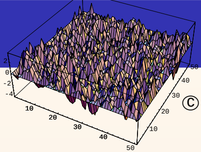
Complexity of Gaussian fields and second moments

Ofer Zeitouni

Weizmann & Courant

July 22, 2015

A Theorem



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Consider the (discrete) Gaussian Free Field on the (planar) box (of side N) with Dirichlet boundary conditions.
Let M_N denote its maximum.

Theorem (Bramson-Ding-Z. '13)

$M_N - EM_N$ converges in law: there exists a constant $c > 0$ and random variable $Z > 0$ so that

$$\lim_{N\to\infty} P(M_N \le EM_N + x) = E(e^{-cZe^{-\sqrt{2\pi}x}})$$

Shifted Gumbel

Also, information on location of maximum, extremal process.. (later)

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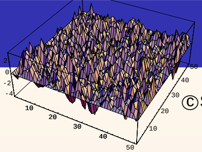

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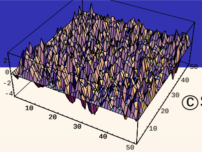

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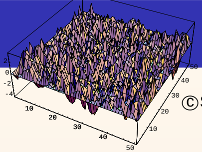

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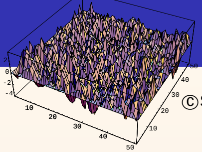

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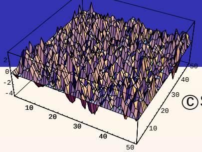

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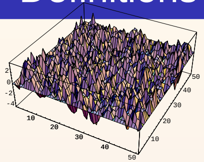

$V_N = ([0, N-1] \cap \mathbb{Z})^2$; $\{w_m\}_{m \geq 0}$ - simple random walk killed at $\tau = \min\{m : w_m \in \partial V_N\}$.

$$G_N(x,y) = E^x\left(\sum_{m=0}^{\tau} \mathbf{1}_{w_m = y}\right)$$

GFF is the zero-mean Gaussian field $\{\mathcal{X}_z^N\}_{z \in V_N}$, covariance G_N. Alternatively, density is

$$\frac{1}{Z_N} e^{-\sum_{x \sim y}(X_x - X_y)^2/2}.$$

GFF on tree = BRW.

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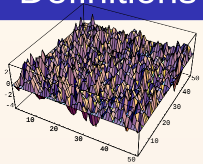

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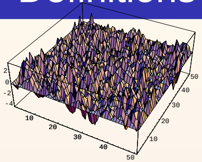

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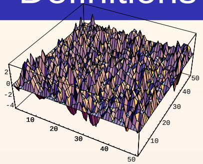

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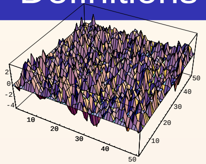

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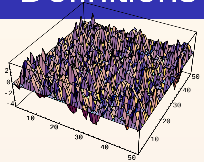

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Law of large numbers

$M_N = \max_{z \in V_N} \mathcal{X}_z^N$

Theorem (Bolthausen-Deuschel-Giacomin '01)

$$\frac{M_N}{\log N} \to 2\sqrt{2/\pi}\,.$$

Value of limit is the same as if correlations were ignored (i.e., maximum of N^2 Gaussian variables of variance $G_N(x,x) \sim (2/\pi)\log N$).

Upper bound by first moment computation:

$Z_N = \sum_{z \in V_N} \mathbf{1}_{\mathcal{X}_z^N > ((2+\epsilon)\sqrt{2/\pi})\log N}$, $EZ_N \to_{N\to\infty} 0$.

Lower bound Directly computing EZ_N^2 (with $\epsilon < 0$) does not work; instead, introduce a branching structure via averaging field on different scales - roughly embedding a BRW in GFF.

Truncated second moment method can also be used for the embedded BRW Daviaud '06 and for the analogous continuous GFF Hu-Miller-Peres '10

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Fluctuations - preliminaries

GFF can be defined on any graph; in particular, sequence of boxes in an infinite graph, e.g. $\mathbb{Z}^d$.

Theorem (Borell-Tsirelson '75)

For a centered Gaussian field $\{X_z\}_{z\in T}$ *and* $X^* = \max_{z\in T} X_z$,

$$P(|X^* - EX^*| > \delta) \le 2e^{-\delta^2/2\sigma_*^2}$$

where $\sigma_*^2 = \max_{z\in T} EX_z^2$.

Thus, if SRW on an infinite graph is transient, then fluctuations of maximum of GFF are of order 1.

On the other hand, GFF on $\mathbb{Z}$ is just discrete analogue of Brownian Bridge - and fluctuations of maximum are of order $\sqrt{N}$ (also, maximum does not converge in probability).

$d = 2$ is critical case!

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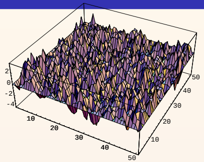

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$$EM_N = 2\sqrt{2/\pi}\log N - (3/4)\sqrt{2/\pi}\log\log N + O(1)\,.$$

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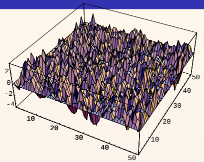

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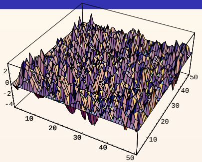

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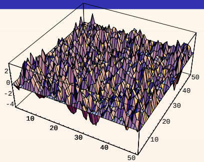

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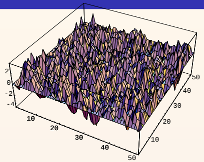

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Proof by comparing GFF to branching random walk and an appropriate modified BRW.

Heart of proof: modified second moment (with a variance reduction step) for obtaining precise tail estimates on the maximum.

Basic decomposition: Write

$$X_v = E(X_v|\mathcal{F}) + (X_v - E(X_v|\mathcal{F})) = X_v^c + X_v^f$$

where $\mathcal{F}$ is boundary of boxes.

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Basic facts: X_v^f is GFF in smaller box, and it is impossible to have two large values of X_v for v, w that are mesoscopically separated.

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Proof by comparing GFF to branching random walk and an appropriate modified BRW.
Heart of proof: modified second moment (with a variance reduction step) for obtaining precise tail estimates on the maximum.
Basic decomposition: Write

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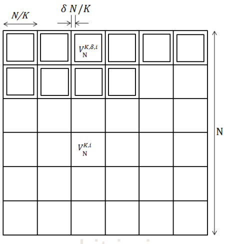

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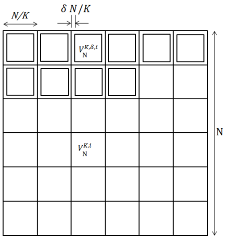

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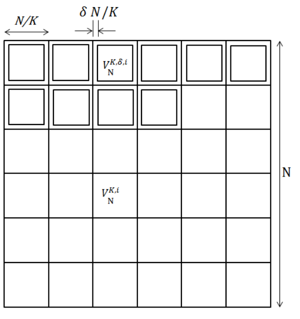

Main lemma:

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where v_i is location of maximum in i-th sub-box of the fine field.

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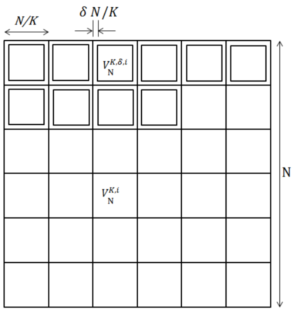

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Let X_{v_i} be maximum in ith box (of arbitrarily slowly increasing K), and set $\zeta_N = \sum_i \delta_{X_{v_i} - m_N}$.

Theorem (Biskup-Louidor '13, '14,'15)

ζ_N converges to a SPPP process of exponential intensity.

Expect full convergence to SDPPP.

Crucial component in the BL proof: local maxima are macroscopically separated, hence by using Gaussian property + linear combination with independent copy,

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2D, log correlated models vs higher dimension

2D was critical: the limiting law of the maximum results is universal in the class of logarithmically correlated fields (Ding-Roy-Z. ’15) (specific models: Madaule ’14; Continuous GFF: Acosta ’15).

Higher dimensional DGFF is easier: less correlation, extremes more Poisson-like.For example, let M_N be maximum of d-dimensional DGFF in box V_N.

Theorem (Chiarini, Cipriani, Hazra ’15)

For $d \geq 3$, the law of $(M_N - EM_N) \cdot \sqrt{\log N}$ converges to a Gumbel distribution.

The proof is an application of Stein’s method (two moments suffice version, of Arratia-Goldstein-Gordon), and uses the Markov property to estimate the law of $\mathcal{X}_V$ conditioned on the outside of a mesoscopic box. The proof actually gives the convergence of the process of extrema to a Poisson process (no random shift, no decoration).

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A (randomly shifted) PPP is characterized by:
Add i.i.d. random variable, then shift (deterministically) and get same process. (Liggett; Aizenman-Ruzmaikina; Arguin)
A decorated PPP of exponential intensity e^{-cx} is characterized by the following property:
Exponentially-c-stable (P. Maillard; Davydov-Molchanov-Zuyev):
$e^a + e^b = c \Rightarrow \xi \stackrel{d}{=} \theta_a \xi_1 + \theta_b \xi_2$.
Identifying the random shift reduces to proving DPPP.
For SDPPP, a (less clean) characterization exists: say ξ exhibits freezing if, for $f \geq 0$ continuous, bounded support,

$$E(e^{-\langle \xi, \theta_y f \rangle}) = g(y - c_f)\,.$$

Motivated by freezing of free energy (Derrida-Spohn, Fyodorov-LeDoussal,..).

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$$H_N(\sigma) = N^{-1/2} \sum_{i,j=1}^{N} J_{i,j} \sigma_i \sigma_j$$

where $\sigma_i \in \{-1, 1\}$, $J_{i,j}$ - i.i.d. standard Gaussian.

Gibbs measure

$$\frac{e^{-\beta H_N(\sigma)}}{Z_N}$$

Covariance

$$E H_N(\sigma) H_N(\sigma') = N(\langle \sigma, \sigma' \rangle / N)^2.$$

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Auffinger, Ben Arous, Cerny: Introduce

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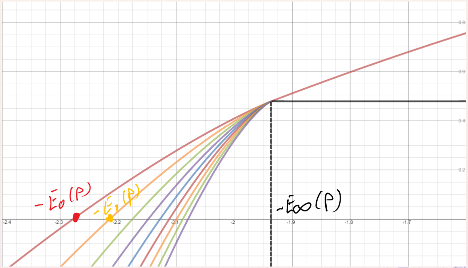

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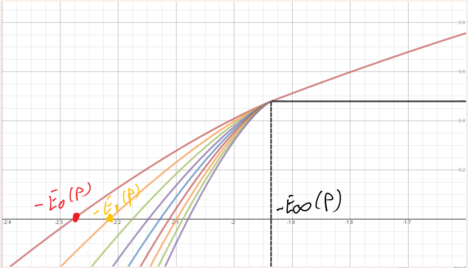

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ABAC: $\lim_{N\to\infty} \frac{1}{N} \log E\,\mathrm{Crit}_N(u) = \Theta_p(u)$.

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Write $f(\sigma) = N^{-1/2} H_{n,p}(\sigma)$. Apply Kac-Rice to $\nabla f(\sigma)$.

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The region $u < -E_\infty$ corresponds to considering e^W with $W = \log \det(z - X)$, where X is GOE and z is outside the spectrum
ABAC use Selberg's formula and intricate computations to evaluate expectation.
Streamlined proof:

- Large deviations at scale N^2 of empirical measure (Ben Arous-Guionnet '97)
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ABAC's Crucial observation: Hessian is a GOE (perturbed by multiple of identity).
The region $u < -E_\infty$ corresponds to considering e^W with $W = \log \det(z - X)$, where X is GOE and z is outside the spectrum
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Can use Kac-Rice for second moment[1]; In view of ABA's remark, why should it work? Surprise!

Theorem (Subag, '15)

For pure $p \geq 7$ spin, it works!

For $-E_0(p) < u < -E_\infty(p)$, $\frac{\mathrm{Crit}_N(u)}{E\mathrm{Crit}_N(u)} \to 1$, in probability.

The computation now involves a pair of deformed GOEs.

Selberg-based computations hard to perform, but the streamlined approach works.

Key point: strong correlation (where overlap of σ, σ' is strictly bounded away from 0) is exponentially negligible.

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Fix $m_N = -E_0 N + c \log n$.

Extremal process: $\mathcal{E}_N := \sum \delta_{H_{N,p}(z_i)+m_N}$, z_i critical points.

Theorem (Subag-Z '15)

The process $\mathcal{E}_N$ converges to a Poisson point process with exponential rate.

Chen-Stein hard to apply directly, because it is hard to evaluate the coupling from a collection of far away critical points.

Instead, follow the Biskup–Louidor idea: use decoupling of second moments to show that $\mathcal{E}_N$ is asymptotically invariant under adding an i.i.d. Gaussian - requires understanding of shift of critical point(s) under perturbation.

Requires a finer analysis of first and second moments of number of critical point $\mathcal{N}(a,b)$ in an interval $[-m_N + a, -m_N + b]$.

Remark:

$$\frac{E(\mathcal{N}(a,b)(\mathcal{N}(a,b)-1))}{(E\mathcal{N}(a,b))^2} \to_{N\to\infty} 1$$

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