Mean Li-Yorke chaos and mean proximality

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- We say $x \in X$ is a **transitive point** if the orbit of x is dense in X.

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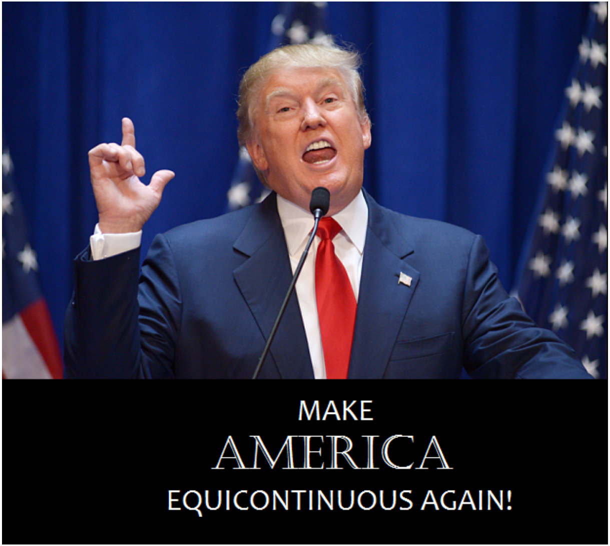
- "*Similar colored should stay with similar colored.* "
- Donald Trump's philosphy

Make

- 

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  Fidel Castro

Chaos/disorder

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- Originally a TDS was defined to be **Devaney chaotic** if it is transitive, sensitive, and has dense periodic points (it was shown later that the sensitivity hypothesis can be removed).

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- Devaney chaos$\Longrightarrow$Li-Yorke Chaos (Huang-Ye '02)

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- One of the motivations of mean definitions are the connections with ergodic theory, e.g.
  A minimal TDS is not mean sensitive iff Continuous eigenfunctions span L^2 (Downarowicz-Glasner '16, Li-Tu-Ye '15).
  An ergodic measure μ has pure point spectrum iff (X, T) is not $\mu-$mean sensitive (GR' 16).

Mean Li-Yorke chaos

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  This is also known as DC2 chaos.

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- "Mean devaney" defined as, transitive *mean* sensitive with dense periodic points does not imply mean Li-Yorke chaos (Falniowski-Kulczycki-Kwietniak-Li '15)

Theorem

Theorem (GR-Jin)

If a TDS (X, T) is mean sensitive and there is a mean proximal pair consisting of a transitive point and a periodic point, then (X, T) is mean Li-Yorke chaotic.

- With this result it is not difficult to construct mean Li-Yorke chaotic systems (with zero entropy).

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- This implies that x and 0^∞ are mean proximal.

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  There are Devaney chaotic systems with positive entropy that are not mean sensitive.
- Topologically weakly mixing TDS are not necessarily mean sensitive (Li-Tu-Ye '15).

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Theorem (GR-Jin)

If (X, T) is mean proximal then it does not have any mean Li-Yorke pairs.

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- When a mental system is suffering with ill health, it may find peace by connecting with more frequency with what is constant in the universe, its unity.
  Francisco Varela y Humberto Maturana *El árbol del conocimiento (1984)*

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- We first prove every invariant measure λ of $(X \times X, T \times T)$ is supported on the diagonal $(\lambda(\Delta_X) = 1)$.
- This implies that every invariant measure μ is supported on a fixed point.
- Nonetheless, since it is proximal, it only contains one fixed point; thus (X, T) is uniquely ergodic.

Local mean equicontinuity

- In '00 Glasner and Weiss introduced the concept of local equicontinuity. A TDS (X, T) is **locally equicontinuous (LE)** if for every $x \in X$ we have that $\overline{orb(x, T)}$ is not sensitive.

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Theorem (GR-Li-Zhang)

LME systems have zero topological entropy. Contrary to LE systems, ergodic measures on LME systems may be supported on non-minimal subsystems