Construction of geometric rough path

Based on a joint work-in-progress with L. Zambotti (UPMC)

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Motivation

Consider the Itô SDE $\mathrm{d}X_t = f(X_t)\,\mathrm{d}B_t$ with $(B_t)_{t\geq 0}$ a standard Brownian motion. The solution map (Itô map) is only measurable in general.

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Worse: *there is no* Banach space $\mathscr{B} \subset C^0$ such that Brownian paths are a. s. in $\mathscr{B}$ and $I(y,x)_t = \int_0^t y_s \dot{x}_s\,\mathrm{d}s$ extends continuously from $C^0 \times C^1$ to $\mathscr{B} \times \mathscr{B}$ (T. Lyons, 1991).

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However: let $\{\rho_\varepsilon\}$ be nice mollifiers and define $B_t^\varepsilon = \int_0^t \rho_\varepsilon(t-s)\,\mathrm{d}B_s$.

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Theorem (Wong–Zakai, 1964)

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There are reasonable approximations for which X^ε *does not* converge.

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Interpretation depends on the regularity of x. OK if $x \in C^1$.

Theorem (Young)

The integral map $I : C^0 \times C^1 \to C^1$ has a unique continuous extension to $C^\alpha \times C^\beta \to C^\beta$ iff $\alpha + \beta > 1$. It is the unique function satisfying $I_0 = 0$ and

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If $x \in C^\alpha$ for $\alpha > \frac{1}{2}$ interpret as $u_t = u_0 + I(f(u), x)_t$.

Problem: Brownian paths fall outside of this scope.

The remainder

If $x \in C^\alpha$ for $\alpha > \frac{1}{2}$ the equation is equivalent to

$$u_t - u_s = f(u_s)(x_t - x_s) + o(|t - s|).$$

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Set $(\delta I)_{st} = I_t - I_s$, $(\Omega R)_{sut} = R_{st} - R_{su} - R_{ut}$. These operators satisfy $\Omega\delta = 0$ and $\ker\Omega = \operatorname{im}\delta$ (exercise).

Characterisation of Young's integral

The existence of I such that

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Gubinelli calls I the *integral*, $A_{st} = y_s(x_t - x_s)$ the *germ*, R the *remainder*.

Beyond Young: The Sewing Map

Theorem (Gubinelli, 2004)

Given a germ A such that $|(\Omega A)_{sut}| \lesssim |u-s|^{\alpha}|t-u|^{\beta}$ *for* $\alpha+\beta>1$, *there exists a remainder R such that* $\Omega R = \Omega A$ *and* $|R_{st}| \lesssim |t-s|^{\alpha+\beta}$.

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Disadvatange: does not solve the problem for low regularity. If $x \in C^{\alpha}$ for $\alpha < \frac{1}{2}$ then $A_{st} = y_s(\delta x)_{st}$ satisfies only $|(\Omega A)_{sut}| \lesssim |u-s|^{\alpha}|t-u|^{\alpha}$.

Beyond Young: Rough paths

If $x \in C^\alpha$ for $\alpha > \frac{1}{2}$ and $\mathrm{d}u_t = f(u_t)\,\mathrm{d}x_t$ for $f \in C^2$ (say), Taylor expansion gives

$$(\delta u)_{st} = f(u_s)(\delta x)_{st} + f(u_s)f'(u_s)\int_s^t (\delta x)_{sr}\,\mathrm{d}x_r + o(|t-s|)$$

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The function $\mathbb{X}_{st} = \int_s^t (\delta x)_{sr}\,\mathrm{d}x_r$ satisfies $(\Omega\mathbb{X})_{sut} = (\delta x)_{su}(\delta x)_{ut}$ and $|\mathbb{X}_{st}| \lesssim |t-s|^{2\alpha}$.

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Definition

Given $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ and $x \in C^\alpha$, we call a pair $(\delta x, \mathbb{X})$ a rough path over x *if $(\Omega\mathbb{X})_{sut} = (\delta x)_{su}(\delta x)_{ut}$ and $|\mathbb{X}_{st}| \lesssim |t-s|^{2\alpha}$.*

If $x \in C^\alpha$ the germ $A_{st} = y_s(\delta x)_{st} + y'_s \mathbb{X}_{st}$ satisfies the hypothesis of the Sewing Lemma for suitable (y, y'), called *controlled paths*.

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Existence? We can set $\mathbb{X}_{st} = \frac{1}{2}(x_t - x_s)^2$. This is a natural choice since $\int_s^t (\delta x)_{sr}\,\mathrm{d}x_r = \frac{1}{2}(x_t - x_s)^2$ if $\alpha > \frac{1}{2}$ (integration by parts).

Relation with generalised Taylor expansions.

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The equation is recast as $(\delta u)_{st} = f(u_s)(\delta x)_{st} + f(u_s) f'(u_s) \mathbb{X}_{st} + o(|t - s|)$ for any $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$. This is called a *Rough differential equation*.

Some remarks

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An example: one-dimensional Brownian Motion

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For every controlled couple (y, y') there is a unique $I \in C^\alpha$ satisfying the previous estimates. If the Stratonovich integral $\int_0^{\cdot} y_s \circ \mathrm{d}B_s$ is well defined, it is equal to I.

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Set $\mathbb{X}'_{st} = \frac{1}{2}[(B_t - B_s)^2 - (t-s)]$. Again, for all $\alpha < \frac{1}{2}$, $|\mathbb{X}'_{st}| \lesssim |t-s|^{2\alpha}$.

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More dimensions

It is interesting to extend the previous setting to functions $x : [0, 1] \to \mathbb{R}^d$.

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Again based on the Taylor expansion

$$(\delta u^i)_{st} = \sum_{j=1}^{d} f_{ij}(u_s)(\delta x^i)_{st} + \sum_{j,k=1}^{d} \sum_{\ell=1}^{m} f_{\ell j}(u_s) \partial_\ell f_{ik}(u_s) \int_s^t (\delta x^j)_{sr} \, \mathrm{d}x^k_r + o(|t - s|).$$

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Therefore, the existence of rough paths over a given path $x : [0,1] \to \mathbb{R}^d$ is not obvious.

Another example: d-dimensional Brownian Motion

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In this case the off-diagonal terms are defined using stochastic calculus. Since $(\Omega\mathbb{X}^{ij})_{sut} \lesssim |u-s|^\alpha |t-u|^\alpha$ we cannot use the Sewing map to find them.

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The problem is to find the antisymmetric part $\mathbf{A}^{ij}$ such that

$$(\Omega \mathbf{A}^{ij})_{sut} = \tfrac{1}{2}\Big((\delta x^i)_{su}(\delta x^j)_{ut} - (\delta x^j)_{su}(\delta x^i)_{ut}\Big).$$

Theorem (Lyons–Victoir, 2007)

Given an α-Hölder path $x : [0,1] \longrightarrow \mathbb{R}^d$, for every $\varepsilon > 0$ there is a geometric rough path of regularity $\alpha - \varepsilon$ over x.

The construction is explicit in the case $d = 2$, $\alpha \in [\frac{1}{3}, \frac{1}{2})$.

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Harder if $\alpha \leq \frac{1}{3}$ since we have to construct not only a 2-tensor (matrix), but also higher lever tensors (Taylor expansion). Algebraic constraints are not as simple as antisymmetry for higher levels.

Theorem (Lyons–Victoir, 2007)

Given an α-Hölder path $x : [0,1] \longrightarrow \mathbb{R}^d$, for every $\varepsilon > 0$ there is a geometric rough path of regularity $\alpha - \varepsilon$ over x.

The construction is explicit in the case $d = 2$, $\alpha \in [\frac{1}{3}, \frac{1}{2})$.

Harder if $\alpha \leq \frac{1}{3}$ since we have to construct not only a 2-tensor (matrix), but also higher lever tensors (Taylor expansion). Algebraic constraints are not as simple as antisymmetry for higher levels.

Theorem (T.–Zambotti, 2018+)

The geometric rough path given by the Lyons–Victoir theorem can be explicitly constructed via iteration.

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The pair $(\delta x, \mathbf{A})$ takes values in $\mathfrak{g}^2(\mathbb{R}^d) = \mathbb{R}^d \oplus \mathfrak{so}(d)$. Lie algebra with bracket $[x + \mathbf{A}, y + \mathbf{B}] = x \otimes y - y \otimes x$ so $(\Omega \mathbf{A})_{sut} = \frac{1}{2}[(\delta x)_{su}, (\delta x)_{ut}]$.

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For Brownian motion, $\mathbf{A}^{ij}_{st} = \frac{1}{2}\int_s^t \Big[(B^i_u - B^i_s)\,\mathrm{d}B^j_u - (B^j_u - B^j_s)\,\mathrm{d}B^i_u\Big]$ a. s. satisfies the above. Relation to construction of the Lévy area process.

The free nilpotent Lie algebra and group

The space $\mathfrak{g}^2(\mathbb{R}^d)$ is the free step-2 nilpotent Lie algebra: $[X, [Y, Z]] = 0$ for any $X, Y, Z \in \mathfrak{g}^2(\mathbb{R}^d)$.

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Algebra structure on $T^{(2)}(\mathbb{R}^d) = \mathbb{R} \oplus \mathbb{R}^d \oplus \mathbb{R}^{d\times d}$ by

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The set $T_1^{(2)} = \{X \in T^{(2)}(\mathbb{R}^d) : \pi_0(X) = 1\}$ is a group with inverse $(1 + x + \mathbf{X})^{-1} = 1 - x - \mathbf{X} + x \otimes x$ and identity $1 \in \mathbb{R}$.

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There is $\exp(x + \mathbf{A}) = 1 + x + \mathbf{A} + \frac{1}{2}x \otimes x$ and $\log(1 + x + \mathbf{X}) = x + \mathbf{X} - \frac{1}{2}x \otimes x$ such that $\exp^{-1} = \log$ and $\exp(\mathfrak{g}^2(\mathbb{R}^d)) = G^2(\mathbb{R}^d)$ is a Lie subgroup of $T_1^{(2)}$.

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General regularity

For Brownian motion $\exp((\delta x)_{st} + \mathbf{A}_{st}) = 1 + (\delta x)_{st} + \mathbb{X}_{st}$ with $\mathbb{X}^{ij}_{st} = \int_s^t (\delta B^i)_{su} \circ \mathrm{d}B^j_u$.

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In general rough path definition gives

$(1 + (\delta x)_{su} + \mathbb{X}_{su})(1 + (\delta x)_{ut} + \mathbb{X}_{ut}) = 1 + (\delta x)_{st} + \mathbb{X}_{st}$ and $1 + (\delta x)_{tt} + \mathbb{X}_{tt} = 1$.

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Definition

A geometric rough path is a path $g : [0,1]^2 \to G^N(\mathbb{R}^d)$ *such that* $g_{tt} = 1$, $g_{su} g_{ut} = g_{st}$ *and* $|\pi_k(g_{st})| \lesssim |t-s|^{k\alpha}$.

The “cutoff” level $N = \lfloor \alpha^{-1} \rfloor$ comes from the Sewing Lemma.

The Baker–Campbell–Hausdorff formula

The product in $G^N(\mathbb{R}^d) = \exp(\mathfrak{g}^N(\mathbb{R}^d))$ is described in terms of $\mathfrak{g}^N(\mathbb{R}^d)$ by the

Theorem (Baker–Campbell–Hausdorff, 1906)

Let $X, Y \in \mathfrak{g}^N(\mathbb{R}^d)$. Then there is a $Z \in \mathfrak{g}^N(\mathbb{R}^d)$, depending on X and Y, such that $\exp(X)\exp(Y) = \exp(Z)$.

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The first terms are (Dynkin, 1947)

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] - \frac{1}{24}[Y,[X,[X,Y]]] + \frac{1}{720}[Y,[Y,[Y,[X,Y]]]] + \frac{1}{720}[X,[X,[X,[X,Y]]]] + \cdots$$

Main result and sketch of proof

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Look for $\mathbf{A} \in \mathfrak{so}(d)$ such that $g^{(2)}_{st} = \exp((\delta x)_{st} + \mathbf{A}_{st})$ satisfies Chen's rule and the analytical constraint.

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This is equivalent to $(\Omega \mathbf{A})_{sut} = \frac{1}{2}[(\delta x)_{su}, (\delta x)_{ut}]$ by the BCH formula.

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To construct **A**, define it first on the dyadics **making some choices**. A Kolmogorov style argument extends **A** to $[0, 1]^2$ with the required regularity.

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Needs a notion of metric on $G^2(\mathbb{R}^d)$.

Higher levels

Iterate: look for a 3-tensor $\mathbf{B}$ such that $g_{st}^{(3)} = \exp((\delta x)_{st} + \mathbf{A}_{st} + \mathbf{B}_{st})$ satisfies the definition.

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The BCH formula and Chen's rule give

$$(\Omega\mathbf{B})_{sut} = \frac{1}{2}[(\delta x)_{su}, \mathbf{A}_{ut}] + \frac{1}{2}[\mathbf{A}_{su}, (\delta x)_{ut}] + \frac{1}{12}[(\delta x)_{su}, [(\delta x)_{su}, (\delta x)_{ut}]] - \frac{1}{12}[(\delta x)_{ut}, [(\delta x)_{su}, (\delta x)_{ut}]].$$

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Define $\mathbf{B}$ on the dyadics keeping this identity and extend as before.

Continue up to level $N = \lfloor \alpha^{-1} \rfloor$. The construction ensures at each step that $g^{(n)} \in G^n(\mathbb{R}^d)$ satisfies the definition.

Main difficulties for the proof

- Need an explicit expression for all the terms in the BCH expansion. A simple enough combinatorial formula was proven by C. Reutenauer in 1986.

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- From that expresion, we need to prove the required estimates in order to use an analytical lemma in Lyons–Victoir.

Perspectives

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- Paths of different Hölder exponents ✓.

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Thanks for your attention