Fluctuation theory of Lévy and Markov additive processes

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Lecture 1: Random walks

Lévy processes are closely related to Random walks

Let $S_n, n \geq 0$ be a random walk

$$S_0 = 0, \quad S_n = \sum_{k=1}^{n} Y_k, \qquad n \geq 1,$$

with $(Y_i)_{i\geq 1}$ i.i.d. $\mathbb{R}^d$-valued r.v.

- $\{S_n, n \geq 0\}$ has independent increments. For any $n, k \geq 0$, the r.v. $S_{n+k} - S_n$ is independent of $(S_0, S_1, \ldots, S_n)$.

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- $\{S_n, n \geq 0\}$ has homogeneous increments. For any $n, k \geq 0$, the r.v. $S_{n+k} - S_n = \sum_{i=n+1}^{n+k} Y_i$ has the same law as $\sum_{i=1}^{k} Y_i = S_k$, that is

$$\mathbb{P}\left(S_{n+k} - S_n \in dy\right) = \mathbb{P}(S_k \in dy), \qquad \text{on } \mathbb{R}^d.$$

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- $\{S_n, n \geq 0\}$ has independent increments. For any $n, k \geq 0$, the r.v. $S_{n+k} - S_n$ is independent of $(S_0, S_1, \ldots, S_n)$.
- $\{S_n, n \geq 0\}$ has homogeneous increments. For any $n, k \geq 0$, the r.v. $S_{n+k} - S_n = \sum_{i=n+1}^{n+k} Y_i$ has the same law as $\sum_{i=1}^{k} Y_i = S_k$, that is

$$\mathbb{P}(S_{n+k} - S_n \in dy) = \mathbb{P}(S_k \in dy), \qquad \text{on } \mathbb{R}^d.$$

$\Rightarrow$ S is Markov chain, its law is totally characterised by the law of Y_1, central in the theory of stochastic processes...

Definamos el máximo pasado $M_0 = 0 \vee Y_0$, y $M_{n+1} = \max\{M_n, S_{n+1}\}$ para $n \geq 0$. Nos interesará el proceso estocástico que se obtiene al *reflejar* la caminata aleatoria en su máximo pasado. Más precisamente, sea Z el proceso estocástico definido por

$$Z_n = M_n - S_n, \qquad n \geq 0.$$

Lemma

Z es una cadena de Markov homogénea con espacio de estados en $\mathbb{R}^+$

Proof.

Veamos que el proceso estocástico$\{Z_n, n \geq 0\}$ lo podemos expresar mediante una recurrencia aleatoria:

$$Z_0 = M_0 - S_0 = \max\{0, Y_0\} - Y_0 = \max\{-Y_0, 0\}$$

y para todo $n \geq 0$,

$$\begin{aligned} Z_{n+1} &= M_{n+1} - S_{n+1} = \max\{M_n, S_{n+1}\} - S_n - Y_{n+1} \\ &= \max\{M_n - S_n - Y_{n+1}, 0\} = \max\{Z_n - Y_{n+1}, 0\} = G(Z_n, Y_{n+1}), \end{aligned}$$

donde $G : \mathbb{R}^2 \to \mathbb{R}$ es la función dada por $G(z, y) = \max\{z - y, 0\}$ para $(x, y) \in \mathbb{R}^2$. □

Denotaremos por τ_1

$$\tau_1 = \inf\{k \geq 1 : S_k > 0\},$$

y $h_1 = S_{\tau_1}$ la posición que toma la caminata en ese instante. Se tiene la siguiente igualdad entre eventos

$$\{\tau_1 = n\} = \{S_1 \leq 0, S_2 \leq 0, \ldots, S_{n-1} \leq 0, S_n > 0\}, \qquad n \geq 1.$$

Por lo que se puede afirmar que τ_1 es un tiempo de paro.
Más aún, la relación anterior nos dice como calcular la ley de probabilidad de τ_1.
Supongamos que $S_0 = x \leq 0$, para todo $n \geq 1$, tenemos que

$$\begin{aligned}\mathbb{P}_x(\tau_1 = n) &= \mathbb{P}_x\left(S_1 \leq 0, S_2 \leq 0, \ldots, S_{n-1} \leq 0, Y_n > -S_{n-1}\right) \\ &= \mathbb{E}_x\left(1_{\{S_1 \leq 0, S_2 \leq 0, \ldots, S_{n-1} \leq 0\}} \overline{F}^-(S_{n-1})\right),\end{aligned}$$

donde $\overline{F}^-(y) = \mathbb{P}(Y_1 > -y)$, con $y \leq 0$.

El r-ésimo instante al cual

$$S_n > S_{n-1}, S_n > S_{n-2}, \ldots, S_n > S_0,$$

es de la forma

$$(\tau_1 + \tau_2 + \cdots + \tau_r, h_1 + h_2 + \cdots + h_r),$$

donde los pares (τ_1, h_1), $(\tau_2, h_2), \ldots, (\tau_r, h_r)$ son independientes e idénticamente. distribuidos.

De manera análoga, la trayectoria de una caminata aleatoria puede ser estudiada a partir de los instantes a los cuales se alcanza un mínimo local, el r-ésimo instante al cual

$$S_n \leq S_{n-1}, S_n \leq S_{n-2}, \ldots, S_n \leq S_0,$$

y la posición que toma la trayectoria en dicho instante es de la forma

$$(\widehat{\tau}_1 + \widehat{\tau}_2 + \cdots + \widehat{\tau}_r, \widehat{h}_1 + \widehat{h}_2 + \cdots + \widehat{h}_r),$$

donde los pares $(\widehat{\tau}_1, \widehat{h}_1)$, $(\widehat{\tau}_2, \widehat{h}_2)$ hasta $(\widehat{\tau}_r, \widehat{h}_r)$ son independientes e idénticamente distribuidos.

Sea $\{Z_n, n \geq 0\}$ la cadena de Markov obtenida al reflejar una caminata aleatoria en $\mathbb{R}$ en su máximo pasado, $Z_n = M_n - S_n$, con la notación introducida anteriormente. Definimos el proceso $\{L_n, n \geq 0\}$ del *tiempo local en* 0 para Z, mediante la relación,

$$L_0 = 0, \qquad L_n = \sum_{k=1}^{n} 1_{\{Z_k=0\}} = \sum_{k=1}^{n} 1_{\{M_k=S_k\}} = \sum_{k=1}^{n} 1_{\{S_1<S_k,\dots,S_{k-1}<S_k\}}, \qquad n \geq 1.$$

Es decir que el proceso L cuenta el número de visitas al estado 0. De hecho se puede verificar que salvo constantes multiplicativas es el único proceso estocástico que:

(a) crece en los instantes en que Z toma el valor 0 y además cumple que

$$L_{n+m} = L_n + L_m \circ \theta_n,$$

donde $L_\cdot \circ \theta_n$ denota el número de ceros en el intervalo $(0, m]$ para la cadena $\{\widetilde{Z}_k = Z_{n+k}, k \geq 0\}$ que es una cadena de Markov con la misma matriz de transición que Z.

(b) si T es un tiempo de paro finito en el cual $Z_T = 0$, entonces se tiene que el proceso estocástico

$$\{(L_{n+T} - L_T, Z_{T+n}), n \geq 0\}$$

tiene la misma distribución que $\{(L_n, Z_n), n \geq 0\}$.

Llamaremos *tiempo local* a cualquier proceso que satisfaga (a) y (b).

Para $n \geq 0$, denotaremos por g_n al último instante anterior a n en el cual S alcanza un nuevo máximo local

$$g_n = \max\{0 \leq k \leq n : M_k = S_k\}.$$

La idea clave aquí es que

$$\mathbb{E}_0 \left(F\left(S_\ell, 0 \leq \ell < g_n\right) G(Z_{g_n+j}, 1 \leq j \leq n - g_n)\right) =$$

$$\mathbb{E}_0 \left(\sum_{k=0}^{n} F\left(S_\ell, 0 \leq \ell < k\right) G(Z_{k+j}, 1 \leq j \leq n-k)\right) 1_{\{S_k > 0, S_k > S_1, \ldots, S_k > S_{k-1}, \tau_1 \circ \Theta_k > n-k\}} \Bigg)$$

$$= \mathbb{E}_0 \left(\sum_{k=0}^{n} F\left(S_\ell, 0 \leq \ell < k\right) 1_{\{S_k > 0, S_k > S_1, \ldots, S_k > S_{k-1}\}} \right.$$

$$\left. \times\ \mathbb{E}_0 \left(G(Z_{k+j}, 1 \leq j \leq n-k) 1_{\tau_1 \circ \Theta_k > n-k\}} | \mathcal{F}_k \right)\right)$$

$$= \sum_{k=0}^{n} \mathbb{E}_0 \left(F\left(S_\ell, 0 \leq \ell < k\right) 1_{\{M_k = S_k\}} \right) \times \mathbb{E}_0 \left(G(S_j, 1 \leq j \leq n-k) 1_{\{\tau_1 > n-k\}} \right)$$

donde $\tau_1 \circ \Theta_k = \min\{j \geq 1 : S_{k+j} - S_k > 0\}$ y para $k \geq 1$ el evento $\{S_k > 0, S_k > S_1, \ldots, S_k > S_{k-1}\}$ es identico a $\{M_k - S_k = 0\}$.

Esta es una ecuación del tipo renovación:

$$\sum_{k=0}^{n} a_k b_{n-k}, \qquad n \geq 0.$$

Si $F \equiv 1$, $a_n = \sum_{k=0}^{n} a_k f_{n-k}$, $n \geq 0$, con $f_j = \mathbb{P}(\tau_1 = j)$. Feller Vol.I Capítulo XIII.

Un proceso importante relacionado con el anterior es el inverso del tiempo local $\{L_n^{-1}, n \geq 0\}$, es decir $L_0^{-1} = 0$, y

$$L_n^{-1} = \inf\{k > 0 : L_k = n\}, \qquad n \geq 1,$$

con la suposición de que $\inf\{\emptyset\} = \infty$. Observemos que $L_1^{-1} = \tau_1$, coincide con el primer instante al cual la caminata aleatoria entra en la semi-recta $(0, \infty)$, que fue definido en la sección anterior. Por lo tanto se puede afirmar que L_1^{-1} es un tiempo de paro. De hecho esta propiedad es compartida por todos los $\{L_n^{-1}\}$. En efecto, sea $n \geq 2$ se tiene que

$$\{L_n^{-1} = \ell\} = \{Z_\ell = 0, \text{exactamente } n-1 \text{ ceros en el vector } (Z_1, \dots, Z_{\ell-1})\},$$

para todo $\ell \geq 1$. Observemos que estos son tiempos de paro para Z y para S ya que Z se obtiene como función de S. De hecho, de lo anterior se verifica fácilmente la igualdad de conjuntos del siguiente lema.

Lemma

Se tiene la igualdad de conjuntos

$$\{L_n^{-1}, 1 \leq n \leq L_\infty\} = \{k \geq 1 : S_k > S_{k-1}, \dots, S_k > S_1, S_k > 0\}.$$

Denotaremos por $S_{L_n^{-1}}$ a la variable aleatoria definida para $\omega \in \Omega$ como

$$S_{L_n^{-1}}(\omega) = \begin{cases} S(\omega)_{L_n^{-1}(\omega)}, & \text{si } L_n^{-1}(\omega) < \infty, \\ \infty, & \text{si } L_n^{-1}(\omega) = \infty. \end{cases}$$

Definimos un nuevo proceso estocástico que toma valores en $\mathbb{R}^+ \times \mathbb{R}^+$ como

$$\{(L_n^{-1}, S_{L_n^{-1}}), n \geq 0\}.$$

Es decir el proceso que nos indica el tiempo al cual se alcanza el máximo por la n-ésima vez y la posición del máximo.

Theorem

El proceso estocástico de **escalera ascendente de tiempo y espacio** *definido por*

$$\{(L_n^{-1}, S_{L_n^{-1}}), n \geq 0\},$$

es una cadena de Markov con espacio de estados $\mathbb{R}^+ \times \mathbb{R}^+ \cup \{\infty, \infty\}$. *De hecho se trata de una caminata aleatoria con coordenadas estrictamente crecientes.*

Vale la pena mencionar que el resultado sigue siendo cierto si se remplaza la caminata aleatoria S por su dual $\widehat{S} = -S$, lo que da lugar a los *procesos de escala descendente.*

Proof.

De hecho una prueba de éste resultado se sigue de la discusión anterior, puesto que

$$(\sum_{i=1}^{n}\tau_i, \sum_{i=1}^{n}h_i) = (L_n^{-1}, S_{L_n^{-1}}), n \geq 1,$$

donde τ_i denota la *duración de la i-ésima excursión fuera de* 0 para Z, y h_i el tamaño del i-ésimo incremento del máximo. □

La ley de los incrementos de L_n^{-1} está dada por

$$\mathbb{P}(L_n^{-1} - L_{n-1}^{-1} = k) = \mathbb{P}(\tau_1 = k | S_0 = 0), \qquad k \geq 1.$$

En el siguiente resultado calcularemos la distribución conjunta de $(S_{\tau_1 - 1}, S_{\tau_1})$. La ley de S_{τ_1} determina la ley de saltos del proceso de escala creciente $S_{L^{-1}}$.

Theorem

Sea V la medida sobre $\mathbb{R}^-$, definida por

$$V(dk) = \mathbb{E}_0\left(\sum_{n=0}^{\tau_1 - 1} 1_{\{S_n \in dk\}}\right), \qquad k \in \mathbb{R}^-.$$

Para toda $f : \mathbb{R}^- \times \mathbb{R}^+ \to \mathbb{R}^+$, acotada y medible se tiene que

$$\mathbb{E}_0\left(f(S_{\tau_1 - 1}, S_{\tau_1})\right) = \int_{k \in \mathbb{R}^-} V(dk)\, \mathbb{E}(f(k, k + Y_1) 1_{\{Y_1 > -k\}}).$$

En particular, se tiene que para todo $y > 0$,

$$\mathbb{P}_0(S_{\tau_1} > y) = \int_{k \in \mathbb{R}^-} V(dk)\, \mathbb{P}(Y_1 > y - k)$$

El siguiente resultado elemental se le suele llamar lema de Dualidad de Feller.

Lemma

Sea $n > 0$ *fijo. Se tiene la igualdad en ley*

$$(-S_k, 0 \leq k \leq n) \stackrel{Law}{=} (S_{n-k} - S_n, 0 \leq k \leq n)$$

En lo subsecuente denotaremos por Γ_p una variable aleatoria geométrica con parametro de éxito p, y densidad de probabilidad

$$\mathbb{P}(\Gamma_p = k) = p(1-p)^k, \qquad k \geq 0,$$

que supondremos independiente de la caminata aleatoria $(S_n, n \geq 0)$. Para $n \geq 0$ denotaremos por g_n al último instante anterior a n en el cual S alcanza un nuevo máximo local

$$g_n = \max\{0 \leq k \leq n : M_k = S_k\}.$$

Denotaremos $G = g_{\Gamma_p}$.

Theorem

(a) *Las v.a.* (G, S_G) *y* $(\Gamma_p - G, S_{\Gamma_p} - S_G)$ *son independientes e infinitamente divisibles.*

(b) *Se tienen las siguiente identidades para* $0 < |s|, |u| < 1$, $\theta, \beta \in \mathbb{R}$,

$$\mathbb{E}\left(s^G \exp\{i\theta S_G\}\right) = \exp\left\{-\sum_{n=1}^{\infty} \int_{(0,\infty)} \frac{q^n}{n} F^{*n}(dy) \left(1 - s^n e^{i\theta y}\right)\right\}$$
$$\mathbb{E}\left(u^{\Gamma_p - G} e^{i\beta(S_{\Gamma_p} - S_G)}\right) = \exp\left\{-\sum_{n=1}^{\infty} \int_{(-\infty,0)} \frac{q^n}{n} F^{*n}(dy) \left(1 - u^n e^{i\beta y}\right)\right\} \tag{1}$$

Idea de la demostración

Condicionando en la esperanza anterior con respecto al valor de Γ_p y asignando un valor a G obtenemos la siguiente sucesión de igualdades.

$$\begin{aligned}
&\mathbb{E}\left(s^G \exp\{i\theta S_G\} u^{\Gamma_p - G} \exp\{i\beta(S_{\Gamma_p} - S_G)\}\right) \\
&= \sum_{n\geq 0} pq^n \, \mathbb{E}\left(s^{g_n} \exp\{i\theta S_{g_n}\} u^{n-g_n} \exp\{i\beta(S_n - S_{g_n})\}\right) \\
&= \sum_{n\geq 0} pq^n \sum_{k=0}^{n} \mathbb{E}\left(s^k \exp\{i\theta S_k\} u^{n-k} \exp\{i\beta(S_n - S_k)\} 1_{\{g_n=k\}}\right) \\
&= \sum_{n\geq 0} pq^n \sum_{k=0}^{n} \mathbb{E}\left(s^k \exp\{i\theta S_k\} u^{n-k} \exp\{i\beta(S_n - S_k)\} 1_{\{S_k>0, S_k>S_1, \ldots, S_k>S_{k-1}} \right. \\
&\qquad \left. \times \, 1_{\{\tau_1 \circ \Theta_k > n-k\}}\right),
\end{aligned} \tag{2}$$

donde $\tau_1 \circ \Theta_k = \min\{j \geq 1 : S_{k+j} - S_k > 0\}$ y para $k \geq 1$ el evento $\{S_k > 0, S_k > S_1, \ldots, S_k > S_{k-1}\}$ es identico a $\{M_k - S_k = 0\}$.

Por independencia y el hecho de que la caminata aleatoria tiene incrementos independientes y estacionarios se tiene la igualdad

$$\sum_{n\geq 0} pq^n \sum_{k=0}^{n} \mathbb{E}\left(s^k \exp\{i\theta S_k\} u^{n-k} \exp\{i\beta(S_n - S_k)\} 1_{\{S_k>0, S_k>S_1,\dots,S_k>S_{k-1}, \tau_1\circ\Theta_k>n-k\}}\right.$$

$$= \sum_{n\geq 0} pq^n \sum_{k=0}^{n} \mathbb{E}\left(s^k \exp\{i\theta S_k\} 1_{\{S_k>0, S_k>S_1,\dots,S_k>S_{k-1}\}}\right.$$

$$\left.\times \mathbb{E}\left(u^{n-k} \exp\{i\beta(S_n - S_k)\} 1_{\{\tau_1\circ\Theta_k>n-k\}} | \mathcal{F}_k\right)\right)$$

$$= \sum_{k\geq 0} q^k \mathbb{E}\left(s^k \exp\{i\theta S_k\} 1_{\{S_k>0, S_k>S_1,\dots,S_k>S_{k-1}\}}\right)$$

$$\times \sum_{n=k}^{\infty} p\, \mathbb{E}\left((qu)^{n-k} \exp\{i\beta S_{n-k}\} 1_{\{\tau_1>n-k\}}\right)$$

$$= \left[\sum_{k\geq 0} q^k \mathbb{E}\left(s^k \exp\{i\theta S_k\} 1_{\{S_k>0, S_k>S_1,\dots,S_k>S_{k-1}\}}\right) \mathbb{P}(\tau_1 > \Gamma_p)\right]$$

$$\times \mathbb{E}\left(u^{\Gamma_p} e^{i\beta S_{\Gamma_p}} | \Gamma_p < \tau_1\right)$$

El resto de la demostración es más tecnico, se basa en la igualdad de Mercator-Newton nos permite afirmar que para $u \in \mathbb{C}$ tal que $|u| < 1$ se tiene la igualdad

$$\log(1-u) = -\sum_{n\geq 1} \frac{u^n}{n}.$$

Corollary

Denotemos $H(dx) = \mathbb{P}(h_1 \in dx)$ *y* $\widehat{H} = \mathbb{P}(\widehat{h}_1 \in dx)$. *Se tiene la igualdad entre medidas:*

$$\sum_{k\geq 1} F^{*k}(dy)\frac{1}{k}|_{y\in(-\infty,0]} = \sum_{n\geq 1}\frac{1}{n}\widehat{H}^{*n}(-dy)|_{y\in(-\infty,0]}, \tag{3}$$

y

$$\sum_{k\geq 1} F^{*k}(dy)\frac{1}{k}|_{y\in(0,\infty)} = \sum_{n\geq 1}\frac{1}{n}H^{*n}(dy)|_{y\in(0,\infty)}. \tag{4}$$

Definimos las medidas $W(dt) = \mathbb{P}(\tau_1 \in dt)$, $\widehat{W}(dt) = \mathbb{P}(\widehat{\tau}_1 \in dt)$, *sobre* $\{1, \ldots, \infty\}$. *Se tienen las identidades*

$$\mathbb{P}(S_k < 0)\frac{1}{k} = \sum_{n\geq 1}\frac{1}{n}\widehat{W}^{*n}(\{k\}), \quad k \geq 1, \tag{5}$$

y

$$\mathbb{P}(S_k > 0)\frac{1}{k} = \sum_{n\geq 1}\frac{1}{n}W^{*n}(\{k\}), \quad k \geq 1. \tag{6}$$

Corollary

Para todo $0 < q < 1$ *se tiene la factorización*

$$(1-q) = \left(1 - \mathbb{E}\left(q^{L_1^{-1}}\right)\right)\left(1 - \mathbb{E}\left(q^{\widehat{L}_1^{-1}}\right)\right)$$

Proof.

La prueba es una simple aplicación de la Formula de Mercator-Newton. □

Corollary

Se tienen las identidades

$$\widehat{\delta_0 - \mu} = (\widehat{\delta_0 - H})(\widehat{\delta_0 - \widehat{H}}) \tag{7}$$

$$\delta_0 - \mu = (\delta_0 - H) * (\delta_0 - \widehat{H}), \tag{8}$$

- Estas factorizaciones de la medida μ son a las que nos referíamos, los factores $(\widehat{\delta_0 - \widehat{H}})$ y $(\widehat{\delta_0 - \widehat{H}})$ se les conoce como los factores de Wiener-Hopf de μ. En general es muy difícil, dada una medida de probabilidad determinar explícitamente sus factores de Wiener-Hopf, sin embargo esto es posible en algunos casos.
- Esta remarcable igualdad nos permite representar cualquier medida de probabilidad en términos de dos medidas de probabilidad, H y $\widehat{H}$ cuyos soportes son disjuntos.

Corollary

Suponga que la caminata aleatoria tiende a menos infinito con probabilidad 1. *Denotemos* $H(x) = \mathbb{P}(h_1 \leq x)$, $x \in \mathbb{R}$. *Definamos la medida de renovación* ψ^+, *asociada al proceso de escalera ascendente, mediante la relación*

$$\psi^+[0,x] = \sum_0^\infty H^{*n}[0,x], \quad x \geq 0,$$

la cual es finita puesto que

$$\psi^+[0,\infty[= \frac{1}{1 - H[0,\infty)},$$

y $H[0,\infty) < 1$ *ya que con probabilidad positiva* S *nunca toma valores positivos. Definamos* $M_\infty = \sup_{n\geq 0} S_n$. *Se tiene la igualdad*

$$\mathbb{P}(M_\infty \leq x) = (1 - H[0,\infty))\psi^+[0,x], \qquad x \geq 0.$$

Lecture 2: Lévy processes

Definition

A $\mathbb{R}^d$-valued stochastic process $\{X_t, t \geq 0\}$ is called a Lévy process if

- it has right continuous left limited paths,
- it has independent increments, i.e. for any $n \geq 1$ and $0 \leq t_1 < t_2 < \cdots < t_n < \infty$ the random variables

$$X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$$

  are independent
- has stationary increment, i.e. for every $s, t \geq 0$ the law of $X_{t+s} - X_t$ is equal to that of X_s.

- Drift process: the deterministic process $\{X_t = at,\, t \geq 0\}$, its characteristic function given by

$$\mathbf{E}(e^{i<\lambda, X_t>}) = \exp\{-(-i\lambda a t)\}, \qquad \lambda \in \mathbb{R}\,.$$

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- Poisson process: let $\{\mathbb{e}_i, i \geq 0\}$ i.i.d.r.v. exponential r.v. $c > 0$, and $S_n = \sum_{i=0}^{n} \mathbb{e}_i, n \geq$ the random walk associated to it. The counting process $\{N_t, t \geq 0\}$ defined by

$$N_t = n \text{ if and only if } S_n \leq t < S_{n+1}, \qquad t \geq 0.$$

The independence and loss of memory imply $\{N_t, t \geq 0\}$ is a Lévy process and N_t follows a Poisson law with parameter tc for $t > 0$.

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$$N_t = n \text{ if and only if } S_n \leq t < S_{n+1}, \qquad t \geq 0.$$

The independence and loss of memory imply $\{N_t, t \geq 0\}$ is a Lévy process and N_t follows a Poisson law with parameter tc for $t > 0$. An useful fact. For $b \neq 0$, the following processes

$$bN_t - bct, \quad t \geq 0,$$

and

$$(bN_t - bct)^2 - bct, \quad t \geq 0,$$

are martingales. (Use $\mathbb{E}(N_t) = ct$, $Var(N_t) = ct$.)

- Compound Poisson process: $\{Y_i, i \geq 0\}$ i.i.d. $\mathbb{R}^d$-valued r.v. with common distribution F, $\{Z_n, n \geq 0\}$ the random walk associated to it, and $\{N_t, t \geq 0\}$ an independent Poisson process. The process

$$X_t = Z_{N_t}, \qquad t \geq 0,$$

is a Lévy process called *Compound Poisson process*. The uni-dimensional law of X is the so called compound Poisson law with parameters (tc, F). The characteristic function of X_t is given by

$$\mathbf{E}(e^{i<\lambda, X_t>}) = \exp\{-t \int_{\mathbb{R}} (1 - e^{i<\lambda, x>}) cF(\mathrm{d}x)\}, \qquad \lambda \in \mathbb{R}.$$

- Compound Poisson process: $\{Y_i, i \geq 0\}$ i.i.d. $\mathbb{R}^d$-valued r.v. with common distribution F, $\{Z_n, n \geq 0\}$ the random walk associated to it, and $\{N_t, t \geq 0\}$ an independent Poisson process. The process

$$X_t = Z_{N_t}, \qquad t \geq 0,$$

is a Lévy process called *Compound Poisson process*. The uni-dimensional law of X is the so called compound Poisson law with parameters (tc, F). The characteristic function of X_t is given by

$$\mathbf{E}(e^{i<\lambda, X_t>}) = \exp\{-t \int_{\mathbb{R}} (1 - e^{i<\lambda, x>}) cF(\mathrm{d}x)\}, \qquad \lambda \in \mathbb{R}.$$

$$\mathbf{E}(e^{i<\lambda, X_t>}) = \sum_{n \geq 0} \mathbb{P}(N_t = n)\, \mathbf{E}(e^{i<\lambda, S_n>} \mid N_t = n)$$

$$= \sum_{n \geq 0} \frac{(ct)^n}{n!} e^{-ct}\, \mathbf{E}(e^{i<\lambda, Y_1>})^n = e^{-ct} \sum_{n \geq 0} \frac{1}{n!} \left(\int_{\mathbb{R}^d \setminus \{0\}} e^{i<\lambda, x>} ctF(dx) \right)^n$$

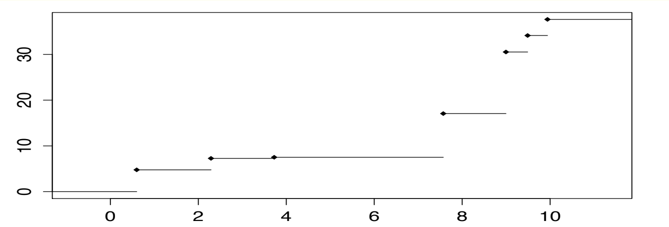

Figure: Monotone Compound Poisson

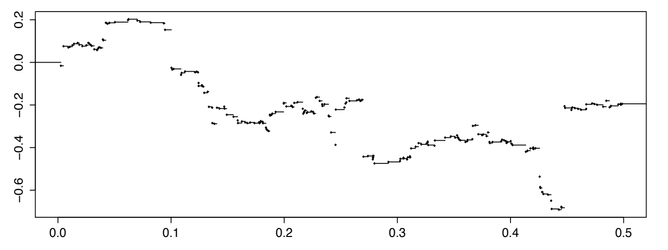

Figure: Non-monotone Compound Poisson

- Standard Brownian Motion: A real valued Lévy process $\{B_t, t \geq 0\}$ is called a standard Brownian motion if for any $t > 0$, B_t follows a Normal law with mean 0 variance t,

$$\mathbb{P}(B_t \in \mathrm{d}x) = \frac{1}{\sqrt{2\pi t}} \exp\{-\frac{x^2}{2t}\}\mathrm{d}x, \quad x \in \mathbb{R};$$

and its characteristic law is given by

$$\mathbb{E}(e^{i\lambda B_t}) = \exp\{-t\frac{\lambda^2}{2}\}, \qquad \lambda \in \mathbb{R}.$$

- Linear Brownian Motion Let $\mathbf{B} = (B^1, B^2, \ldots, B^d)^T$ be d-independent standard Brownian motions, $a \in \mathbb{R}^d$, and σ a $d \times d$-matrix. The process $X_t = at + \sigma \mathbf{B}_t$, $t \geq 0$, is a Lévy process. For each $t \geq 0$, X_t is a Gaussian vector with mean, $\mathbb{E}(X_t^i) = a^{(i)}t$, and covariance matrix

$$\mathbb{E}\left((X_t^{(i)} - a^{(i)}t)(X_t^{(j)} - a^{(j)}t)\right) = \sigma\sigma^T_{i,j}, \qquad i,j \in \{1,\ldots,d\}.$$

Its Fourier transform $\mathbb{E}\left(\exp\{i < \lambda, X_t >\}\right) = \exp\{-t\Psi(\lambda)\}$, is

$$\Psi(\lambda) = - < a, \lambda > + \frac{1}{2}\lambda^T \Sigma \lambda = - < a, \lambda > + \frac{1}{2}||\sigma\lambda||^2, \qquad \lambda \in \mathbb{R}^d.$$

With $\Sigma = \sigma\sigma^T$ the covariance matrix; it is positive definite ($x^T \Sigma x \geq 0$, $x \in \mathbb{R}^d$) and symmetric. $X_t \sim N(at, t\Sigma)$.

Theorem (Lévy–Khintchine's formula)

Let $\{X_t, t \geq 0\}$ *be a* $\mathbb{R}^d$ *valued Lévy process. For* $t > 0$*, the law of* X_t *is infinitely divisible. Furthermore*

$$\mathbb{E}(e^{i<\lambda, X_t>}) = e^{-t\Psi(\lambda)}, \qquad \lambda \in \mathbb{R}^d,$$

where $\Psi : \mathbb{R}^d \to \mathbb{C}$ *is the characteristic exponent of* X *and*

$$\begin{aligned}\Psi(\lambda) = & -i< a, \lambda > + ||Q\lambda^T||^2/2 \\ & + \int_{\{x\in\mathbb{R}^d, |x|\in(-1,1)\setminus\{0\}\}} (1 - e^{i<\lambda,x>} + i < \lambda, x >)\Pi(dx) \\ & + \int_{\{x\in\mathbb{R}^d, |x|\in(-1,1)^c\}} (1 - e^{i<\lambda,x>})\Pi(dx)\end{aligned}$$

with $a \in \mathbb{R}^d$, Q *a* $d \times d$ *matrix, and* Π *is a measure on* $\mathbb{R}^d \setminus \{0\}$ *such that* $\int_{\mathbb{R}\setminus\{0\}} (1 \wedge ||x||^2)\Pi(dx) < \infty$. a, Q, Π *are the linear term, Gaussian term and* Π *is the Lévy measure, respectively. The matrix* $\Sigma = Q^T Q$ *is the covariance matrix. The triplet* (a, Σ, Π) *characterizes the law of* X *under* $\mathbb{P}$.

Remark

Π is only required to be σ-finite, nevertheless it is necessarily finite over any set that does not contain a ball of radius $r > 0$ around 0.

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Indeed, the fact that the function $x \mapsto \frac{x^2}{1+x^2}$ is increasing implies that

$$\begin{aligned}\Pi(z \in \mathbb{R}^d : ||z|| > r) &\leq \frac{1+r^2}{r^2} \int_{\mathbb{R}^d} \frac{||z||^2}{1+||z||^2} \Pi(dz) \\ &\leq \frac{1+r^2}{r^2} \int_{\mathbb{R}^d} 1 \wedge ||z||^2 \Pi(dz) < \infty\end{aligned}$$

- Compound Poisson process

$$\Pi(dx) = cF(dx),$$

$$\mathbf{E}(e^{i<\lambda, X_t>}) = \exp\{-t\int_{\mathbb{R}}(1-e^{i<\lambda,x>})cF(\mathrm{d}x)\}, \qquad \lambda \in \mathbb{R}.$$

- Lineal Brownian motion $\Pi \equiv 0$, $\mathbb{E}\left(\exp\{i < \lambda, X_t >\}\right) = \exp\{-t\Psi(\lambda)\}$,

$$\Psi(\lambda) = - < a, \lambda > + \frac{1}{2}\lambda^T \Sigma \lambda = - < a, \lambda > + \frac{1}{2}||\sigma\lambda||^2, \qquad \lambda \in \mathbb{R}^d.$$

With $a \in \mathbb{R}^d$ and $\Sigma = \sigma\sigma^T$ the covariance matrix, which is positive definite and symmetric.

Stable Lévy processes

Lévy processes with the scaling property: $\exists \alpha > 0$ such that $\forall c > 0$

$$(cX_{tc^{-\alpha}},\ t \geq 0) \stackrel{\text{Law}}{=} (X_t, t \geq 0).$$

In this case we X is said α-stable. This is equivalent to require that the characteristic exponent Ψ, satisfy

$$\Psi(k\lambda) = k^{\alpha}\Psi(\lambda),$$

for all $k > 0$ and for all $\lambda \in \mathbb{R}^d$. Then

$$\Psi(\lambda) = ||\lambda||^{\alpha}\Psi\left(\frac{\lambda}{||\lambda||}\right), \quad \lambda \in \mathbb{R}^d.$$

When $d = 1$

$$\Psi(\lambda) = |\lambda|^{\alpha}(\mathrm{e}^{\pi i u\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\lambda>0)} + \mathrm{e}^{-\pi i u\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\lambda<0)}), \qquad \lambda \in \mathbb{R}.$$

for $\lambda \in \mathbb{R}$, where $\rho = \mathbb{P}(X_1 > 0)$.

$\rho = \mathbb{P}(X_1 > 0)$ and the Lévy measure

When $\alpha = 2$, X is a Brownian motion and hence $\rho = 1/2$. When $\alpha = 1$ the self-similarity holds only when $\rho = 1/2$.
The parameter ρ is bound to $0 < \alpha\rho, \alpha(1-\rho) \leq 1$.
For $0 < \alpha < 1$, $\rho \in [0,1]$ and for $1 < \alpha < 2$, $\rho \in [1 - \frac{1}{\alpha}, \frac{1}{\alpha}]$, and $\rho = \alpha^{-1}$, if the process X has no positive jumps, and $\rho = 1 - 1/\alpha$ if it has no negative jumps.
When $0 < \alpha < 2$, we have that $Q = 0$ and its Lévy measure is given by

$$\Pi(\mathrm{d}x) = \begin{cases} \frac{c_+ \mathrm{d}x}{x^{1+\alpha}} & \text{if } x > 0,\ \alpha \neq 1 \\ \frac{c_- \mathrm{d}x}{|x|^{1+\alpha}} & \text{if } x < 0,\ \alpha \neq 1 \\ \frac{c\mathrm{d}x}{|x|^2}, & \text{if } \alpha = 1,\ \rho = 1/2; \end{cases}$$

with

$$c_+ = \Gamma(1+\alpha)\frac{\sin(\alpha\pi\rho)}{\pi}, \qquad c_- = \Gamma(1+\alpha)\frac{\sin(\alpha\pi(1-\rho))}{\pi},$$

for some $\rho \in [0,1]$,
Necessarily $0 < \alpha < 2$ because of the integrability condition on Π.

Subordinators

Definition

A Lévy process is a **subordinator** if it has non-decreasing paths.

Lemma

If X *is a subordinator the characteristic exponent* Ψ, *can be extended analytically to the semi-plan* $\Im(z) \in [0, \infty[$. *Then the law of a subordinator characterized by the Laplace exponent* $\phi(\lambda) : \mathbb{R}^+ \to \mathbb{R}^+$ *defined by*

$$\mathbf{E}(e^{-\lambda X_1}) = e^{-\phi(\lambda)}, \qquad \lambda \geq 0,$$

where $\phi(\lambda) = \Psi(i\lambda)$. *Moreover* $Q = 0$, $\Pi(-\infty, 0) = 0$, $\int_0^\infty 1 \wedge x\Pi(dx) < \infty$, *and there is* $a \geq 0$ *s.t.*

$$\phi(\lambda) = a\lambda + \int_0^\infty (1 - e^{-\lambda x})\Pi(dx), \qquad \lambda \geq 0.$$

Example (Gamma Subordinator)

$(X_t, t \geq 0)$ is a Gamma(b, a) subordinator if its one dimensional law is Gamma(at, b), that is

$$\mathbb{P}(X_t \in dx) = \frac{b^{at}}{\Gamma(at)} x^{at-1} e^{-bx} dx, \qquad x \geq 0.$$

Its Laplace transform takes the form

$$\mathbb{E}(e^{-\lambda X_t}) = \left(\frac{b}{b+\lambda}\right)^{at} = \exp\left\{-t(a \log((b+\lambda)/b)\right\}, \qquad \lambda \geq 0.$$

Frullani's formula, establishes

$$\log(x/y) = \int_0^\infty (e^{-yt} - e^{-xt}) \frac{dt}{t}, \qquad x, y > 0.$$

Then

$$\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda t}) \frac{ae^{-bt}}{t} dt, \qquad \lambda \geq 0.$$

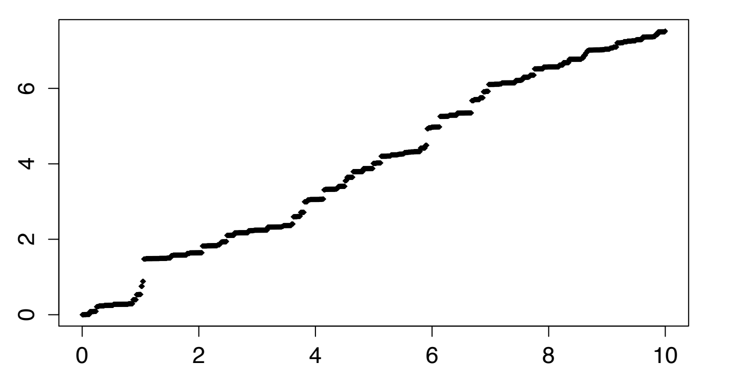

Figure: A Gamma subordinator

Example

Let X be an α-stable subordinator with index $\alpha \in (0,1)$. Its Laplace exponent is

$$\phi(\lambda) = \lambda^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1 - e^{-\lambda x}) \frac{dx}{x^{1+\alpha}}, \qquad \lambda \geq 0.$$

(Integration by parts)

Lemma

For $t > 0$, $\mathbb{E}(X_t) = \infty$. *Furthermore,*

$$\mathbb{E}(X_t^\beta) < \infty \quad \textit{if and only if } \beta < \alpha,$$

if and only if

$$\int_1^\infty x^\beta \frac{dx}{x^{1+\alpha}} < \infty.$$

The strong Markov property

We will denote by $\mathcal{F}_t = \sigma(X_s, s \leq t) \vee \mathcal{N}$, for $t \geq 0$, with $\mathcal{N}$ the null-sets of $\mathbb{P}$.

Lemma

A Lévy process is a strong Markov process. We have that for every T finite stopping time the pre-T-process $(X_s, s \leq T)$ is independent of the post-T-process, $(\widetilde{X}_s = X_{s+T} - X_T, s \geq 0)$, and the latter has the same law as $(X_u, u \geq 0)$.

Remark

For $x \in \mathbb{R}$, we will denote by $\mathbb{P}_x$ the push forward measure of the transform $x + X$. This is the law of X started at $x_0 = x$.

Idea of Proof.

For T deterministic, it is enough to show that for for $m \geq 1$, and $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq t$ and $0 \leq s_1 \leq \cdots \leq s_m$ the vectors

$$(X_{t_1}, \ldots, X_{t_n}) \quad \text{y } (\widetilde{X}_{s_1}, \ldots, \widetilde{X}_{s_m})$$

are independent and the second has the same law as $(X_{s_1}, \ldots, X_{s_m})$.

□

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$$\begin{aligned} &\mathbb{E}\left(\exp\left\{i\left(\sum_{j=1}^{n}\lambda_j X_{t_j} + \sum_{k=1}^{m}\beta_k \widetilde{X}_{s_k}\right)\right\}\right) \\ &= \mathbb{E}\left(\exp\left\{i\left(\sum_{j=1}^{n}\lambda_j X_{t_j}\right)\right\}\right)\mathbb{E}\left(\exp\left\{i\left(\sum_{k=1}^{m}\beta_k X_{s_k}\right)\right\}\right), \end{aligned} \tag{9}$$

for any $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, $(\beta_1, \ldots, \beta_m) \in \mathbb{R}^m$, $n, m \geq 1$.

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for any $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, $(\beta_1, \ldots, \beta_m) \in \mathbb{R}^m$, $n, m \geq 1$.
The argument for T taking countably many values is done by considering the events $\{T = a_i\}$. General T by approximation. □

Definition

Let $(\Theta, \mathcal{B}, \rho)$ a space of σ-finite measure. A family of $\mathbb{N} \cup \{\infty\}$-valued random variables $(N(B), B \in \mathcal{B})$ is called a Poisson measure with intensity measure ρ, if

(i) $N(B)$ $\sim$Poisson $\rho(B)$, with the assumption that $\rho(B) = 0$ iff $N(B) = 0$ a.s. and $\rho(B) = \infty$ iff $N(B) = \infty$.

(ii) if $B_j \in \mathcal{B}$, $j \in \{1, \ldots, n\}$ are disjoint sets then $N(B_1), \ldots, N(B_k)$ are independent.

(iii) For $\omega \in \Theta$ the set function $B \mapsto N(B)(\omega)$ is a measure on $(\Theta, \mathcal{B})$.

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As a consequence:

$$\mathbb{E}(N(B)) = \rho(B) = Var(N(B)), \qquad B \in \mathcal{B}.$$

For a $t > 0$ we denote $X_{t-} = \lim_{s\uparrow t} X_s$, which exists and is finite by the assumption of having càdlàg paths, and $\Delta X_t = X_t - X_{t-}$.
For $B \in \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ we define

$$J(B,\omega) = \#\{s > 0 : (s, \Delta X_s) \in B\}, \qquad \omega \in \Omega.$$

Theorem (Lévy-Itô decomposition-I)

Let X be a $\mathbb{R}^d$ valued Lévy process with characteristics (a, Σ, Π) and Λ denote the Lebesgue measure on $[0, \infty)$. We have :

(i) *The familly $(J(B), B \in \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ is a Poisson random measure with intensity measure $\Lambda \otimes \Pi$.*

For $\omega \in \Omega$ the measure $J()(\omega)$ can be written as

$$J(B)(\omega) = \sum_{t \geq 0} 1_{\{(t,\Delta_t(\omega)) \in B\}},$$

where $\Delta_t(\omega)$ are the spatial coordinates in $\mathbb{R}^d \setminus \{0\}$ of the points t for which $J(\{t\} \times \mathbb{R}^d \setminus \{0\})(\omega) = 1$, that is those t for which $\Delta_t(\omega) = X_t(\omega) - X_{t-}(\omega) \neq 0$.
We will call

$$((t, \Delta_t), t \geq 0)$$

the Poisson point process of jumps of X.

Lemma (Campbell's formula)

For $f : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ *we have that for* $t > 0$

$$\sum_{s \leq t} |f(s, \Delta_s)|,$$

is finite a.s. if and only if $\int_0^t ds \int_{\mathbb{R}^d \setminus \{0\}} 1 \wedge |f(s,y)| \Pi(dy) < \infty$. *In that case*

$$\mathbb{E}\left(\sum_{s \leq t} f(s, \Delta_s) \right) = \int_0^t ds \int_{\mathbb{R}^d \setminus \{0\}} f(s,y) \Pi(dy),$$

and the exponential formula holds

$$\mathbb{E}\left(\exp\{i\lambda \sum_{s \leq t} f(s, \Delta_s)\} \right) = \exp\{-\int_0^t ds \int_{\mathbb{R}^d \setminus \{0\}} \left(1 - e^{i\lambda f(s,y)}\right) \Pi(dy)\}.$$

If f *is positive the above formula remains valid if* $i\lambda$ *is replaced by* $-\lambda$.

A first consequence

If Π is a a Lévy measure on $(0,\infty)$ such that

$$\int_{(0,\infty)} 1 \wedge x\Pi(dx) < \infty,$$

then for any $a \geq 0$ the process

$$X_t = at + \sum_{s \leq t} \Delta_s, \qquad t \geq 0,$$

is finite a.s., has independent and stationary increments, the paths are non decreasing and according to the exponential formula its Laplace transform is given by

$$\mathbb{E}(e^{-\lambda X_t}) = \exp\{-at - t\int_0^\infty (1-e^{-\lambda x})\Pi(dx)\}, \qquad \lambda \geq 0.$$

Every subordinator can be build in this way.

Lemma (Compensation or Master formula)

Let $(t, \Delta_t, t \geq 0)$ *the Poisson point process of jumps of* X. *For* H *measurable, left continuous and positive valued functional, the identity*

$$\mathbb{E}\left(\sum_{t>0} H((X_u, u<t), \Delta_t)\right)$$

$$= \mathbb{E}\left(\int_0^\infty dt \int_{\mathbb{R}^d \setminus \{0\}} \Pi(dy) H((X_u, u<t), y)\right),$$

holds. If $\mathbb{E}\left(\int_0^t ds \int_{\mathbb{R}^d \setminus \{0\}} \Pi(dy) H\ \ ((X_u, u<s), y)\right) < \infty,\ \forall t > 0,$ *the process*

$$\int_{s\in(0,t]} \int_{\mathbb{R}^d \setminus \{0\}} H((X_u, u<s), y)(J(dsdy) - ds\Pi(dy)), \qquad t \geq 0,$$

is a *Martingale,*

Lemma (Compensation or Master formula)

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holds. If $\mathbb{E}\left(\int_0^t ds \int_{\mathbb{R}^d\setminus\{0\}} \Pi(dy) H^2((X_u, u<s), y)\right) < \infty, \forall t>0$, *the process*

$$\int_{s\in(0,t]}\int_{\mathbb{R}^d\setminus\{0\}} H((X_u, u<s), y)(J(dsdy) - ds\Pi(dy)), \qquad t \geq 0,$$

is a square integrable Martingale, with quadratic variation $\int_{s\in(0,t]}\int_{\mathbb{R}^d\setminus\{0\}} H^2((X_u, u<s), y)ds\Pi(dy)$

An application to first passage

We will assume that X is a subordinator with characteristics (b, Π).
Let $x > 0$ and $\tau_x^+ = \inf\{t > 0 : X_t > x\}$, the first passage time above level x for X and (U_x, O_x) be the undershoot and overshoot of X at level x,

$$O_x = X_{\tau_x^+} - x, \qquad U_x = x - X_{\tau_x^+ -}.$$

We are interested by the distribution of the random variables (τ_x, U_x, O_x).
The *potential* or *renewal* measure of X is defined as the measure

$$V(dy) := \mathbb{E}\left(\int_0^\infty ds 1_{\{X_s \in dy\}}\right), \qquad y \geq 0.$$

This measure is characterised by its Laplace transform, which is given by

$$\int_{[0,\infty)} V(dy) e^{-\lambda y} = \frac{1}{\phi(\lambda)}, \qquad \lambda > 0.$$

Theorem

For any $f : \mathbb{R}^2 \to \mathbb{R}^+$ *measurable*

$$\mathbb{E}(f(U_x, O_x)1_{\{U_x>0\}}) = \int_0^x V(dy) \int_{(0,\infty)} \Pi(dz) f(x-y, y+z-x) 1_{\{z>x-y\}}$$

For every $f : \mathbb{R}^+ \to \mathbb{R}^+$

$$\mathbb{E}(f(\tau_x^+)1_{\{U_x>0\}}) = \int_0^\infty f(t)\, \mathbb{E}\left(\overline{\Pi}(x - X_t), X_t < x\right).$$

Proof.

On $X_{\tau_x^+} > x$, τ_x^+ is the unique instant where $X_{t-} < t$ and $X_t > x$, hence

$$\begin{aligned}&\mathbb{E}\left(f(\tau_x, U_x, O_x)1_{\{U_x>0\}}\right)\\&= \mathbb{E}\left(\sum_{t>0} f(t, x - X_{t-}, (X_t - X_{t-}) + X_{t-} - x)1_{\{X_t > x > X_{t-}\}}\right)\end{aligned}$$

Now, we apply the compensation formula to get

$$= \mathbb{E}\left(\int_0^\infty dt \int_{(0,\infty)} \Pi(dy) f(t, x - X_{t-}, y + X_{t-} - x)1_{\{y > x - X_{t-} > 0\}}\right).$$

The set of discontinuities has zero Lebesgue measure

$$= \mathbb{E}\left(\int_0^\infty dt \int_{(0,\infty)} \Pi(dy) f(t, x - X_t, y + X_t - x)1_{\{y > x - X_t > 0\}}\right).$$

Specialize to time or space. □

If X is α-stable subordinator

$$\phi(\lambda) = \lambda^{\alpha},$$

the Lévy measure is $\Pi(dx) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{dx}{x^{1+\alpha}}$. The renewal measure has Laplace transform

$$\int_{[0,\infty)} V(dy) e^{-\lambda y} = \frac{1}{\phi(\lambda)} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-\lambda y} y^{\alpha-1} dy.$$

Thus $V(dy) = \frac{1}{\Gamma(\alpha)} y^{\alpha-1} dy$, $y \geq 0$.

Corollary

For any $x > 0$ the random variables U_x/x and O_x/U_x are independent, its law do not depend of x, the former has a $Beta(1-\alpha, \alpha)$ distribution and the latter has a Pareto distribution on $(0, \infty)$ of parameter α.

$$\mathbb{E}\left(f\left(\frac{U_x}{x}\right) g\left(O_x/U_x\right)\right) \propto \int_0^1 du u^{-\alpha}(1-u)^{\alpha-1} f(u) \int_0^{\infty} \frac{dv}{(1+v)^{1+\alpha}} g(v),$$

the normalising constant is $c_{\alpha} = \frac{\alpha}{\Gamma(1-\alpha)\Gamma(\alpha)}$.

From now on $d = 1$

For $x \in \mathbb{R}$, we denote by $\mathbb{P}_x$ the push forward measure of the transformation $x + X$ under $\mathbb{P}$. Let $\widehat{X}$ be the *dual Lévy process*, defined by

$$\widehat{X}_t = -X_t, \quad t \geq 0, \qquad \text{under } \mathbb{P}.$$

The process $\widehat{X}$ is a Lévy process with characteristic exponent

$$\widehat{\Psi}(\lambda) = \Psi(-\lambda), \quad \lambda \in \mathbb{R}.$$

Lemma

For each $t > 0$, fixed, the time reversed process $\{X_{(t-s)-} - X_t,\ 0 \leq s \leq t\}$, has the same law as the dual process $\{\widehat{X}_s,\ s \leq t\}$ under $\mathbb{P}$.

Proof.

The time reversed process has independent increments, has cádlág paths. The law of $\{X_{(t-s)-} - X_t$ equals that of $-X_s$, for any $0 \leq s \leq t$, under $\mathbb{P}$. □

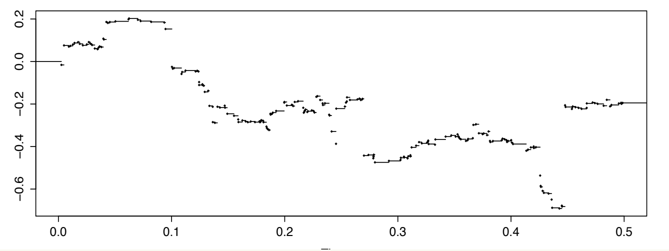

Figure: Seen from right to left the jumps change of sign, the increments are still independent and stationary.

For $t > 0$, denote the running supremum by $S_t = \sup\{0 \vee X_s, s \leq t\}$ and the running infimum by $I_t = \inf\{0 \wedge X_s, s \leq t\}$, for $t > 0$.

Lemma

For each $t > 0$ fixed, the pairs of variables $(S_t, S_t - X_t)$ and $(X_t - I_t, -I_t)$ have the same law.

Proof.

Take $\widetilde{X}_t = X_t$ and $\widetilde{X}_s = X_t - X_{(t-s)-}$ $0 \leq s < t$. Notice
$(S_t, S_t - X_t) = (\widetilde{X}_t - \widetilde{I}_t, -\widetilde{I}_t)$ a.s. By the duality lemma X and $\widetilde{X}$ have the same law. □

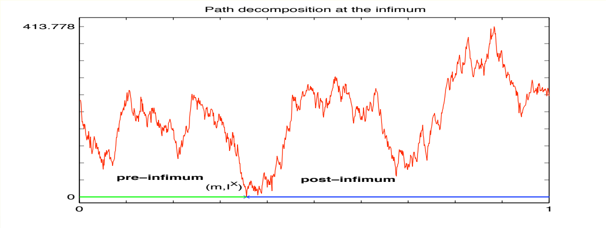

Figure: $(X_t - I_t, -I_t)$

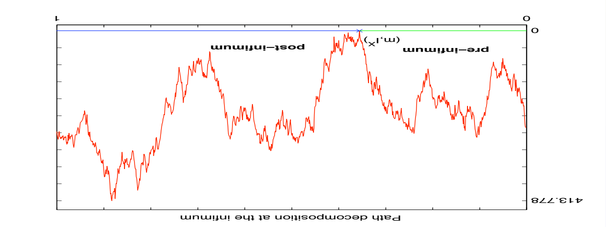

Figure: $(\widetilde{S}_t, \widetilde{S}_t - \widetilde{X}_t)$

The Wiener-Hopf factorisation-I

Let $\tau = \mathbb{e}$ be an exponential time of parameter q, and independent of X. Recall $S_t = \sup\{0 \vee X_s, s \leq t\}$ and $I_t = \inf\{0 \wedge X_s, s \leq t\}$,

$$g_t = \sup\{s < t : X_s = S_s\}, \qquad t > 0.$$

The Wiener-Hopf factorisation states

$$(\tau, X_\tau) = (g_\tau, S_\tau) \underbrace{+}_{independent} (\tau - g_\tau, X_\tau - S_\tau),$$

$$(\tau - g_\tau, X_\tau - S_\tau) \stackrel{\text{Law}}{=} (\widehat{g}_\tau, -\widehat{S}_\tau),$$

and provides a characterisation of the law of these r.v.

Theorem

The joint law of $(g_\tau, \tau - g_\tau, S_\tau, S_\tau - X_\tau)$ *is determined by*

(i) *The pairs* (g_τ, S_τ) *and* $(\tau - g_\tau, S_\tau - X_\tau)$ *are independent and infinitely divisible*

(ii) *For all* $\alpha, \beta > 0$,

$$\mathbb{E}\left(\exp\{-\alpha g_\tau - \beta S_\tau\}\right) = \exp\left(\int_0^\infty \frac{\mathrm{d}t}{t} \int_{[0,\infty[} (e^{-\alpha t - \beta x} - 1) e^{-qt}\, \mathbb{P}(X_t \in \mathrm{d}x)\right),$$

and

$$\mathbb{E}\left(\exp\{-\alpha(\tau - g_\tau) - \beta(S_\tau - X_\tau)\}\right) = \exp\left(\int_0^\infty \frac{\mathrm{d}t}{t} \int_{]-\infty,0]} (e^{-\alpha t - \beta x} - 1) e^{-qt}\, \mathbb{P}(X_t \in \mathrm{d}x)\right).$$

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The proof is based in excursion theory

The characteristic function of X_τ can be written as

$$\mathbb{E}(\exp\{i\lambda X_\tau\}) = \frac{q}{q+\Psi(\lambda)} = \Psi_+(\lambda)\Psi_-(\lambda),$$

where

$$\Psi_+(\lambda) = \mathbb{E}(\exp\{i\lambda S_\tau\}), \qquad \Psi_-(\lambda) = \mathbb{E}(\exp\{-i\lambda(S_\tau - X_\tau)\}), \qquad \lambda \in \mathbb{R}.$$

- If X is a Brownian motion

$$\frac{q}{q+\frac{\lambda^2}{2}} = \frac{\sqrt{q}}{\sqrt{q}-i\lambda}\frac{\sqrt{q}}{\sqrt{q}+i\lambda},$$

Lemma

Assume X is a real valued Lévy process. The process X reflected in the supremum $R_t = S_t - X_t$, $t \geq 0$, is a Markov process in the filtration $(\mathcal{F}_t, t \geq 0)$ and it has the Feller property.

Proof.

Let T be a finite stopping time and $s \geq 0$. We have the identity

$$\begin{aligned} S_{T+s} &= S_T \vee \sup\{X_{T+u}, 0 \leq u \leq s\} \\ &= X_T + (S_T - X_T) \vee \sup\{X_{T+u} - X_T, 0 \leq u \leq s\}. \end{aligned} \tag{10}$$

We can write

$$S_{T+s} - X_{T+s} = (S_T - X_T) \vee \sup\{X_{T+u} - X_T, 0 \leq u \leq s\} - (X_{T+s} - X_T).$$

The Markov property of X implies that the conditional law of $S_{T+s} - X_{T+s}$ given $\mathcal{F}_T$ is the same as that of $(x \vee S_s) - X_s$ under $\mathbb{P}$ with $x = S_T - X_T \geq 0$, which is the law of $S_s - X_s$ under $\mathbb{P}_{-x}$. □

Local time

For simplicity we will assume that 0 is regular upwards and downwards i.e. $\tau_0^+ = \inf\{t > 0 : X_t > 0\}$ is such that $\mathbb{P}(\tau_0^+ > 0) = 0 = \widehat{\mathbb{P}}(\tau_0^+ > 0)$. Equivalently the first return to 0 for R is zero a.s.

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General theory on Markov processes establishes that there exists a local time at 0 for R, i.e. a non-decreasing adapted process $(L_t, t \geq 0)$ such that:

- $L_0 = 0$ when $R_0 = 0$;
- $L_{t+s} = L_t + \underbrace{L_s \circ \theta_t}_{\text{shift at } t}$, for $s, t \geq 0$;
- L is the unique, up to multiplicative constants, functional that grows at the times where $R = 0$;
$$\int_0^\infty 1_{\{R_s \neq 0\}} dL_s = 0, \quad \text{a.s.}$$
- if T is a random time such that on $\{T < \infty\}$, $R_T = 0$, a.s. and the conditional law of $\{(L_{T+t} - L_T, R_t), t \geq 0\}$, given $\{T < \infty\}$, is the same as that of $\{(L_t, R_t), t \geq 0\}$ under $\mathbf{P}(|R_0 = 0)$.

- If X is a Brownian motion

$$L_t = \lim_{\epsilon \to 0+} \frac{1}{\epsilon} \int_0^t 1_{\{S_s - X_s \leq \epsilon\}} ds, \qquad t \geq 0,$$

  the limit holds uniformly over bounded intervals in probability.
- The same holds if X has no-negative jumps.
- In general there exists a function $\widehat{V}$, s.t.

$$L_t = \lim_{\epsilon \to 0+} \frac{1}{\widehat{V}(\epsilon)} \int_0^t 1_{\{S_s - X_s \leq \epsilon\}} ds, \qquad t \geq 0,$$

  the limit holds uniformly over bounded intervals in probability.
- There exists a constant $\delta \geq 0$ such that

$$L_t = \delta \int_0^t 1_{\{R_s = 0\}} ds, \qquad t \geq 0.$$

Theorem (Itô's excursion theory for R)

For $t > 0$, let $L_t^{-1} = \inf\{s > 0 : L_s > t\}$, with the assumption that $\inf \emptyset = \infty$.

- *The process of excursions $(\mathbf{e}_t, t \geq 0)$*

$$\mathbf{e}_t = \begin{cases} X_{L_{t-}^{-1}} - X_{L_{t-}^{-1}+s}, & 0 \leq s \leq L_t^{-1} - L_{t-}^{-1}, \quad \text{if } L_t^{-1} - L_{t-}^{-1} > 0 \\ \Delta, & \text{if } L_t^{-1} - L_{t-}^{-1} = 0, \end{cases}$$

is a Poisson point process with values in $\mathbb{D}^\dagger$, and characteristic measure $\overline{n}$. (Itô, 1971)

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- *Under $\overline{n}$ the process of coordinates has lifetime ζ, bears the Markov property with the same semigroup as $\widehat{X}$ killed at the time $\tau_0^- = \inf\{t > 0 : X_t < 0\}$, that is*

$$\begin{aligned} &\overline{n}\left(F(\mathbf{e}_u, u \leq t) f(\mathbf{e}_{t+s}), t + s < \zeta\right) \\ &= \overline{n}\left(F(\mathbf{e}_u, u \leq t) \widehat{\mathbb{E}}_{\mathbf{e}_t}\left(f(X_s), s < \tau_0^-\right), t < \zeta\right), \end{aligned}$$

  for any F, f measurable bounded functionals.

Theorem (Master formula)

Let $\mathcal{G}$ denote the left extrema of the excursion intervals, and for $g \in \mathcal{G}$, $d_g = \inf\{t > g : R_t = 0\}$.

$$\mathbb{E}\left(\sum_{g \in \mathcal{G}} F(X_s, s < g) H \underbrace{(S_g - X_{g+u}, u \leq d_g - g)}_{\text{excursion at time } g}\right)$$
$$= \mathbb{E}\left(\int_0^\infty dL_t F(X_s, s < t) \overline{n}\left(H(\epsilon_u, u \leq \zeta)\right)\right);$$

and

$$\mathbb{E}\left(\int_0^\infty dt F(X_s, s < t) f(X_t) 1_{\{X_t = S_t\}}\right) = \delta\, \mathbb{E}\left(\int_0^\infty dL_t F(X_s, s < t) f(X_t)\right),$$

where F, G, f *are test functionals, and the stochastic process* $(\omega, t) \mapsto F(X_s(\omega), s < t)$, *is adapted and left continuous.*

The stationary measure of the reflected process $R_t = S_t - X_t,\ t \geq 0$.

$$\mathbb{E}\left(F\left(X_t, 0 \leq t < g_s\right) H(X_{g_s+t} - X_{g_s}, 0 \leq t \leq s - g_s)\right)$$

$$= \mathbb{E}\left(\sum_{g \in \mathcal{G}} F\left(X_t, 0 \leq t < g\right) H(X_{g+t} - X_g, 0 \leq t \leq s - g) 1_{\{0 \leq g < s < d_g\}}\right) =$$

$$\mathbb{E}\left(\int_0^s \mathrm{d}L_u F\left(X_t, 0 \leq t < u\right) \int_{\mathbb{D}} \overline{n}(\mathrm{d}e) H(-e(t), 0 \leq t \leq s - u) 1_{\{s-u<\zeta\}}\right),$$

Lemma

The processes $(X_t, 0 \le t < g_\tau)$ *and* $(X_{g_\tau + t} - X_{g_\tau}, 0 \le t \le \tau - g_\tau)$ *are independent.*

Proof.

□

Lemma

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Proof.

By the compensation formula, for $s > 0$,

$$\mathbb{E}\left(F\left(X_t, 0 \le t < g_s\right) H(X_{g_s+t} - X_{g_s}, 0 \le t \le s - g_s)\right)$$

$$= \mathbb{E}\left(\sum_{g \in \mathcal{G}} F\left(X_t, 0 \le t < g\right) H(X_{g+t} - X_g, 0 \le t \le s - g) 1_{\{0 \le g < s < d_g\}}\right) =$$

$$\mathbb{E}\left(\int_0^s \mathrm{d}L_u F\left(X_t, 0 \le t < u\right) \int_{\mathbb{D}} \overline{n}(\mathrm{d}e) H(-e(t), 0 \le t \le s - u) 1_{\{s-u<\zeta\}}\right),$$

Notice that for s fixed there is no independence. □

Integrate w.r.t. $qe^{-q}ds$ to get

$$\mathbb{E}\left(F\left(X_t, 0 \le t < G_\tau\right) H(X_{G_\tau+t} - X_{G_\tau}, 0 \le t \le \tau - G_\tau)\right) =$$

$$\mathbb{E}\left(\int_0^\infty \mathrm{d}L_u e^{-qu} F\left(X_t, 0 \le t < u\right)\right)\left(\int_{\mathbb{D}} \overline{n}(\mathrm{de}) H(-e(t), 0 \le t \le \tau) 1_{\{\tau<\zeta\}}\right),$$

Conclude by normalising to get probability measures.

$(L_t, t \geq 0)$, the local time at 0 of the reflected process $S - X = (S_t - X_t,\ t \geq 0)$. We define the right continuous inverse of L by

$$L_t^{-1} = \inf\{s > 0 : L_s > t\}, \quad t \geq 0.$$

- upward ladder time process $(L_t^{-1}, t \geq 0)$,
- upward ladder height process $(H_t \equiv S_{L_t^{-1}}, t \geq 0)$.

The ladder process (L^{-1}, H) is a bivariate subordinator (possibly killed),

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The ladder process (L^{-1}, H) is a bivariate subordinator (possibly killed), whose Laplace exponent κ is given by

Fristedt's formula

for $\lambda, \mu \geq 0$,

$$\begin{aligned}\kappa(\lambda,\mu) &= -\log \mathbf{E}(\exp\{-\lambda L_1^{-1} - \mu H_1\}) \\ &= c\exp\left(\int_0^\infty \frac{\mathrm{d}t}{t}\int_{[0,\infty[}(e^{-t} - e^{-\lambda t - \mu x})\,\mathbf{P}(\xi_t \in \mathrm{d}x)\right).\end{aligned}$$

Draw the ladder height process

Lemma

The r.v. (g_τ, S_τ) *is infinitely divisible and its Laplace transform is*

$$\mathbb{E}\left(\exp\{-\alpha g_\tau - \beta S_\tau\}\right) = \kappa(q,0)/\kappa(\alpha+q,\beta), \qquad \alpha,\beta > 0.$$

Proof.

Master Formula! □

- Π will be the Lévy measure and for $x > 0$, $\overline{\Pi}^{+}(x) = \Pi(x, \infty)$.

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$$V(ds, dx) = \int_0^\infty dt \cdot \mathbb{P}(L_t^{-1} \in ds, H_t \in dx)$$

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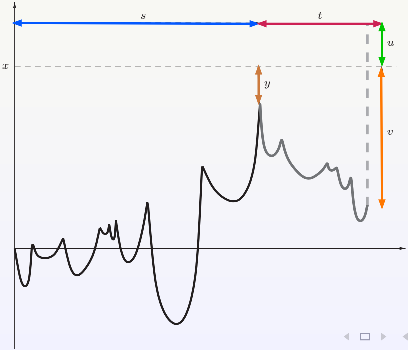
- The same construction can be done for $-X$ giving us the descending ladder height process $(\widehat{L}^{-1}, \widehat{H})$ and associated potential measure $\widehat{V}(ds, dx)$.
- The ladder processes has (amongst other things) hidden information about the distribution of $\overline{X}_t$, τ_x^+ and

$$g_t = \sup\{s < t : X_s = \overline{X}_s\}.$$

The quintuple law at first passage



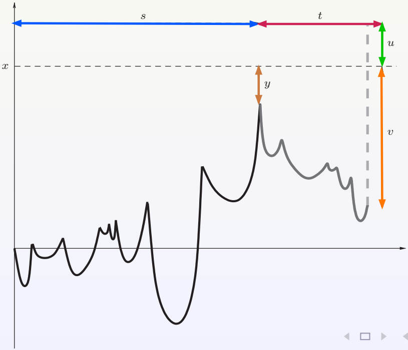
The quintuple law at first passage

Theorem (Doney and Kyprianou 2006)

For each $x > 0$ *we have on* $u > 0$, $v \geq y$, $y \in [0, x]$, $s, t \geq 0$,

$$\mathbb{P}(\tau_x^+ - g_{\tau_x^+ -} \in dt,\ g_{\tau_x^+ -} \in ds,\ X_{\tau_x^+} - x \in du,\ x - X_{\tau_x^+ -} \in dv,\ x - \overline{X}_{\tau_x^+ -} \in dy)$$
$$= V(ds, x - dy)\widehat{V}(dt, dv - y)\Pi(du + v)$$

where the equality holds up to a normalising multiplicative constant.



Suppose that X is a two-sided strictly stable process with index $\alpha \in (1,2)$ and positivity parameter $\rho = \mathbb{P}(X_t \geq 0) \in (0,1)$, then the following facts are known:

- Its jump measure is given by

$$\Pi(dx) = 1_{(x>0)} \frac{c_+}{x^{1+\alpha}} dx + 1_{(x<0)} \frac{c_-}{|x|^{1+\alpha}} dx$$

- Its renewal measures $V(dx) := V(\mathbb{R}_+, dx)$ and $\widehat{V}(x) := \widehat{V}(\mathbb{R}_+, dx)$ are known

$$V(dx) = \frac{x^{\alpha\rho-1}}{\Gamma(\alpha\rho)} dx \text{ and } \widehat{V}(dx) = \frac{x^{\alpha(1-\rho)-1}}{\Gamma(\alpha(1-\rho))} dx.$$

Corollary

The random variables $r^{-1}(U(r), O(r))$ *have a joint p.d.f.*

$$p_{\alpha\rho}(u,v) = \frac{\alpha\rho \sin \alpha\rho\pi}{\pi} (1-u)^{\alpha\rho-1} (u+v)^{-1-\alpha\rho},$$

for $0 < u < 1, v > 0$, *if* $\alpha\rho \in (0,1)$; *and is the Dirac mass at* $(0,0)$ *if* $\alpha\rho = 1$.

Lemma (Vigon 2002, Équations amicales)

Let $\widehat{V}(\mathrm{d}y) = \widehat{V}([0,\infty) \times \mathrm{d}y)$

$$\overline{\Pi}_H(x) = \int_0^\infty \widehat{V}(\mathrm{d}y)\overline{\Pi}^+(x+y),$$

$$\overline{\Pi}^+(x) = \int_{]x,\infty[} \Pi_H(\mathrm{d}y)\overline{\Pi}_{\widehat{H}}(y-x) + \widehat{d}\overline{p}(x) + \widehat{k}\overline{\Pi}_H(x),$$

where $\overline{p}(x)$ *is the density of the measure* Π_H*, which exists if* $\widehat{d} > 0$.

Proof.

Notice that

$$\overline{\Pi}_H(x) = \overline{n}\left(\epsilon_\zeta < -x, \zeta < \infty\right).$$

In the event where the excursion ends by a jump, ζ is the unique time where $\epsilon_{t-} > 0 > \epsilon_t$, this equals

$$\overline{n}\left(\sum_{0<t} 1_{\{\epsilon_{t-}>0>-x>\epsilon_{t-}+\epsilon_t-\epsilon_{t-}\}}\right),$$

By the Poissonian structure of the jumps and the compensation formula

$$\overline{n}\left(\int_0^\zeta \mathrm{d}t 1_{\{\epsilon_{t-}>0\}}\overline{\Pi}^+(x+\epsilon_{t-})\right) = \int_0^\infty \widehat{V}(\mathrm{d}y)\overline{\Pi}^+(x+y).$$

□

Spectrally negative LP

We will assume $\Pi(0,\infty)=0$, and that X is not monotone.

- $\mathbb{E}(e^{\beta X_1})<\infty$ because $\underbrace{\int_1^\infty e^{\beta x}\Pi(dx)}_{=0}+\int_{-\infty}^{-1} e^{\beta x}\Pi(dx)<\infty$
- Ψ is well defined and analytical on $\{\Im(z)\leq 0\}$, $\mathbb{E}(\exp\{\lambda X_1\})=e^{\psi(\lambda)}$,

$$\psi(\lambda)=-\Psi(-i\lambda)=a\lambda+\frac{\sigma^2}{2}\lambda^2+\int_{(-\infty,0)} e^{\lambda x}-1-\lambda x 1_{\{x<-1\}}\Pi(dx).$$

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- By Hölder's inequality ψ is convex on $[0,\infty)$, $\psi(0)=0$, $\psi(\infty)=\infty$ and $\mathbb{E}_0(X_1)=\psi'(0+)$.

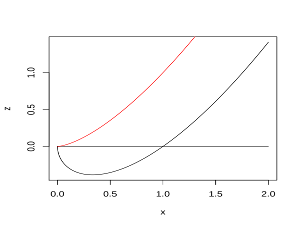

Figure: Typical shape of ψ. Black $\psi'(0+)<0$, Red $\psi'(0+)\geq 0$.

Lemma

For $q \geq 0$, let $\Phi(q)$ be the largest solution to $\psi(\lambda) = q$. The continuous increasing process $S_t = \sup\{X_s, s \leq t\}$ is the local time at 0 for the process reflected R. Its right continuous inverse

$$\tau_x^+ = \inf\{t > 0 : X_t > x\}, \qquad x \geq 0,$$

is subordinator with Laplace exponent Φ,

$$\mathbb{E}\left(\exp\{-\beta\tau_x^+\}\right) = \exp\{-x\Phi(\beta)\}, \qquad \beta \geq 0.$$

If X drifts towards $-\infty$, τ^+ is killed with rate $\Phi(0)$.

Proof.

The process $M_t = \exp\{\Phi(\beta)X_t - t\beta\}$ is a Martingale (the Wald martingale of $\Phi(\beta)$). So is the process $M_{t\wedge\tau_x^+}$, and it is bounded by $e^{\Phi(\beta)x}$. By a Dominated convergence argument we get

$$1 = \mathbb{E}\left(e^{\beta x}e^{-\beta\tau_x^+}\right), \qquad x \geq 0.$$

□

When X is Brownian motion it is a consequence of the reflection principle that τ^+ is an $1/2$-stable subordinator.

- The absence of positive jumps implies that the upward ladder height process $H_t = S_{L_t^{-1}} = t$, $t \geq 0$.

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- The downward ladder heigh process has Laplace exponent

$$\widehat{\kappa}(\alpha, \beta) = \frac{\alpha - \Psi(\beta)}{\Phi(\alpha) - \beta}, \qquad \alpha, \beta > 0.$$

Scale functions

For each $q \geq 0$, the, so-called, q-scale function $W^{(q)} : \mathbb{R} \mapsto [0, \infty)$ is defined by $W^{(q)}(x) = 0$ for $x < 0$ and elsewhere continuous and increasing satisfying

$$\int_0^\infty e^{-\beta x} W^{(q)}(x) dx = \frac{1}{\psi(\beta) - q}$$

for all β sufficiently large ($\psi(\beta) > q$).

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Scale functions are fundamental to virtually **all** fluctuation identities concerning spectrally negative Lévy processes.
Let $\tau_a^- = \inf\{t > 0 : X_t < a\}$, $\tau_b^+ = \inf\{t > 0 : X_t > b\}$, $a, b \in \mathbb{R}$. We have the classical identity

$$\mathbb{E}_x(e^{-q\tau_a^+} \mathbf{1}_{(\tau_a^+ < \tau_0^-)}) = \frac{W^{(q)}(x)}{W^{(q)}(a)}$$

for $q \geq 0$, $0 \leq x \leq a$.

Applications in:

- ruin theory (first appearance in Tackács (1966), Zolotarev (1964)),

$$\mathbb{P}_x(\tau_0^- < \infty) = 1 - \frac{W(x)}{W(\infty)}, \qquad W(\infty) = 1/\psi'(0+) \in (0, \infty).$$

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- fluctuation theory of Lévy processes,
- optimal stopping,
- optimal control,
- queuing and storage models,
- branching processes,
- insurance risk and ruin,
- credit risk,
- fragmentation.

Proof of the two sided exit formula $q=0$ and $X_t \to \infty$ a.s.

$$\mathbb{P}_x(\tau_a^+ < \tau_0^-) = \frac{W(x)}{W(a)}.$$

For $y \geq 0$ let $h_y = \sup\{(S-X)\tau_{y-}^+ + t, 0 \leq t < \tau_y^+ - \tau_{y-}^+\}$,

$$\mathbb{P}(\tau_{a-x}^+ < \tau_{-x}^-) = \mathbb{P}\left(\#\{h_y > y+x,\ y \in [0,a-x]\} = 0\right),$$

By the Poissonnian structure of the excursions this is equal to

$$\exp\left\{-\int_{[0,\infty)} dy 1_{\{y\in[0,a-x]\}} \int_{\mathbb{D}} \overline{n}(de) 1_{\{h(e)>y+x\}}\right\}$$
$$= \exp\{-\int_x^{a+x} dy \overline{n}(h>y)\}.$$

Make $a \to \infty$, to get that

$$\mathbb{P}(-I_\infty \leq x) = \mathbb{P}(\tau_{-x}^- = \infty) = \exp\{-\int_x^\infty dy \overline{n}(h>y)\},$$

and verify that this has the right Laplace transform.
For a general X and q use a change of measure. □

Let X be a spectrally negative real valued Lévy process, with unbounded variation, Laplace exponent ψ,

$$\mathbb{E}\left(\exp\{\lambda X_t\}\right) = \exp\{t\psi(\lambda)\}, \qquad \lambda \geq 0,$$

and with absolutely continuous one-dimensional distributions.
For $q \geq 0$, $W^{(q)}$ is the q-scale function of X, the continuous function with Laplace transform

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x)dx = \frac{1}{\psi(\lambda) - q}, \qquad \lambda > 0 \ s.t. \ \psi(\lambda) > q.$$

For $q \in \mathbb{C}$

$$W^{(q)}(x) = \sum_{n \geq 0} q^n W^{*n+1}(x), \qquad x \geq 0.$$

For $a > 0$ fixed, let $\tau = \inf\{t > 0 : X_t \notin [0, a]\}$, and consider the law of the process killed at τ,

$$P_t f(x) = \mathbb{E}_x \left(f(X_t), t < \tau \right), \qquad t \geq 0, x \in [0, a].$$

Theorem (Bertoin (1997))

Let $\rho = \inf\{q > 0 : W^{(-q)}(a) = 0\}$. *We have*

(i) $\lim_{t\to\infty} e^{\rho t} P_t f(x) = W^{(-\rho)}(x) \int_{[0,a]} f(y) W^{(-q)}(a-y) dy$, *for* $x \in [0, a]$, *for* f *continuous and bounded.*

(ii) $\Pi(dx) = W^{(-\rho)}(a-x)dx$ *is* ρ*-invariant,* $e^{\rho t}\Pi P_t f = \Pi f$.

(iii) $P_t W^{(-\rho)}(x) = e^{-\rho t} W^{(-\rho)}(x)$, $x \in [0, a]$.

Uses Tuominen and Tweedie (1979) theory of discrete skeletons. Results for general Lévy process where obtained by Kolb and Savov (2014) using spectral theory. Both approaches provide (abstract) exponential rate of convergence.

Bertoin's approach is based in knowledge of the q-resolvent density

$$u_q(x,y)dy = \int_0^\infty e^{-qt}\,\mathbb{P}_x\left(X_t \in dy, t < \tau\right), \quad y \in [0,a].$$

According to Suprum (1980)

$$u_q^{[0,a]}(x,y) = \frac{W^{(q)}(x)W^{(q)}(a-y)}{W^{(q)}(a)} - W^{(q)}(x-y), \qquad x,y \in [0,a].$$

ρ is also determined by

$$\rho = \inf\{q > 0 : u_{-q}^{[0,a]}(x,y) = \infty\},$$

and it is independent of x, y.

The assumption of unbounded variation and SNLP implies that any $x \in \mathbb{R}$ is regular for X.
For $x \in [0, a]$, general theory of Markov processes guarantees the existence of a local time at x, $(L_t^x, t \geq 0)$, for X killed at its first exit from $[0, a]$, and it is normalized s.t.

$$u_q(x, y) = \mathbb{E}_x \left(\int_0^\tau e^{-qt} dL_t^y \right), q \geq 0.$$

For each $x \in [0, a]$, Itô's excursion theory guarantee's the existence of an excursion measure N_x s.t. the last exit formula holds: for f measurable and positive and $t > 0$

$$\mathbb{E}_x \left(f(X_t), t < \tau \right) = \mathbb{E}_x \left(\int_0^t dL_s^x 1_{\{s<\tau\}} N_x(f(\epsilon_{t-s}), t - s < \zeta < \tau) \right).$$

(This is a consequence of the master formula in excursion theory.)

The measure

$$\int_{[0,\infty)} U^x(ds)g(s) = \mathbb{E}_x\left(\int_0^\infty dL_s^x 1_{\{s<\tau\}} g(s)\right),$$

is the renewal measure of the subordinator *inverse of the local time* $L_t^{x,-1} = \inf\{u > 0 : L_u^x > t\}$, $t > 0$,

$$\int_{[0,\infty)} U^x(ds)g(s) = \mathbb{E}_x\left(\int_0^\infty du g(L_u^{x,-1}) 1_{\{L_u^{x,-1}<\infty\}}\right).$$

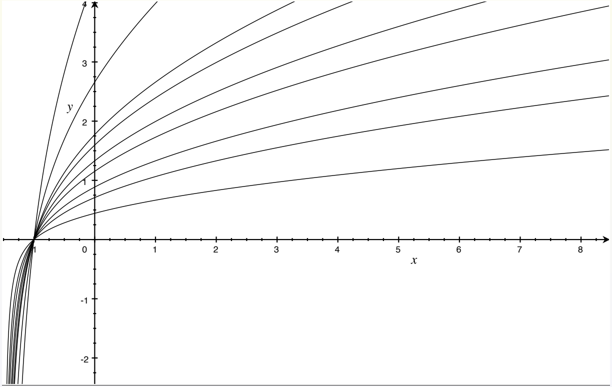
$L^{x,-1}$ is a subordinator killed at an exponential time with Laplace exponent

$$\phi_x(\lambda) = \frac{1}{u_\lambda(x,x)} = \frac{W^{(q)}(a)}{W^{(q)}(x)W^{(q)}(a-x)}, \qquad \lambda \geq 0.$$

$$\phi_x(\lambda) = N_x(\zeta > \tau) + N_x((1-e^{-\lambda\zeta})1_{\{\zeta<\tau\}}), \qquad \lambda \geq 0.$$

For $x \in [0,a]$ $\phi_x(q) = \dfrac{W^{(q)}(a)}{W^{(q)}(x)W^{(q)}(a-x)}$, $\qquad q \geq 0$;

$$\rho = \inf\{\lambda > 0 : \phi_x(-\lambda) = 0\}, \quad x \in (0,a).$$



Standard renewal theory (Cramér's estimate) guarantees that if $\exists\beta > 0$ s.t. $\phi_x(-\beta) = 0$ and $\overline{\beta}_x = \sup\{\lambda > 0 : |\phi_x(-\lambda)| < \infty\}$, then under some conditions on g

$$\left|\int_0^t U^x(ds)e^{\beta s}e^{\beta(t-s)}g(t-s) - \frac{1}{\phi_x'(-\beta)}\int_0^\infty e^{\beta s}g(s)ds\right| \leq Ke^{-\delta t},$$

for $t \to \infty$, for any $\delta < \overline{\beta}_x - \beta$. The convergence is uniform in the sense

$$\sup_{\substack{|g|\leq f \\ \int_0^\infty f(s)ds<\infty}} ||\int_0^t U^x(ds)e^{\beta s}e^{\beta(t-s)}g(t-s) - \frac{1}{\phi_x'(-\beta)}\int_0^\infty e^{\beta s}g(s)ds|| \leq K_f e^{-\delta t}$$

Recall

$$\rho = \beta = \inf\{\lambda > 0 : \phi_x(-\lambda) = 0\}.$$

This arguments gives

$$e^{\rho t}\,\mathbb{E}_x\left(f(X_t), t<\tau\right) = \int_0^t U^x(ds)e^{\rho s}e^{\rho(t-s)}N_x(f(\epsilon_{t-s}), t-s<\zeta<\tau)$$
$$\xrightarrow[t\to\infty]{} \frac{1}{\phi_x'(-\beta)}\int_0^\infty e^{\rho s}N_x(f(\epsilon_s), s<\zeta<\tau)ds.$$

The rate of convergence is exponential of order at least

$$\overline{\beta}_x = \sup\{\lambda>0 : \frac{W^{(-\lambda)}(a)}{W^{(-\lambda)}(x)W^{(-\lambda)}(a-x)}<\infty\},$$

and the convergence is uniform over bounded functions.

We have weak convergence of $e^{\rho t} P_t$ towards the measure

$$\mu_x(dy) := \frac{1}{N_x^{[0,a]}\left(\zeta e^{\rho\zeta}, \zeta < \tau\right)} \int_0^\infty ds e^{\rho s} N_x^{[0,a]}\left(1_{\{X_s \in dy\}}, s < \zeta\right),$$

whose total mass

$$h(x) := \mu_x 1 = \frac{N_x^{[0,a]}(e^{\rho\zeta} - 1, \zeta < \tau)}{\rho N_x^{[0,a]}\left(\zeta e^{\rho\zeta}, \zeta < \tau\right)} = \frac{N_x(\tau < \zeta)}{\rho N_x^{[0,a]}\left(\zeta e^{\rho\zeta}, \zeta < \tau\right)},$$

s.b. a ρ-invariant function,

$$e^{\rho t} P_t h(x) = h(x),$$

and we should also have that the

$$e^{\rho s} \mu_x P_s f = \mu_x f.$$

According to Kolb and Savov (2014), for $\epsilon > 0$ and t large enough

$$\sup_{|f|\leq 1} \sup_{x\in[0,a]} ||e^{\rho t} P_t f(x) - \frac{1}{\phi'_x(-\beta)} \int_0^\infty e^{\rho s} N_x(f(\epsilon_s), s < \zeta < T) ds|| \leq \exp\{-t(\Re\rho_2 - \rho - \epsilon)\},$$

with ρ_2 the second eigenvalue of P_t. The above argument allow us to deduce the lower bound for the spectral gap

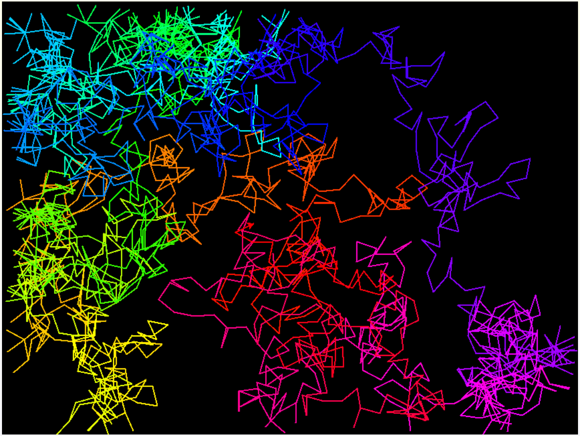
$$\overline{\beta}_{a/2} = \inf\{\lambda > 0 : W^{(-\lambda)}(a/2) W^{(-\lambda)}(a/2) = 0\} \leq \Re\rho_2 - \rho.$$

Conclusion

- The approach is relatively elementary and provides an easy way to determine a lower bound for the spectral gap.
- The method works for any Markov process with local times, it is either needed to have a good understanding of the resolvent or of the length of the excursions that do not exceed a given height. (Other examples are Ornstein-Uhlenbeck type processes, Reflected Lévy processes, spectrally one-sided spectrally negative Lévy processes...)
- The last exit formula seems to have been unexploited. It allows to derive results about the rate of convergence to a stationary measure when the process is recurrent, results about polynomial rate of convergence when there are no exponential moments for the inverse local time, etc..

Lecture 3: Markov additive processes

The random walker carries many bottles with different spirits, each spirit fixes a different dynamic. He choses a bottle, drinks a sip of it and walks; after a random time (he does not remember when he took the last sip), he choses again a bottle, drinks a sip and continues walking... He dies if he drinks from the adulterated bottle Δ.[1]



[1] Any resemblance to reality is pure coincidence.

Let E be a non-empty set, Δ is some isolated state and $E \cup \{\Delta\}$

Definition (Neveu (1961), Çinlar (1972))

A *Markov additive process (MAP)* is an $E \times \mathbb{R}$-valued strong Markov process $\{(((J,\xi)_t, t \geq 0), \mathbb{P}_{\varphi,x}\}$ with cemetery state (Δ, ∞), lifetime ζ, such that

(i) the paths of $((J,\xi)_t, t \geq 0)$ are right continuous on $[0,\infty)$, have left-limits and are quasi-left continuous on $[0,\zeta)$;

(ii) J is a strong Markov process: the future is independent of the past given the present;

(iii) for any $(\varphi, z) \in E \times \mathbb{R}$, $t, s \geq 0$ and $f : E \times \mathbb{R} \to \mathbb{R}$ measurable and positive

$$\mathbb{P}_{\varphi,z}\left(f(J_{t+s}, \xi_{t+s} - \xi_t), t+s < \zeta | \mathcal{F}_t\right) = \mathbb{P}_{J_t,0}\left(f(J_s, \xi_s), s < \zeta\right) 1_{\{t<\zeta\}};$$

if at time t, we observe the random walker from its actual position, the movement looks the same as the original one with starting dynamic the one valid at time t.

E countable

When E is finite or countable the process J is a Markov chain that describes the phases of the process and ξ evolves as a concatenation of independent Lévy processes $(\xi^i, i \in E)$ shifted by an independent sequence of r.v. $(U^n_{i,j}, (i,j) \in E \times E, n \geq 1)$.

- J starts in state j, ξ moves as ξ^j,
- at an exponential time T_1 of parameter q_j, J jumps to a new position, say k, with probability $p_{j,k}$, and stays there for an exponential time of parameter q_k,
- at time T_1, ξ jumps from position ξ_{T_1-} to position $\xi_{T_1-} + U^1_{j,k}$ and from there it evolves as ξ^k
- and so on

Notice ξ has jumps coming from each Lévy process and from J. Conditionally on J, ξ has jumps at the fixed times $(T_n, n \geq 1)$.

E finite

When E is finite, the dynamics of (J, ξ) are determined by:

- the infinitesimal generator of J, say $Q = (q_{i,j}, i, j \in E)$, $(q_{i,i} = 0, i \in E)$.
- the laws of $(U_{i,j}, i, j \in E)$ say $F_{i,j}(dy) = \mathbb{P}(U_{i,j} \in dy)$,
- the characteristic exponents of $(\xi^j, j \in E)$, say $(\Psi_j, j \in E)$
  $\mathbb{E}\left(\exp\{\lambda \xi^j_t\}\right) = \exp\{t\Psi_j(\lambda)\}$,

The transition semigroup of (J, ξ) is characterised through its matrix exponent $K(\lambda) = (K_{i,j}(\lambda), i, j \in E)$ as

$$F^{(t)}_{i,j}(\lambda) = \mathbb{E}_i\left(\exp\{\lambda \xi_t\} 1_{\{J_t = j\}}\right) = \exp\{tK(\lambda)\}_{i,j}, \qquad i, j \in E, t \geq 0,$$

where

$$K(\lambda) = Q + (\Psi_j(\lambda))_{\mathsf{diag}} + \left(q_{i,j}\left(\mathbb{E}\left(\exp\{\lambda U_{i,j}\}\right) - 1\right)\right)_{i,j \in E}.$$

See Asmussen's book *Applied Probability and Queues*.

Lemma (Kaspi (1982), Palmowski et al. (2011), Ivanovs (2015) Kyprianou et al. (2016))

Assume that the duality condition is satisfied and that E *is finite. Let* $\mathbf{e}_q$ *an independent exponential r.v. of parameter* q, $\underline{G}_q = \sup\{s < \mathbf{e}_q : \xi_s = \underline{\xi}_s\}$.

(i) *The pairs of random variables* $(\underline{G}_q, \underline{\xi}_{\mathbf{e}_q})$ *and* $(\mathbf{e}_q - \underline{G}_q, \xi_{\mathbf{e}_q} - \underline{\xi}_{\mathbf{e}_q})$ *are conditionally independent given* $(J_{\underline{G}_q -}, J_{\underline{G}_q})$.

(ii) *The conditional laws are characterised through the* q*-potentials of the excursion measures from the supremum and the infimum, respectively.*

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Can this result be extended to non-countable case? What else can be said?

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Can this result be extended to non-countable case? What else can be said? Yes, by means of the theory of exit systems of Maissoneuve.

Lemma (Maissoneuve (1982), Çinlar and Kaspi (1982), Kaspi (1983))

There exists an additive functional $\underline{\mathcal{L}}$, *with* 1-*potential smaller than* 1, *carried by* $E \times \mathbb{R} \times 0$ *and a kernel* P^* *from* $(E \times \mathbb{R} \times \mathbb{R}, \mathcal{E} \times \mathcal{R} \times \mathcal{R})$ *into* $(\Omega, \mathcal{F})$ *such that*

- $P^*_{\varphi,x,y}\left(1 - e^{-R}\right) \leq 1$ *for all* (φ, x, y)
- $\mathbb{E}.\left(\sum_{g \in G} Z_g H \circ \theta_g\right) = \mathbb{E}.\left(\int_0^\infty d\underline{\mathcal{L}}_s Z_s P^*_{J_s,\xi_s,U_s}(H)\right)$, *where* Z *is any positive predictable process,* H *any measurable and bounded functional and* G *is the set of left end points of intervals that are contiguous to* M.

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The excursion measure $\underline{N}_j$ *is the image measure of* $P^*_{j,0,0}$ *under the mapping that stops the path at the end of the first excursion:* $(J, \xi, \xi - I)_{\cdot \wedge R}$.
Under $\underline{N}_j$ *the excursion process has the Markov property with the same semigroup as* (J, ξ) *killed when it passes below zero.*

Theorem (Time of minimum and last exit formula)

Let $U_j^-(d\varphi, dr, dz)$ *be the potential measure of the downward ladder process*

$$U_j^-(d\varphi, dr, dz) = \mathbb{E}_j\left(\int_0^\infty ds 1_{\{J_{\underline{\tau}_s}\in d\varphi, \underline{\tau}_s \in dr, \underline{H}_s \in dz, \underline{\tau}_s < \infty\}}\right), \varphi \in E, r, z \in [0,\infty),$$

and $\underline{G}_t = \sup\{s < t : \xi_s = \underline{\xi}_s\}$.

(i) *The downward ladder measure characterizes* $(J_{\underline{\tau}}, \underline{\tau}, \underline{H})$

(ii) *For any* $f : E \times \mathbb{R} \to \mathbb{R}^+$ *measurable*

$$\mathbb{E}_{j,x}\left(f(J_t, \xi_t) 1_{\{\underline{G}_t < t\}}\right) = \int_{E\times[0,t]\times\mathbb{R}^+} U_j^-(d\varphi, dr, dz) \underline{N}_\varphi\left(f(J_{t-r}, x - z + \xi_{t-r}) 1_{\{t-r<\zeta\}}\right)$$

(iii) *Let* $\tau_0^- = \inf\{t > 0 : \xi_t < 0\}$.

$$\mathbb{E}_{j,x}\left(f(J_t, \xi_t) 1_{\{\underline{G}_t < t\}} 1_{\{t < \tau_0^-\}}\right) = \int_{E\times[0,t]\times[0,x)} U_j^-(d\varphi, dr, dz) \underline{N}_\varphi\left(f(J_{t-r}, x - z + \xi_{t-r}) 1_{\{t-r<\zeta\}}\right)$$

The down(up)ward ladder measure is a key element in the fluctuation theory, is it possible to obtain this measure or its marginals explicitly? In some cases, yes, one may

(i) determine and invert the Laplace transform

(ii) for stable processes, via some explicit identities where the measure plays a roll.

Theorem (Kyprianou, R. Şengül (2016))

If (J,ξ) is the MAP associated to a 1 dimensional stable process, the up(down)ward ladder height potential densities are as follows.
$\alpha\in(0,1)$: *for* $x\geq 0$,

$$\boldsymbol{u}^+(x) \propto \begin{pmatrix} \frac{\Gamma(1-\alpha\hat\rho)}{\Gamma(\alpha\rho)}(1-e^{-x})^{\alpha\rho-1}(1+e^{-x})^{\alpha\hat\rho} & \frac{\Gamma(1-\alpha\hat\rho)}{\Gamma(\alpha\rho)}(1-e^{-x})^{\alpha\rho}(1+e^{-x})^{\alpha\hat\rho-1} \\ \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat\rho)}(1-e^{-x})^{\alpha\hat\rho}(1+e^{-x})^{\alpha\rho-1} & \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat\rho)}(1-e^{-x})^{\alpha\hat\rho-1}(1+e^{-x})^{\alpha\rho} \end{pmatrix}$$

and

$$\boldsymbol{u}^-(x) \propto \begin{pmatrix} \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat\rho)}(e^{x}-1)^{\alpha\hat\rho-1}(e^{x}+1)^{\alpha\rho} & \frac{\sin(\alpha\pi\hat\rho)\Gamma(1-\alpha\rho)}{\sin(\alpha\pi\rho)\Gamma(\alpha\hat\rho)}(e^{x}-1)^{\alpha\hat\rho}(e^{x}+1)^{\alpha\rho-1} \\ \frac{\sin(\alpha\pi\rho)\Gamma(1-\alpha\hat\rho)}{\sin(\alpha\pi\hat\rho)\Gamma(\alpha\rho)}(e^{x}-1)^{\alpha\rho}(e^{x}+1)^{\alpha\hat\rho-1} & \frac{\Gamma(1-\alpha\hat\rho)}{\Gamma(\alpha\rho)}(e^{x}-1)^{\alpha\rho-1}(e^{x}+1)^{\alpha\hat\rho} \end{pmatrix}.$$

While for $\alpha=1$ *and symmetric: for* $x\geq 0$,

$$\boldsymbol{u}^+(x)=\boldsymbol{u}^-(x)\propto\begin{pmatrix}(1-e^{-x})^{-1/2}(1+e^{-x})^{1/2} & (1-e^{-x})^{1/2}(1+e^{-x})^{-1/2}\\ (1-e^{-x})^{1/2}(1+e^{-x})^{-1/2} & (1-e^{-x})^{-1/2}(1+e^{-x})^{1/2}\end{pmatrix}.$$

The constant vector c^+, *resp.* c_-, *is determined by solving* $M^{\pm}c^{\pm}=\mathbf{1}$, *for the matrices* $M^{\pm}:=\int_{[0,\infty)}dx\boldsymbol{u}^{\pm}(x)$.

Theorem (Kyprianou, R. Şengül (2016))

$\alpha \in (1,2)$: *for* $x \geq 0$,

$$\boldsymbol{u}^{+}(x)$$
$$\propto \frac{\alpha-1}{2}\begin{pmatrix}(1-e^{-x})^{\alpha\rho-1}(1+e^{-x})^{\alpha\hat\rho} & (1-e^{-x})^{\alpha\rho}(1+e^{-x})^{\alpha\hat\rho-1}\\ (1-e^{-x})^{\alpha\hat\rho}(1+e^{-x})^{\alpha\rho-1} & (1-e^{-x})^{\alpha\hat\rho-1}(1+e^{-x})^{\alpha\rho}\end{pmatrix}$$
$$-\frac{(\alpha-1)^2}{2(\lambda+\alpha-1)}\begin{pmatrix}(1-e^{-x})^{\alpha\rho-1}(1+e^{-x})^{\alpha\hat\rho-1} & (1-e^{-x})^{\alpha\rho-1}(1+e^{-x})^{\alpha\hat\rho-1}\\ (1-e^{-x})^{\alpha\hat\rho-1}(1+e^{-x})^{\alpha\rho-1} & (1-e^{-x})^{\alpha\hat\rho-1}(1+e^{-x})^{\alpha\rho-1}\end{pmatrix}$$

and

$$\boldsymbol{u}^{-}(x)$$
$$\propto \frac{\alpha-1}{2}\begin{pmatrix}(e^{x}-1)^{\alpha\hat\rho-1}(e^{x}+1)^{\alpha\rho} & \frac{\sin(\alpha\pi\hat\rho)}{\sin(\alpha\pi\rho)}(e^{x}-1)^{\alpha\rho}(e^{x}+1)^{\alpha\rho-1}\\ \frac{\sin(\alpha\pi\rho)}{\sin(\alpha\pi\hat\rho)}(e^{x}-1)^{\alpha\rho}(e^{x}+1)^{\alpha\hat\rho-1} & (e^{x}-1)^{\alpha\rho-1}(e^{x}+1)^{\alpha\hat\rho}\end{pmatrix}$$
$$-\frac{(\alpha-1)^2}{2(\lambda+\alpha-1)}\begin{pmatrix}(e^{x}-1)^{\alpha\hat\rho-1}(e^{x}+1)^{\alpha\rho-1} & \frac{\sin(\alpha\pi\hat\rho)}{\sin(\alpha\pi\rho)}(e^{x}-1)^{\alpha\hat\rho-1}(e^{x}+1)^{\alpha\rho-1}\\ \frac{\sin(\alpha\pi\rho)}{\sin(\alpha\pi\hat\rho)}(e^{x}-1)^{\alpha\rho-1}(e^{x}+1)^{\alpha\hat\rho-1} & (e^{x}-1)^{\alpha\rho-1}(e^{x}+1)^{\alpha\hat\rho-1}\end{pmatrix}.$$

Idea of proof for $\alpha \in (0,1)$

Let $X_{\underline{m}}$ be the point of closest reach of the origin. We observe that

$$\mathbb{P}_{e^x}(X_{\underline{m}} > 1) = \mathbb{P}_{1,x}(\tau_0^-(\xi) = \infty; J_{\underline{G}_\infty} = 1) = U^-_{1,1}(0,x)\underline{N}_1(\zeta = \infty),$$

with

$$U^-_{1,1}(0,x) = \int_{E\times[0,\infty)\times[0,x)} U^-_1(d\varphi, dr, dz)1_{\{\varphi=1\}}$$

Lemma (Blumenthal, Getoor and Ray (1961); Kyprianou, Pardo and Watson (2014))

Assume $\alpha \in (0,1)$. *Let* $\tau_{(-1,1)} := \inf\{t \geq 0 : |X_t| < 1\}$. *We have that, for* $x > 1$,

$$\mathbb{P}_x(\tau_{(-1,1)} = \infty) = \Phi(x),$$

$$\Phi(x) = \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha)} \int_0^{(x-1)/(x+1)} t^{\alpha\hat{\rho}-1}(1-t)^{-\alpha}\, dt.$$

Furthermore, the invariance under translation and the scaling property implies

$$\mathbb{P}_x(\tau_{(-u,v)} = \infty) = \Phi\left(\frac{2x+u-v}{u+v}\right)$$

Let $\underline{m}^+$ and $\underline{m}^-$ be the times when X is at the closest point to the origin on the positive and negative side of the origin, respectively. Thus

$$\mathbb{P}_x(|X_{\underline{m}^-}| > u; X_{\underline{m}^+} > v) = \mathbb{P}_x(\tau_{(-u,v)} = \infty) = \Phi\left(\frac{2x+u-v}{u+v}\right),$$

The point of closest reach of the origin $X_{\underline{m}}$ on $(0,\infty)$ has a law

$$\frac{\mathbb{P}_x(X_{\underline{m}} \in dz)}{dz} = -\frac{\partial}{\partial v}\mathbb{P}(|X_{\underline{m}^-}| > z; X_{\underline{m}^+} > v)|_{v=z}.$$

From there we easily determine the value of

$$U^-_{1,1}(0,x)\underline{N}_1(\zeta = \infty) = \mathbb{P}_{x,1}(\tau_0^-(\xi) = \infty; J_{\underline{G}_\infty} = 1) = \mathbb{P}_{e^x}(X_{\underline{m}} > 1).$$

Differentiating we get u^-.
u^+ is obtained by using the some duality relations. For $\alpha \in (1,2)$ we introduce the point of furthest reach up to first hitting of 0. For $\alpha = 1$ we use results on the two sided exit problem.

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