Averaging, homogenization, and large deviation methods for the study of randomly perturbed dynamical systems and PDEs

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1. Quasilinear parabolic PDEs with small diffusion term (joint work with M. Freidlin and L. Tcheuko).

2. Averaging of randomly perturbed dynamical systems with ergodic components (joint work with D. Dolgopyat).

3. Averaging of deterministically perturbed dynamical systems with ergodic components (joint work with D. Dolgopyat and M. Freidlin).

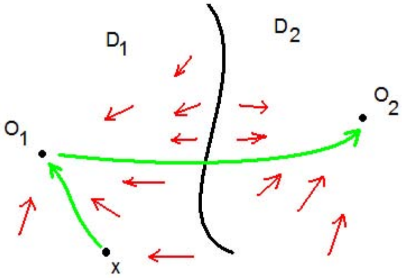
4. Transition from homogenization to averaging in cellular flows (joint work with M. Hairer, Z. Pajor-Gyulai).

Part 1. Quasilinear PDEs - long time behavior

First, let's look at the well-known linear case.

In terms of SDEs: $\dot{X}_t^x = b(X_t^x), \ \ X_0^x = x \in \mathbb{R}^d$;

$$dX_t^{x,\varepsilon} = b(X_t^{x,\varepsilon})dt + \varepsilon\sigma(X_t^{x,\varepsilon})dW_t, \ \ X_0^{x,\varepsilon} = x.$$



In terms of PDEs:

$$\frac{\partial u^\varepsilon}{\partial t} = \frac{\varepsilon^2}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j} + b(x)\cdot\nabla_x u^\varepsilon,$$

$$u^\varepsilon(0,x) = g(x), \ x \in \mathbb{R}^d.$$

Relationship: $u^\varepsilon(t,x) = \mathrm{E}g(X_t^{x,\varepsilon})$.

Action functional and Quasi-potential

Action functional:

$$S_{0,T}(\varphi) = \frac{1}{2}\int_0^T \sum_{i,j=1}^{d} a^{ij}(\varphi_t)(\dot{\varphi}_t^i - b_i(\varphi_t))(\dot{\varphi}_t^j - b_j(\varphi_t))dt, \quad T \geq 0,$$

if $\varphi \in C([0,T], \mathbb{R}^d)$ is absolutely continuous,

$$S_{0,T}(\varphi) = +\infty, \quad \text{otherwise}.$$

$a^{ij} = (a^{-1})_{ij} = ((\sigma\sigma^*)^{-1})_{ij}$.

Quasi-potential:

$$V_{mn} = V(O_m, O_n) =$$

$$\inf_T \{S_{0,T}(\varphi) : \varphi(0) = O_m, \varphi(T) = O_n\}.$$

τ^{ε}_{mn} - the time it takes the process to go from O_m to a small neighborhood of O_n.

$$\tau^{\varepsilon}_{mn} \sim \exp(V_{mn}/\varepsilon^2),$$

Consider the process (and solution of PDE) at times $t(\varepsilon)$ with $\ln(t(\varepsilon)) \sim \lambda/\varepsilon^2$. Suppose, for example, that $x \in D_1$ and $V_{12} < V_{21}$.

If $\lambda < V_{12}$, then $u^{\varepsilon}(t(\varepsilon), x) \to g(O_1)$.
If $\lambda > V_{12}$, then $u^{\varepsilon}(t(\varepsilon), x) \to g(O_2)$.

Quasi-linear problem:

$$\frac{\partial u^\varepsilon}{\partial t} = \frac{\varepsilon^2}{2} \sum_{i,j=1}^{d} a_{ij}(x, u^\varepsilon) \frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j} + b(x) \cdot \nabla_x u^\varepsilon,$$

$$u^\varepsilon(0, x) = g.$$

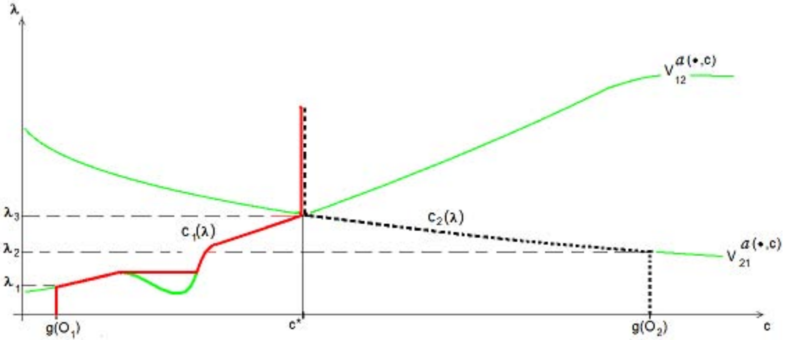
equivalent to the system

$$dX_s^{t,x,\varepsilon} = b(X_s^{t,x,\varepsilon})dt + \varepsilon\sigma(X_s^{t,x,\varepsilon}, u^\varepsilon(t-s, X_s^{t,x,\varepsilon}))dW_s,$$

$$u^\varepsilon(t,x) = \mathrm{E} g(X_t^{t,x,\varepsilon}).$$

Construct $\widetilde{V}_{12}$ using $a_{ij}(x, g(O_1))$.
If $\lambda < \widetilde{V}_{12}$, then still $u^\varepsilon(t(\varepsilon), x) \to g(O_1)$.
If $\lambda > \widetilde{V}_{12}$, then new effects appear for u^ε and the processes.

Result in the non-linear case (2 equilibriums, for simplicity):

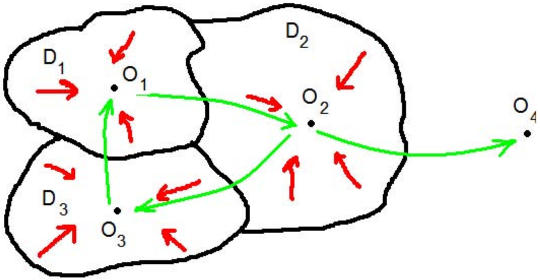


Theorem: (Freidlin-Koralov)

$$\lim_{\varepsilon \downarrow 0} u^{\varepsilon}(\exp(\lambda/\varepsilon^2), x) = c_n(\lambda), \quad x \in D_n.$$

Multiple equilibriums

(joint work with L. Tcheuko)



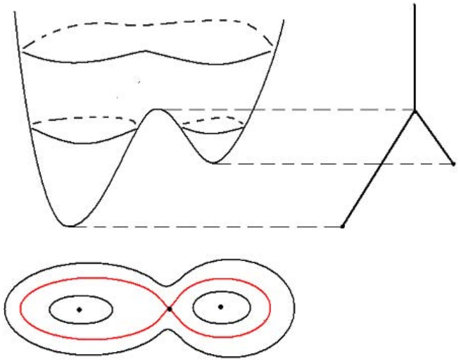
Result: As above, there are limits of u^{ε}, $x \in D_n$, in each exponential time scale. Again, they are determined by $v_{mn}(c) = V_{mn}^{a(x,c)} = V^{a(x,c)}(O_m, O_n)$.
The main difficulty - the hierarchy of cycles may evolve in time (i.e., depends on the time scale).

Part 2. Averaging of locally Hamiltonian flows.

Incompressible flow:

$$\dot{x}(t) = v(x(t)), \quad x(0) = x_0 \in \mathbb{R}^2 \text{ or } x_0 \in M.$$

(a) Hamiltonian flows.



Perturbation:

$$dX_t^\varepsilon = \frac{1}{\varepsilon} v(X_t^\varepsilon)dt + \sigma(X_t^{\varkappa,\varepsilon})dW_t \quad \text{(random)},$$

$$dX_t^\varepsilon = \frac{1}{\varepsilon} v(X_t^\varepsilon)dt + b(X_t^{\varkappa,\varepsilon})dt \quad \text{(deterministic)}.$$

The dynamics consists of the fast motion (with speed of order $1/\varepsilon$) along the unperturbed trajectories together with the slow motion (with speed of order 1) in the direction transversal to the unperturbed trajectories.

Averaging - consider $h : \mathbb{R}^2 \to \mathbb{G}$. Then

$$h(X_t^\varepsilon) \to Y_t \quad \text{as} \quad \varepsilon \downarrow 0.$$

Locally (away from the vertices of the graph):

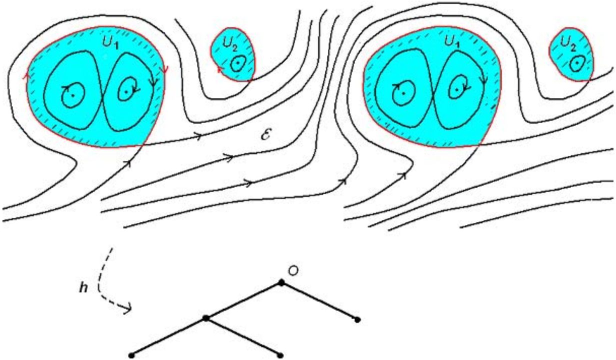
$$\frac{dY_t}{dt} = \frac{\widetilde{b}(Y_t)}{T(Y_t)}, \quad \text{(deterministic)}, \quad \text{where}$$

$$T(h) = \int_{\gamma(h)} \frac{dl}{|\nabla H|}, \quad \widetilde{b}(h) = \int_{\gamma(h)} \frac{\langle b, \nabla H \rangle}{|\nabla H|} dl \quad ,$$

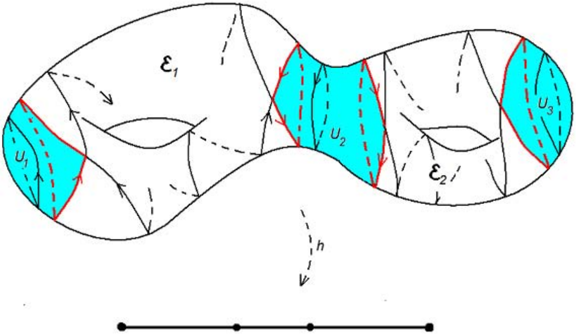
$$dY_t = \overline{\sigma}(Y_t) dW_t + \overline{b}(Y_t) dt \quad \text{(random perturbations)}.$$

Behavior at the vertices. Random perturbations - Freidlin and Wentzell. Deterministic perturbations - regularization required. (Brin and Freidlin).

(b) Locally Hamiltonian flows (there are regions where the unperturbed dynamimcs is ergodic). Example:
$H = H_0(x_1, x_2) + \alpha x_1 + \beta x_2$, α/β - irrational.



M - manifold with an area form,
v - incompressible vector field,
X_t^ε - process with generator $L^\varepsilon = \frac{1}{\varepsilon} L_v + L_D$.



Unperturbed dynamics:

$U_1, ..., U_m$ - periodic sets
$\mathcal{E}_1, ..., \mathcal{E}_n$ - 'ergodic components'
Flow on $\mathcal{E}_i$ is isomorphic to a special flow over an interval exchange transformation.
Graph:
- Each edge corresponds to one of U_k
- Three types of vertices:
(a) Those corresponding to $\mathcal{E}_i$,
(b) Those corresponding to saddle points,
(c) Those corresponding to equilibriums (but not saddles).
The flow is Hamiltonian on U_k with a Hamiltonian H. Denote:
h_k - coordinate on I_k.

Theorem 1 (Dolgopyat, Koralov) *The measure on on* $C([0,\infty),\mathbb{G})$ *induced by the process* $Y_t^\varepsilon = h(X_t^\varepsilon)$ *converges weakly to the measure induced by the process with the generator* $\mathcal{L}$ *with the initial distribution* $h(X_0^\varepsilon)$.

The limiting process is described via its generator $\mathcal{L}$, which is defined as follows.

Let $L_k f(h_k) = a_k(h_k) f'' + b_k(h_k) f'$

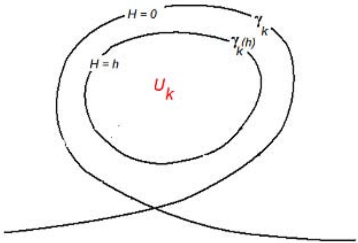
be the differential operator on the interior of the edge I_k (coefficients are defined below).

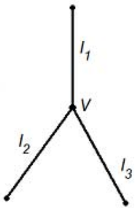
For $f \in D(\mathcal{L})$, we define $\mathcal{L}f = L_k f$ in the interior of each edge, and as the limit of $L_k f$ at the endpoints of I_k.

$D(\mathcal{L})$ consists of $f \in C(\mathbb{G}) \cap C^2(I_k)$ such that
(a) $\lim_{h_k \to 0} L_k f(h_k)$ exist and are the same for all edges entering the same vertex V.
(b) At vertices corresponding to $\mathcal{E}_i$:

$$\sum_{k=1}^{n} p_k^V \lim_{h_k \to 0} f'(h_k) = \lim_{h_k \to 0} L_k f(h_k).$$

(the same with 0 in the right hand side for vertices corresponding to saddles).





Coefficients:

In local coordinates in U_k ($\omega = dxdy$):

$$dX_t^\varepsilon = \frac{1}{\varepsilon} v(X_t^\varepsilon)dt + u(X_t^\varepsilon)dt + \sigma(X_t^\varepsilon)dW_t.$$

Then,

$$a_k(h_k) = \frac{1}{2} T^{-1}(h_k) \int_{\gamma_k(h_k)} \frac{\langle \alpha \nabla H, \nabla H \rangle}{|\nabla H|} dl \quad \text{and}$$

$$b_k(h_k) = \frac{1}{2} T^{-1}(h_k) \int_{\gamma_k(h_k)} \frac{2\langle u, \nabla H \rangle + \alpha \cdot H''}{|\nabla H|} dl,$$

where $\alpha = \sigma\sigma^*$.

$$p_k^V = \pm \frac{1}{2} \int_{\gamma_k} \frac{\langle \alpha \nabla H, \nabla H \rangle}{|\nabla H|} dl.$$

Ingredients of the proof.

(1) Assume (temporarily) that the area measure λ is invariant for the process for each ε.
For the limit Y_t of $Y_t^\varepsilon = h(X_t^\varepsilon)$, we should have

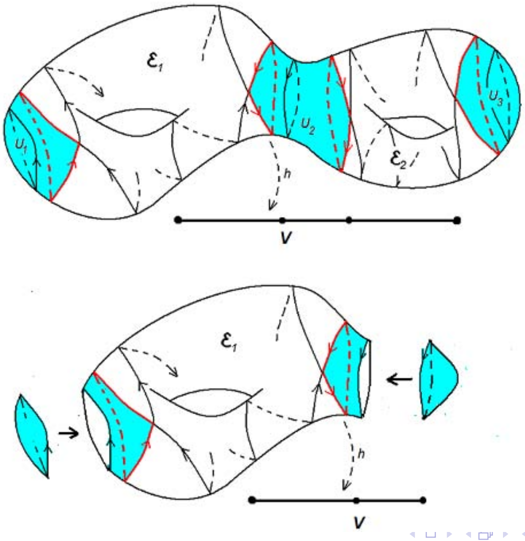
$$\mathbb{E}[f(Y_T) - f(Y_0) - \int_0^T \mathcal{L}f(Y_s)ds] = 0.$$

Need to prove the following lemma.
Lemma1 *For each function $f \in D(\mathcal{L})$ and each $T > 0$ we have*

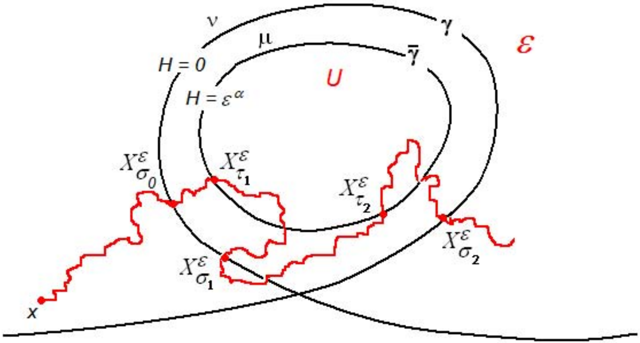
$$\mathbb{E}_x[f(h(X_T^\varepsilon)) - f(h(X_0^\varepsilon)) - \int_0^T \mathcal{L}f(h(X_s^\varepsilon))ds] \to 0$$

uniformly in $x \in \mathbb{T}^2$ as $\varepsilon \to 0$.

(2) Localization (can deal with a star-shaped graph with one accessible vertex)



(3) Need: $\mathbb{E}_x[f(h(X_T^\varepsilon)) - f(h(X_0^\varepsilon)) - \int_0^T \mathcal{L}f(h(X_s^\varepsilon))ds] \to 0$.



Split $[0, T]$ into intervals: $[0, \sigma_0]$, $[\sigma_0, \tau_1]$, $[\tau_1, \sigma_1]$, $[\sigma_1, \tau_2]$, ...

On intervals $[\tau_n, \sigma_n]$ (inside periodic component) - averaging (Freidlin-Wentzell) with small modifications.

On intervals $[\sigma_n, \tau_{n+1}]$ (getting from the ergodic component into the periodic component):

$$\mathbb{E}_x[f(h(X^\varepsilon_{\tau_{n+1}})) - f(h(X^\varepsilon_{\sigma_n})) - \int_{\sigma_n}^{\tau_{n+1}} \mathcal{L}f(h(X^\varepsilon_s))ds] \approx$$

$$\mathbb{E}_\nu[f(h(X^\varepsilon_\tau)) - f(h(X^\varepsilon_0)) - \int_0^\tau \mathcal{L}f(h(X^\varepsilon_s))ds] \approx$$

$$f'(0)\varepsilon^\alpha - \mathbb{E}_\nu\tau \cdot \mathcal{L}f(0).$$

- How can we calculate $\mathbb{E}_\nu\tau$?
- Why can we assume that we start with the invariant measure ν?

If λ is invariant: $\frac{\mathbb{E}_\nu \tau}{\lambda(\mathcal{E})} \approx \frac{\mathbb{E}_\mu \sigma}{\lambda(U)}$, so

$$\mathbb{E}_\nu \tau \approx \frac{\lambda(\mathcal{E})}{\lambda(U)} \cdot \mathbb{E}_\mu \sigma \approx \mathrm{const} \cdot \varepsilon^\alpha.$$

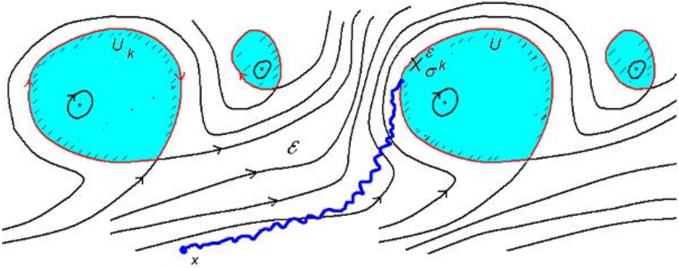
If λ is not invariant: consider

$$d\widetilde{X}_t^\varepsilon = \frac{1}{\varepsilon} v(\widetilde{X}_t^\varepsilon)dt + \widetilde{u}(\widetilde{X}_t^\varepsilon)dt + \sigma(\widetilde{X}_t^\varepsilon)dW_t,$$

(replace u by some $\widetilde{u}$ so that λ is invariant for the new process). By the Girsanov Theorem:

$$\widetilde{\nu} \approx \nu, \quad \mathbb{E}_{\widetilde{\nu}} \widetilde{\tau} \approx \mathbb{E}_\nu \tau.$$

So, $\mathbb{E}_\nu \tau \approx \frac{\lambda(\mathcal{E})}{\lambda(U)} \cdot \mathbb{E}_{\widetilde{\mu}} \widetilde{\sigma}$ (the gluing conditions are the same as for the measure-preserving process).



(4) Why does $\mathbb{E}_x \sigma^k \to 0$ as $\varepsilon \downarrow 0$? (time to reach U^k)
Let $u^\varepsilon(t, y)$, $y \in M \setminus U_k$, be the probability that the process starting at y does not reach U_k before time t.

$$\frac{\partial u^\varepsilon(t, y)}{\partial t} = \left(L_D + \frac{1}{\varepsilon} L_v \right) u^\varepsilon$$

$$u^\varepsilon(0, y) = 1, \; y \in M \setminus U_k, \quad u^\varepsilon(t, y) = 0, \;\; t > 0.$$

(a) **Lemma** (Zlatos): All $H_0^1(M \setminus U_k)$-eigenvalues for $v\nabla$ are zero on $\mathcal{E}$ implies that the $L^2(\mathcal{E})$-norm (and so $L^1(\mathcal{E})$-norm) of $u^\varepsilon(t,\cdot)$ tends to zero as $\varepsilon \downarrow 0$ for each $t > 0$.

(b) A uniform bound on fundamental solution doesn't get affected by adding an incompressible drift term.

(a) and (b) imply that $\mathbb{E}_x\sigma \to 0$. With some effort possible to show that $\mathbb{E}_x\sigma^k \to 0$.

Recall

$$dX_t^{\varkappa,\varepsilon} = \frac{1}{\varepsilon} v(X_t^{\varkappa,\varepsilon})dt + b(X_t^{\varkappa,\varepsilon})dt +$$

$$\varkappa u(X_t^{\varkappa,\varepsilon})dt + \sqrt{\varkappa}\sigma(X_t^{\varkappa,\varepsilon})dW_t.$$

Let $Y_t^{\varkappa,\varepsilon} = h(X_t^{\varkappa,\varepsilon})$ be the corresponding process on the graph $\mathbb{G}$. We demonstrated that the distribution of $Y_t^{\varkappa,\varepsilon}$ converges, as $\varepsilon \downarrow 0$, to the distribution of a limiting process, which will be denoted by $Z_t^{\varkappa}$. $Z_t^{\varkappa}$, in turn, converges to the distribution of a limiting Markov process on $\mathbb{G}$ when $\varkappa \downarrow 0$.

The limiting process Z_t can be described as follows. It is a Markov process with continuous trajectories which moves deterministically along an edge I_k of the graph with the speed

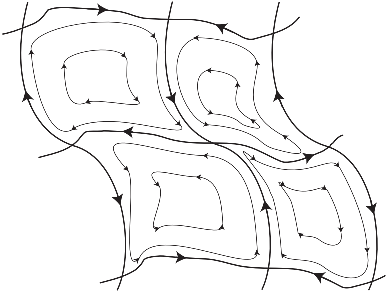
$$\overline{b}_k(h_k) = \frac{1}{2}(T_k(h_k))^{-1} \int_{\gamma_k(h_k)} \frac{2\langle b, \nabla H\rangle}{|\nabla H|} dl.$$

If the process reaches V corresponding to an ergodic component, then it either remains at V forever or spends exponential time in V and then continues with deterministic motion away from V along a randomly selected edge (with probabilities which can be specified). The same if V corresponds to a saddle point, but no exponential delay.

Theorem 2 (Dolgopyat, Freidlin, Koralov) *The measure on on* $C([0,\infty),\mathbb{G})$ *induced by the process* $Z_t^{\varkappa}$ *converges weakly to the measure induced by the process* Z_t *with the initial distribution* $h(X_0^{\varepsilon})$*.*

The process Z_t is defined by the deterministic system. The stochastic perturbations are used just for regularization purposes.

Part 4. From homogenization to averaging.



Cellular flow: Let v be an incompressible periodic vector field.

$$v(x) = \nabla^{\perp} H = (-H'_{x_2}, H'_{x_1})$$

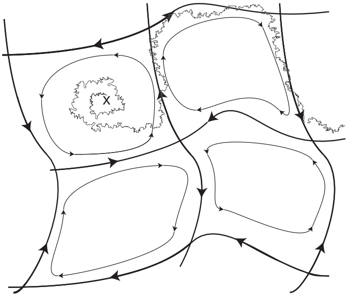
- Critical points are non-deg.
- $\{H(x) = 0\}$ forms a lattice
- H is periodic

Note that $\int_{\mathbb{T}^2} v(x)dx = 0$

Homogenization vs Averaging (in probabilistic terms)

Consider the process

$$dX_t^{x,\varepsilon} = \frac{1}{\varepsilon} v(X_t^{x,\varepsilon})dt + dW_t, \qquad X_0^{x,\varepsilon} = x \in \mathbb{R}^2$$



- Homogenization: Behavior of $X_t^{x,\varepsilon}$ as $t \uparrow \infty$ (ε - fixed).
- Averaging: Behavior of $H(X_t^{x,\varepsilon})$ as $\varepsilon \downarrow 0$ (t - fixed).

Homogenization

Simplest homogenization result: $X^{x,\varepsilon}_{ct}/\sqrt{t} \Rightarrow W^{A(\varepsilon)}_c$, as $t \to \infty$, $W^{A(\varepsilon)}_c$-Brownian motion with diffusion matrix $A(\varepsilon)$.

Why: Define u^ε_i, $i = 1, 2$, as solutions of

$$(\frac{1}{\varepsilon} v\nabla + \frac{1}{2}\Delta)(u^\varepsilon_i + x_i) = 0, \quad u^\varepsilon_i - \text{periodic}.$$

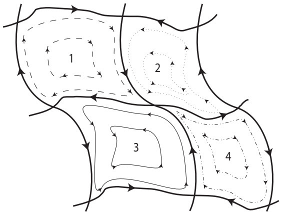
Apply Ito's formula to $(u^\varepsilon_i + x_i)(X^{x,\varepsilon}_t)$, get the result.

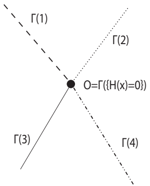
The matrix $A(\varepsilon)$ is expressed in terms of u^ε_i.

Averaging - 'Speeding up time'

- Process moves fast along the level curves.
- Possible to keep track of the motion across the curves, rather than the location on the curve.

Let $\Gamma : \mathbb{R}^2 \to G$ be a projection onto a graph: $\Gamma(x) = (H(x), i)$





Process $\Gamma(X_t^{x,\varepsilon})$ on G

$\downarrow$

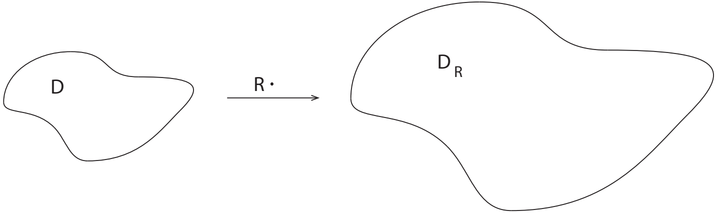
Averaged process
Diffusion on edges +
Gluing conditions at O.

Theorem (Khasminskii, Freidlin-Wentzell)

$$(\Gamma(X_t^{x,\varepsilon}))_{t\geq 0} \overset{\varepsilon\to 0}{\Rightarrow} (Y_t^{\Gamma(x)})_{t\geq 0} \qquad \text{in } C([0,\infty], G)$$

Homogenization vs Averaging (in PDE terms)

Let $D \subseteq \mathbb{R}^2$ be a domain and let $D_R = \{Rx | x \in D\}$.

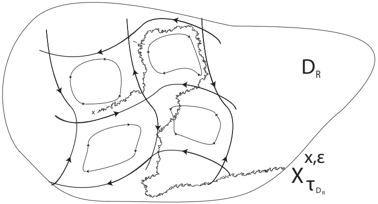


Dirichlet problem:

$$\frac{1}{2}\Delta u^{\varepsilon,R}(x) + \frac{1}{\varepsilon} v(x) \nabla u^{\varepsilon,R}(x) = -f\left(\frac{x}{R}\right) \text{ in } D_R, \qquad u^{\varepsilon,R}|_{\partial D_R} = 0$$

Two parameters: R- size of domain, $1/\varepsilon$ - speed of flow.

Using stochastic representation



$$\text{Homogenization}: \lim_{R\to\infty} R^{-2} u^{\varepsilon,R}(x)$$

$$= \lim_{R\to\infty} R^{-2}\mathrm{E}\int_0^{\tau_{\partial D_R}} f\left(\frac{X_t^{x,\varepsilon}}{R}\right) dt$$

$$= \mathrm{E}\int_0^{\tau_{\partial D}} f\left(W_t^{A(\varepsilon)}\right) dt = \bar{u}^{\varepsilon}(0),$$

where
$$\sum_{i,j=1}^{2} A_{ij}(\varepsilon)\partial_{x_i x_j}\bar{u}^{\varepsilon} = f, \quad \bar{u}^{\varepsilon}|_{\partial D} = 0.$$

Averaging : $\lim_{\varepsilon\downarrow 0} u^{\varepsilon,R}(x)$ is given by an ODE on one edge of the graph.

Two parameter asymptotics ($R \uparrow \infty$, $\varepsilon \downarrow 0$)

Averaging and homogenization are still useful notions.

Iyer, Komorowski, Novikov, Ryzhik (2012)

$$R^{4-\alpha} \geq \frac{1}{\varepsilon}$$

$\downarrow$

Homogenization regime

$$R^4 \log^2 R \leq c \frac{1}{\varepsilon \log^2(\varepsilon)}$$

$\downarrow$

Averaging regime

- PDE methods: Multi-scale expansion and construction of appropriate sub and supersolutions to estimate the principal Dirichlet eigenvalue.
- Intermediate regime:
  - $R \approx \varepsilon^{-1/4}$
  - Only numerical results (until now) (Iyer, Zygalakis 2012)

Two parameter asymptotics ($R \uparrow \infty$, $\varepsilon \downarrow 0$)

- Homogenization regime ($R \gg \varepsilon^{-1/4}$):

  Motion experiences many cells, which results in an effective behavior

- Averaging regime ($R \ll \varepsilon^{-1/4}$):

  Effectively only the initial cell is visited, quickly reaching the boundary of D after hitting the separatrix.

Our result: Derive a limit theorem for

$$dX_t^{x,\varepsilon} = \frac{1}{\varepsilon} v(X_t^{x,\varepsilon})dt + dW_t, \qquad X_0^{x,\varepsilon} = x, \quad \varepsilon \downarrow 0.$$

- Implies PDE results in both the averaging and the homogenization regimes.
- Transition regime is described (turns out to be $R \sim \varepsilon^{-1/4}$).
- No symmetries are assumed.

Towards a limit theorem for $X_t^{x,\varepsilon}$

Averaging: behavior of $\Gamma(X_t^{x,\varepsilon})$ as $\varepsilon \downarrow 0$.
We want: asymptotics of $X_t^{x,\varepsilon}$ itself as $\varepsilon \downarrow 0$.

Consider the displacement $X_t^{x,\varepsilon} - x$:

- Main contribution: Close to the separatrix.
- Time within cell: Wasted.

When ε is small:

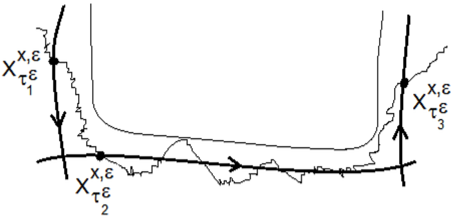
- Nearly all time is spent in the cell.
- Short time, but many changes around the separatrix
  $\Rightarrow$ Need to measure how much time is spent around a separatrix

A step back - asymptotics of $A(\varepsilon)$

Recall that $X_{ct}^{x,\varepsilon} \sim \sqrt{t} W_c^{A(\varepsilon)}$, as $t \to \infty$.
Theorem: (Fannjiang, Papanicolau 1994, Koralov 2004)

$$A(\varepsilon) = (A(0) + o(1))\varepsilon^{-1/2} \text{ as } \varepsilon \downarrow 0.$$

(In PDE terms this corresponds to $R \uparrow \infty$ first, then $\varepsilon \downarrow 0$.)



$X^{x,\varepsilon}_{\tau_0^\varepsilon}, X^{x,\varepsilon}_{\tau_1^\varepsilon}, X^{x,\varepsilon}_{\tau_2^\varepsilon}, \ldots$ converges to a Markov chain.

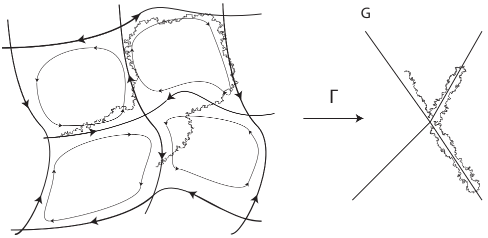
$\mathrm{E}(\tau_n^\varepsilon - \tau_{n-1}^\varepsilon) \sim \varepsilon^{\frac{1}{2}}$ (but due to rare events, and that's why we need $T \to \infty$ first).

Approx. $T\varepsilon^{-\frac{1}{2}}$ transitions, $\sqrt{T}\varepsilon^{-\frac{1}{4}}$ displacement in time T.

Measuring separatrix time

Key question: How much time is spent around the separatrix?

$$\text{formally}: \quad dY^y_t = b(Y^y_t)dt + a(Y^y_t)dW, \qquad Y^y_0 = y$$



$$\Gamma(X^{x,\varepsilon}_\cdot) \overset{\varepsilon\to 0}{\Rightarrow} Y^{\Gamma(x)}_\cdot$$

(Freidlin-Wentzell)

Local time $L^y_t(x)$ for the process on the graph:

$$\int_0^t f(Y^y_s)a^2(Y^y_s)ds = 2\int_{-\infty}^{\infty} f(z)L^y_t(z)dz \qquad \forall f \text{ Borel}$$

continuous, nondecreasing, constant on $\{Y_t \neq x\}$.

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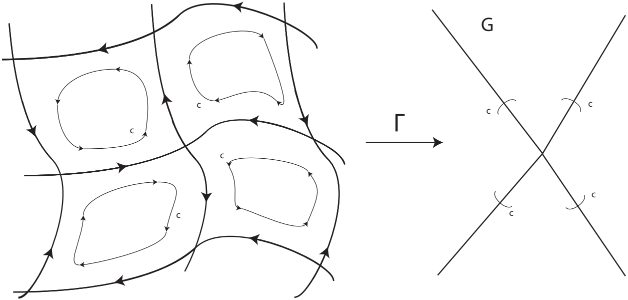

Variant of Levy's downcrossing lemma

on the graph : $cD_t^y(0,c) \overset{c \to 0}{\to} L_t^y(0)$ for each t.

on the plane $\to D_t^{x,\varepsilon}(0,c) \overset{D}{\approx} D_t^{\Gamma(x)}(0,c) \leftarrow$ on the graph

Main result

Let $\tilde{W}^Q$ be a two dimensional Brownian motion with covariance matrix Q.

Theorem (Hairer, Koralov, Pajor-Gyulai, 2014)

There is a non-degenerate Q depending on the geometry such that

$$\varepsilon^{1/4} X_{\cdot}^{x,\varepsilon} \Rightarrow \tilde{W}^Q_{L_{\cdot}^{\Gamma(x)}(0)} \quad \text{as } \varepsilon \downarrow 0.$$

In other words, the limit is a random independent time-change of the Brownian motion

Outline of the proof

- Obtain a limit theorem for the displacement during an upcrossing (small c - fixed, $\varepsilon \downarrow 0$).
- Argue that downcrossing times and spatial displacement during upcrossings are asymptotically independent.
- Use the averaging principle to control the number of downcrossings.
- Put these together to obtain the convergence of one dimensional distributions.
- Convergence of finite dimensional distributions follows by strong Markov property.
- Prove tightness.

Displacement and downcrossing time

We need to capture what happens between two downcrossings.

Let $\mu_0^\varepsilon = 0$, $\sigma_0^\varepsilon = \tau_0^\varepsilon$,

$$\mu_n^\varepsilon = \inf\{t \geq \sigma_{n-1}^\varepsilon : X_t^{x,\varepsilon} \in \partial V^c\} \qquad \sigma_n^\varepsilon = \inf\{t \geq \mu_n^\varepsilon : X_t^{x,\varepsilon} \in \mathcal{L}\}$$

and

$$S_n^{x,c,\varepsilon} = X_{\sigma_n^\varepsilon}^{x,\varepsilon} - X_{\sigma_{n-1}^\varepsilon}^{x,\varepsilon} \qquad T_n^{x,c,\varepsilon} = \sigma_n^\varepsilon - \mu_n^\varepsilon$$

Lemma

There is a non-degenerate matrx Q such that for small $c > 0$,

$$\varepsilon^{1/4} S_n^{x,c,\varepsilon} \stackrel{\mathcal{D}}{\to} \sqrt{c}(1 + a(c))\sqrt{\xi} N(0, Q) \qquad \varepsilon \downarrow 0$$

where $\xi \sim EXP(1)$ and $N(0, Q) \sim \mathcal{N}(0, Q)$ while $a(c) \stackrel{c \to 0}{\to} 0$.

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Lemma

For fixed n and c, the random vectors

$$(T_0^{x,\varepsilon}, \varepsilon^{1/4} S_1^{x,c,\varepsilon}, T_1^{x,c,\varepsilon}, \ldots T_{n-1}^{x,c,\varepsilon}, \varepsilon^{1/4} S_n^{x,c,\varepsilon})$$

converge, as $\varepsilon \downarrow 0$, to a random vector with independent components.

One dimensional distributions

Let $t > 0$, $f \in C_b(\mathbb{R}^2)$ and $\eta > 0$. We want

$$|\mathrm{E}f(\varepsilon^{1/4} X_t^{x,\varepsilon}) - \mathrm{E}f(\tilde{W}^Q_{L_t^{\Gamma(x)}})| < \eta \qquad \varepsilon \leq \varepsilon_0$$

Take $Z = \sqrt{\xi} N(0, Q)$ and $Z_1^c, Z_2^c, ... \sim \sqrt{c}(1 + a(c))Z$ indep.

Lemma (CLT)

Suppose that N_δ are $\mathbb{N}$-valued random variables independent of the family $\{Z_i^\delta\}$ that satisfy $\mathrm{E}N_\delta \leq C/\delta$ for some $C > 0$. Let $f \in C_b(\mathbb{R}^2)$ and let $\tilde{W}_t^Q$ be a Brownian motion with covariance Q, independent of $\{N_\delta\}$. Then

$$\mathrm{E}f(Z_1^\delta + ... + Z_{N_\delta}^\delta) - \mathrm{E}f(\tilde{W}^Q_{\delta N_\delta}) \to 0 \ \ as \ \delta \downarrow 0,$$

The rest is approximation.

PDE results - Averaging regime

Averaging regime: $\varepsilon^{1/4}R \to 0$

- Exiting initial cell $\to L_t^{\Gamma(x)}(0) \approx \delta > 0$ quickly

$$|X_t^\varepsilon| \approx |\tilde{W}^Q_{\delta/\sqrt{\varepsilon}}| \approx \delta^{1/2}\varepsilon^{-1/4} > R$$

- D_R has been reached!

$$\lim_{\varepsilon\downarrow 0, R\uparrow\infty} u^{\varepsilon,R}(x) = \mathrm{E} \lim_{\varepsilon\downarrow 0, R\uparrow\infty} \int_0^{\tau_{\mathcal{L}}(X_\cdot^{x,\varepsilon})} f\left(\frac{X_t^{x,\varepsilon}}{R}\right) dt = f(0)\mathrm{E}\tau_{\mathcal{L}}(X_\cdot^{x,\varepsilon})$$

PDE results - Homogenization regime

Homogenization regime: : $\varepsilon^{1/4} R \to \infty$

- Many cells are visited $\to$ ergodic behavior.
- $L_t^{\Gamma(x)} \approx ct$

$$\lim_{\varepsilon \downarrow 0, R \uparrow \infty} (\varepsilon^{1/2} R^2)^{-1} u^{\varepsilon,R}(x) = \mathrm{E} \int_0^{\tau_{\partial D}(\tilde{W}^{cQ}_{\cdot})} f(\tilde{W}^{cQ}_t) dt$$

This is the solution of a constant coefficient PDE at the origin.

PDE results - transition regime

Transition regime: $\varepsilon^{1/4} R \to C \in (0, \infty)$

- Several cells are visited, but not enough for an ergodic behavior to set in.
- Behavior of local time is not universal.

$$\lim_{\varepsilon \downarrow 0, R \uparrow \infty} u^{\varepsilon, R}(x) = \mathrm{E} \int_0^{\tau_{\partial D_C}(\tilde{W}^Q_{L^{\Gamma(x)}_\cdot})} f(\tilde{W}^Q_{L_t^{\Gamma(x)}}) dt$$

This is a mixture of the two regimes!