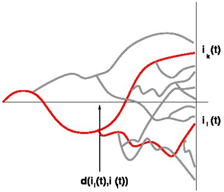
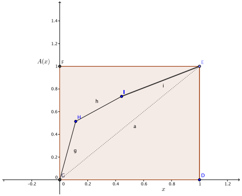
- Canonical tree-distance: $d(\mathbf{i}_\ell(t), \mathbf{i}_k(t)) \equiv$ time of most recent common ancestor of $\mathbf{i}_\ell(t)$ and $\mathbf{i}_k(t)$
- For fixed time horizon t, define Gaussian process, $(x_k^t(s), k \leq n(t), s \leq t)$, with covariance

$$\mathbb{E} x_k^t(r) x_\ell^t(s) = tA(t^{-1} d(\mathbf{i}_k(r), \mathbf{i}_\ell(s)))$$

for $A : [0,1] \to [0,1]$, increasing.





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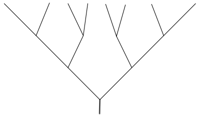

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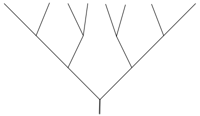

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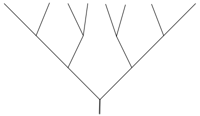

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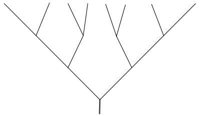

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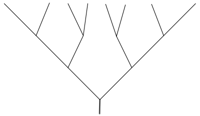

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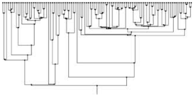

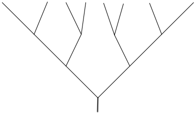

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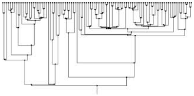

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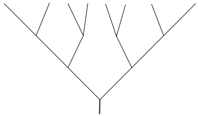

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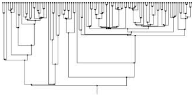

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- General A: variable speed BBM [Derrida-Spohn '88, Fang-Zeitouni '12]

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- Is there a limiting extremal process, $\mathcal{P}$, such that

$$\sum_{k \leq n(t)} \delta_{u_t^{-1}(x_k(t))} \to \mathcal{P}?$$

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With $u_t(x) = t\sqrt{2r} - \frac{\ln(rt)}{2\sqrt{2r}} + \frac{x}{\sqrt{r}} + \frac{\ln(n(t)/\mathbb{E}n(t))}{\sqrt{2r}}$, where $n(t)/\mathbb{E}n(t) \to RV$, a.s.

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- $$\sum_{k\leq n(t)} \delta_{u_t^{-1}(x_k(t))} \to \text{PPP}\left(\frac{1}{4\pi} e^{-\sqrt{2}x} dx\right)$$

where PPP(μ) denotes the Poisson Point Process with intensity μ.

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Universality 1: the order of the maximum

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Note in particular that as long as $A(s) \leq s$, for all $s \leq 1$, then $\bar{A}(s) = s$, and the order of the maximum is the same as in the REM.

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Note the special role of the linear function $A(s) = s$

Branching Brownian motion

(BBM) is a classical object in probability, combining the standard models of random motion and random genealogies into one: Each particle of the Galton-Watson process performs Brownian motion independently of any other. This produces an immersion of the Galton-Watson process in space.

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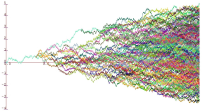

Picture by **Matt Roberts**, Bath

BBM is the canonical model of a spatial branching process.

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One of the simplest reaction-diffusion equations is the Fisher-Kolmogorov-Petrovsky-Piscounov (F-KPP) equation:

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Fischer used this equation to model the evolution of biological populations. It accounts for:

- birth: v,
- death: $-v^2$,
- diffusive migration: $\partial_x^2 v$.

F-KPP equation and BBM

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Lemma (McKeane '75, Ikeda, Nagasawa, Watanabe '69)

Let $f : \mathbb{R} \to [0,1]$ *and* $\{x_k(t) : k \leq n(t)\}$ *BBM.*

$$u(t,x) = \mathbb{E}\left[\prod_{k=1}^{n(t)} f(x - x_k(t))\right]$$

Then $v \equiv 1 - u$ *is the solution of the F-KPP equation with initial condition* $v(0,x) = 1 - f(x)$.

Travelling waves

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Theorem (KPP '37,......, Bramson '78)

The equation

$$\frac{1}{2}\omega'' + \sqrt{2}\omega' - \omega^2 + \omega = 0.$$

has a unique solution satisfying $0 < \omega(x) < 1$, $\omega(x) \to 0$, *as* $x \to +\infty$, *and* $\omega(x) \to 1$, *as* $x \to -\infty$, *up to translation.*

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For suitable initial conditions,

$$u(t, x + m(t)) \to \omega(x),$$

where $m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \ln t$, *where* ω *is one of the stationary solutions.*

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  and

- the Laplace functional $u(t,x) = \mathbb{E}\exp(-\sum_{k\leq n(t)} \phi(x_k(t)))$
  Allows to characterise the extremal process...

The derivative martingale

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$Z \overset{(d)}{=} \lim_{t\to\infty} Z(t)$, where $Z(t)$ is the derivative martingale,

$$Z(t) = \sum_{k\leq n(t)} \{\sqrt{2}t - x_k(t)\} e^{-\sqrt{2}\{\sqrt{2}t - x_k(t)\}}$$

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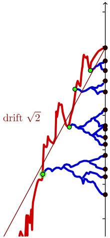
Cluster process:

$$\Delta(t) \equiv \sum_k \delta_{x_k(t)-\max_{j\leq n(t)} x_j(t)}.$$



conditioned on the event $\left\{\max_{j\leq n(t)} x_j(t) > \sqrt{2}t\right\}$ converges in law to point process, Δ.

[Chauvin, Rouault '90]

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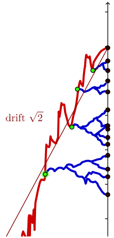

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[Chauvin, Rouault '90]

$$\mathcal{E} \equiv \sum_{i,j\in\mathbb{N}} \delta_{p_i+\Delta_j^{(i)}}, \qquad \Delta^{(i)} \text{ iid copies of } \Delta$$

The extremal process

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Theorem (Arguin-B-Kistler '11, Aidékon, Brunet, Berestycki, Shi '11)

The point process $\mathcal{E}_t \equiv \sum_{i=1}^{n(t)} \delta_{x_i(t)-m(t)} \to \mathcal{E}$.

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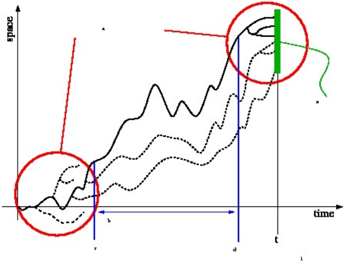

Interpretation:

p_i: positions of maxima of clusters with recent common ancestors.

$\Delta^{(i)}$: positions of members of clusters seen from their maximal one

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$$\mathbb{E}\left[\exp\left(-\int \phi(y)\mathcal{E}_t(dy)\right)\right] \to \mathbb{E}\left[\exp\left(-C(\phi)Z\right)\right]$$

for any $\phi \in \mathcal{C}_c(\mathbb{R})$ non-negative, where

$$C(\phi) = \lim_{t\to\infty}\sqrt{\frac{2}{\pi}}\int_0^\infty \left(1 - u(t, y + \sqrt{2}t)\right) y e^{\sqrt{2}y} dy$$

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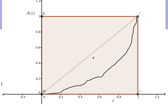
Then show that the limit is the Laplace functional of the process $\mathcal{E}$ described above.



Variable speed BBM.....below the straight line...

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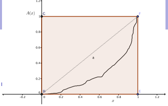

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Assume that $A(x) < x, \forall x \in (0,1)$, $A'(0) = a^2 < 1$, $A'(1) = b^2 > 1$.

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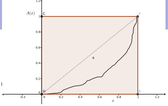

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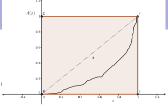

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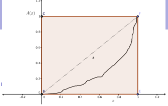

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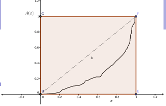

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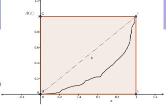

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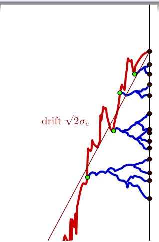
- Δ: are as in BBM but with the conditioning on the event $\{\max_k x_k(t) \geq \sqrt{2}bt\}$.



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Elements of the proof:

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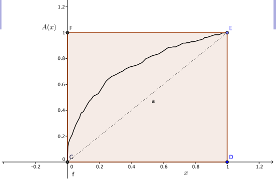
1) Explicit construction for the case of two speeds:
2) Gaussian comparison for general A.

Above the straight line



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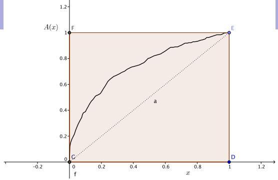

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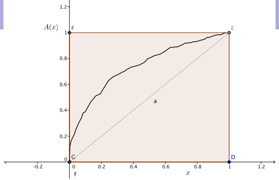

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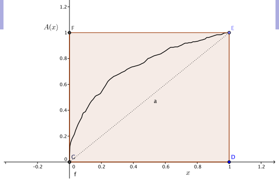

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- If A is piecewise linear, it is quite easy to get the full picture: Cascade of BBM processes.

- If A is strictly concave, Fang and Zeitouni '12 and Maillard and Zeitouni '13 have shown that the correct rescaling is

$$m(t) = C_\sigma t - D_\sigma t^{1/3} - \sigma^2(1) \ln t + f_t$$

(with explicit constants C_σ and D_σ), and $|f_t|$ bounded and

$$\mathbb{P}\left[M_T - m(t) \leq x\right) \to \phi(x/\sigma(0)),$$

and ϕ a traveling wave solution to the F-KPP equation.

Adding an extra dimension...

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Chose an embedding $\gamma : \{1, \dots, n(t)\} \to \mathbb{R}_+$, such that

$$|(\gamma(i_k(t)) - \gamma(i_j(t))| \sim e^{-d(i_k(t), i_j(t))}$$

Define for $u \in \mathbb{R}_+$, $r < t$,

$$Z(r, t, u) \equiv \sum_{k: \gamma(i_k(r)) \leq u} \{\sqrt{2}t - x_k(t)\} e^{-\sqrt{2}\{\sqrt{2}t - x_k(t)\}}.$$

Then

$$\lim_{r \uparrow \infty} \lim_{t \uparrow \infty} Z(r, t, u) \to Z(u)$$

Full convergence

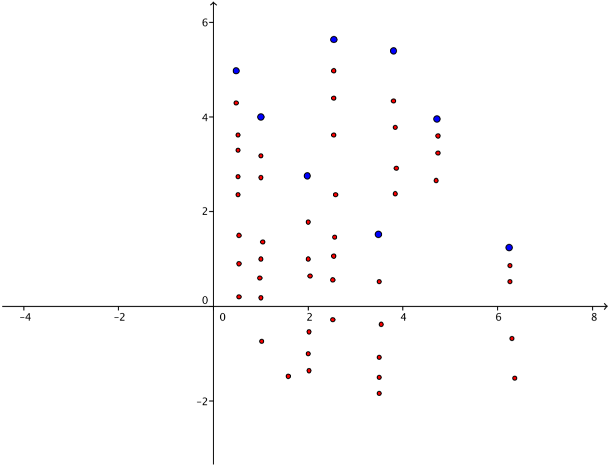
Theorem (B, Hartung '14)

The point process $\mathcal{E}_t \equiv \sum_{k=1}^{n(t)} \delta_{(\gamma(i_k(t)), x_i(t)-m(t))} \to \widetilde{\mathcal{E}}$ *on* $\mathbb{R}_+ \times \mathbb{R}$, *where*

$$\widetilde{\mathcal{E}} \equiv \sum_{i,j} \delta_{(q_i,p_i)+(0,\Delta_j^{(i)})},$$

with (q_i, p_i) *atoms of a Cox process on* $\mathbb{R}_+ \times \mathbb{R}$ *with intensity measure* $Z(du) \times Ce^{-\sqrt{2}x}dx$, *and* $\Delta_j^{(i)}$ *as before.*

Adding another dimension



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- Statistics of zeros of Riemann zeta-function [Fyodorov, Keating '12]

Thank you for your attention!