Criterion for the exponential convergence of conditioned processes

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1. Absorbed processes and quasi-stationary distributions

1a. Absorbed processes

Example 1 : Process on $[0, +\infty[$ absorbed at 0

$\rightarrow$ $\partial = 0$ unique absorbing point

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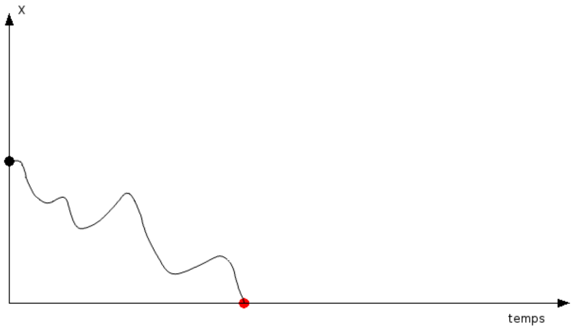

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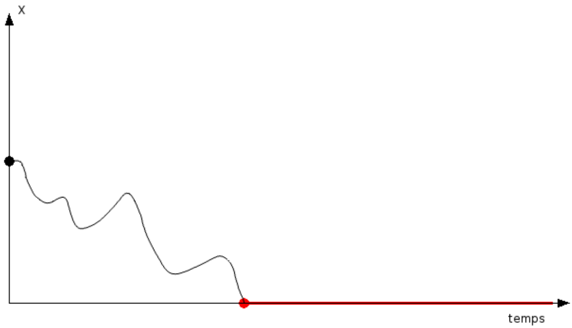

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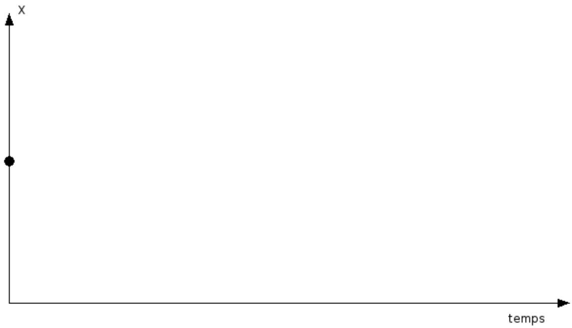

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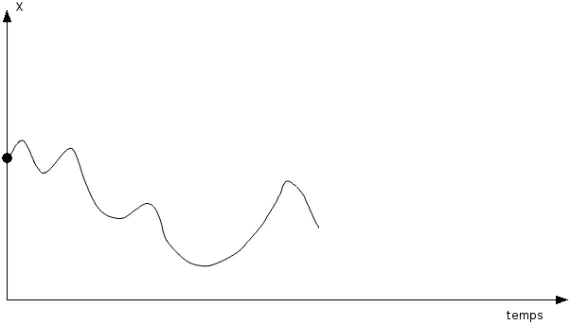

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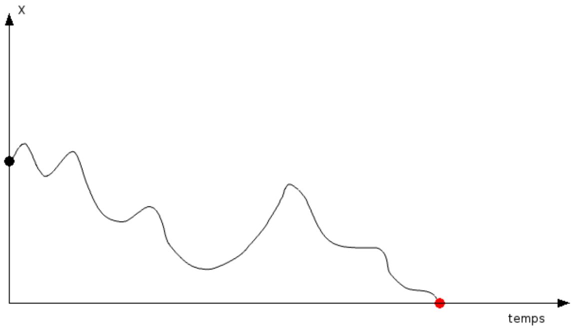

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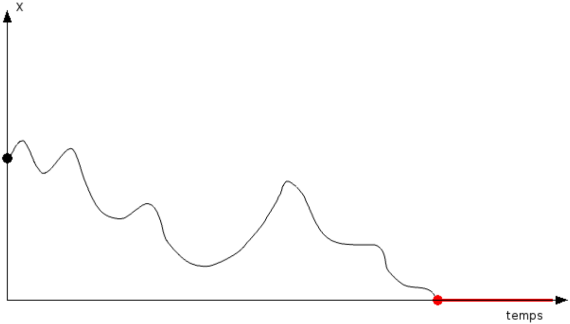

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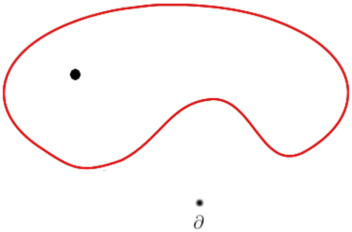

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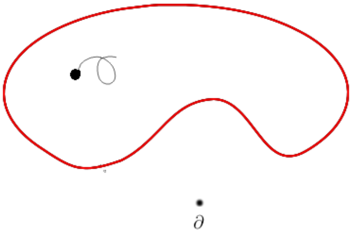

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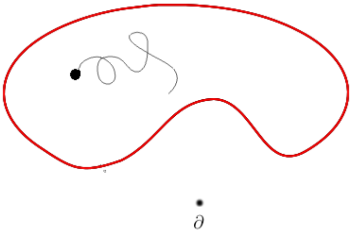

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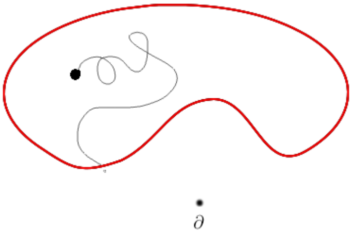

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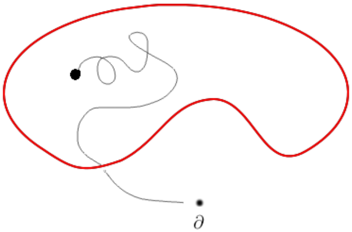

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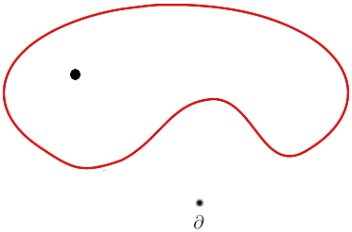

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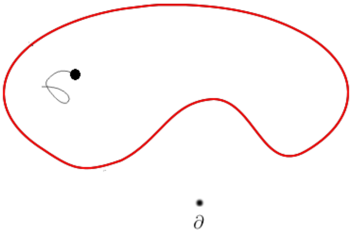

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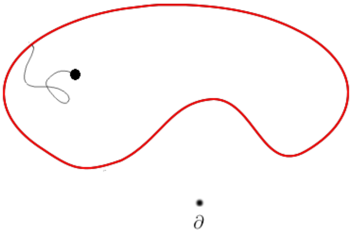

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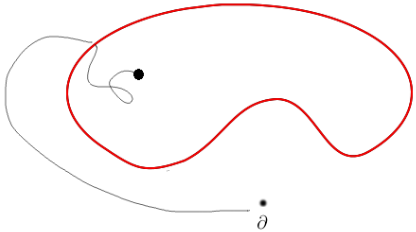

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1a. Absorbed processes

Let $(X_t)_{t\in[0,+\infty[}$ evolving $E \cup \{\partial\}$, where $\partial \notin E$ is absorbing.

Denoting by $\tau_\partial = \inf\{t \geq 0,\ X_t = \partial\}$ the hitting time of ∂,

$$X_t = \partial,\ \forall t \geq \tau_\partial \text{ almost surely.}$$

A class of examples : Stochastic population models

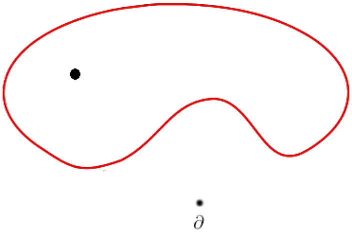
- sub-critical Galton-Watson processes,
- birth and death processes,
- stochastic Lotka-Volterra system of SDEs,
- Wright-Fisher process on $[0,1]$.

1b. Quasi-stationary distributions

In most interesting cases,

$$\mathbb{P}_{x_0}(X_t \in \cdot) \xrightarrow[t\to\infty]{} \delta_\partial,\ \forall x_0 \in E \cup \{\partial\}.$$

Question : What about the distribution of the process conditioned not to be absorbed, namely $\mathbb{P}_{x_0}(X_t \in \cdot \mid t < \tau_\partial)$?



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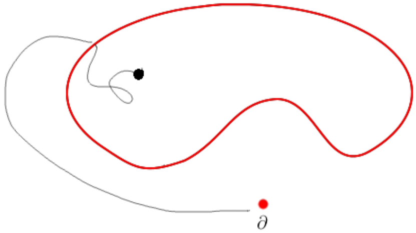

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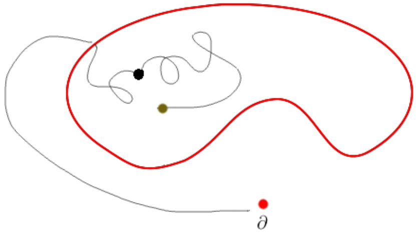

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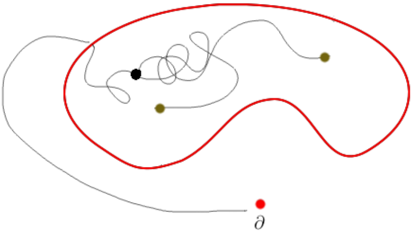

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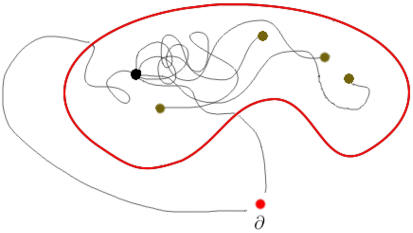

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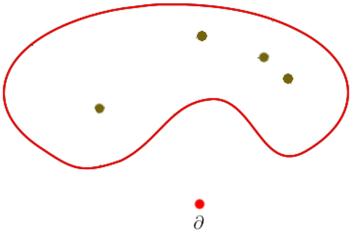


1b. Quasi-stationary distributions

Definition

A quasi-stationary distribution (QSD) is a probability measure α on E such that

$$\alpha = \lim_{t\to\infty} \mathbb{P}_\mu\left(X_t \in \cdot | t < \tau_\partial\right)$$

for some initial distribution μ on E.

Proposition

A probability measure α is a QSD if and only if, for any $t \geq 0$,

$$\alpha = \mathbb{P}_\alpha(X_t \in .| t < \tau_\partial).$$

→ Surveys and book

- Méléard, V. 2012, Van Doorn, Pollett 2013
- Collet, Martínez, San Martín 2013

1b. QSDs : an elementary example

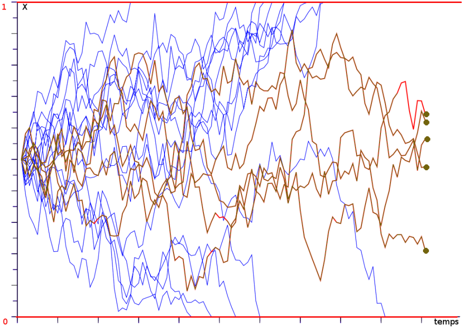
Let Y be a process evolving in E (without absorption) and τ an independent exponential clock. One sets

$$X_t = \begin{cases} Y_t \text{ if } t < \tau, \\ \partial \notin E \text{ if } t \geq \tau. \end{cases}$$

Proposition

- → The law of X conditioned on non-absorption and the law of Y are the same.
- → Hence, the QSDs of X are the stationary distributions of Y.

1b. QSDs : BM on $E =]0{,}1[$, $\partial = \{0{,}1\}$



→ $\alpha(dx) = \pi \sin(\pi x)\, dx$ is the unique QSD

→ for any probability measure μ sur E,

$$\alpha = \lim_{t\to\infty} \mathbb{P}_\mu \left(X_t \in \cdot | t < \tau_\partial \right).$$

- 1d SDE's : Hening & Kolb 2016 for a nearly complete treatment of the existence/uniqueness problem.

2. Basic properties

2a. Exponential absorption time

Proposition

If α is a QSD, then $\exists \lambda_0 > 0$ such that

$$\mathbb{P}_\alpha(t < \tau_\partial) = e^{-\lambda_0 t} \text{ and } \mathbb{E}_\alpha(e^{\theta \tau_\partial}) < +\infty,\ \forall \theta \in [0,\lambda_0[.$$

$\rightarrow$ Partial counterparts

- Ferrari, Kesten, Martinez, Picco (1995)
- Bourget, Chaumont and Sapoukhina (2013)

$\rightarrow$ Galton-Watson process :

- Critical case : $\mathbb{E}_x(\tau_\partial) = \infty$ for all $x \in \mathbb{N} \Rightarrow$ no QSD.
- Sub-critical case : Yaglom (1947), Athreya, Ney (1972)

2b. Absorption time and position are independent

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Proof : For all $f : E \to \mathbb{R}_+$ and $t \geq 0$,

$$\mathbb{E}_\alpha \left(f(X_{\tau_\partial -}) \mathbf{1}_{t<\tau_\partial} \right) = \mathbb{E}_\alpha \left(f(X_{\tau_\partial -}) \mid t < \tau_\partial \right) \mathbb{P}_\alpha(t < \tau_\partial).$$

But $f(X_{\tau_\partial -}) = f(X_{\tau_\partial -}) \circ \theta_t$, hence

$$\begin{aligned} \mathbb{E}_\alpha \left(f(X_{\tau_\partial -}) \mid t < \tau_\partial \right) &= \mathbb{E}_\alpha \left[\mathbb{E}_{X_t} \left(f(X_{\tau_\partial -}) \right) \mid t < \tau_\partial \right] \\ &= \mathbb{E}_\alpha \left(f(X_{\tau_\partial -}) \right). \end{aligned}$$

2c. Absorption rate admits a limit

Proposition

If α is a QSD such that $\alpha = \lim_{t\to\infty} \mathbb{P}_\mu(X_t \in \cdot \mid t < \tau_\partial)$, then

$$\text{absorbtion rate}(t) \stackrel{\text{def}}{=} \mathbb{P}^\partial_\mu(\tau_\partial \in]t, t+1] | \tau_\partial > t) \xrightarrow[t\to\infty]{} e^{-\lambda_0}.$$

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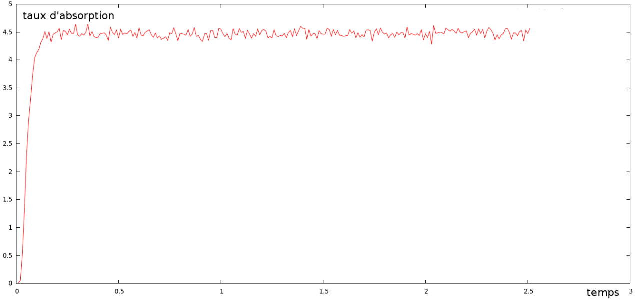
Brownian motion on $E =]0,1[$ absorbed at $\partial = \{0,1\}$.



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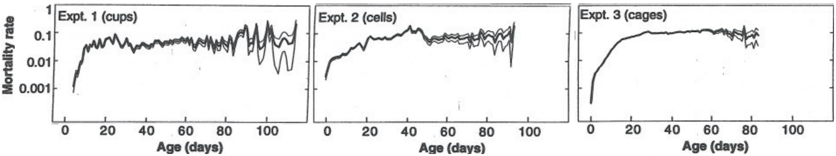
Slowing of Mortality Rates at Older Ages In Large Medfly Cohorts (1992) Carey et al.



See also Steinsaltz & Wachter 2006

3. Uniqueness of QSDs and exponential convergence

Définition

Let α be a QSD. The domain of attraction of α is the set of initial distributions μ such that

$$\lim_{t\to\infty} \mathbb{P}_\mu(X_t \in .|t < \tau_\partial) = \alpha.$$

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In the general case :

- **Existence** of a QSD is not true
- Existence does not imply **uniqueness** of a QSD
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Question : how to guarantee the above properties ?

Let X evolving in $E \cup \{\partial\}$ absorbed at ∂.

→ Assumption A1 (Doeblin condition)

There exists a probability measure ν and $c_1 > 0$ such that

$$\mathbb{P}_x(X_1 \in \cdot \,|\, 1 < \tau_\partial) \geq c_1 \nu(\cdot), \; \forall x \in E.$$

→ Assumption A2 (Harnack inequality)

$$\frac{\mathbb{P}_\nu(t < \tau_\partial)}{\mathbb{P}_x(t < \tau_\partial)} \geq c_2 > 0, \; \forall x \in E, t \geq 0.$$

Theorem (Champagnat, V. 2016)

A1 and A2 ⇔ there exists $C > 0$, $\gamma > 0$ and $\alpha \in \mathcal{M}_1(E)$ such that, for all $\mu \in \mathcal{M}_1(E)$,

$$\left\| \mathbb{P}_\mu(X_t \in \cdot | t < \tau_\partial) - \alpha \right\|_{TV} \leq C e^{-\gamma t}.$$

One can choose $C e^{-\gamma t} = (1 - c_1 c_2)^{\lfloor t \rfloor}$.

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Comments and other examples...

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- We used this procedure in several situations
  - general one dimensional diffusion processes
  - multi-dimensional diffusion processes
  - birth and death processes with catastrophes
  - multi-dimensional birth and death processes
  - branching/dying Brownian motions
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- The uniform convergence is a feature but also a limitation...

4. The multi-dimensional diffusion case

Framework

We consider a diffusion process X evolving in an open domain $D \subset \mathbb{R}^d$ and absorbed at the boundary ∂D, solution to the SDE

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt, \ X_0 \in D,$$

where σ and b are sufficiently regular.

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Existing tools to prove existence, uniqueness, convergence.

- Spectral theory tools
  - Pinsky 1985. Sufficient spectral conditions
  - Cattiaux, Méléard 2010. Auto-adjoint operator
  - Knobloch, Partzsch 2010. Two sided estimates + spectral conditions (included in Birkhoff 1957)
- Probabilistic tools (using A1 and A2)
  - Del Moral, V 2016. Bounded domain with enough regularity
  - Champagnat, Coulibaly, V. 2016 Two sided estimates or regularity of the domain

Result with D bounded)

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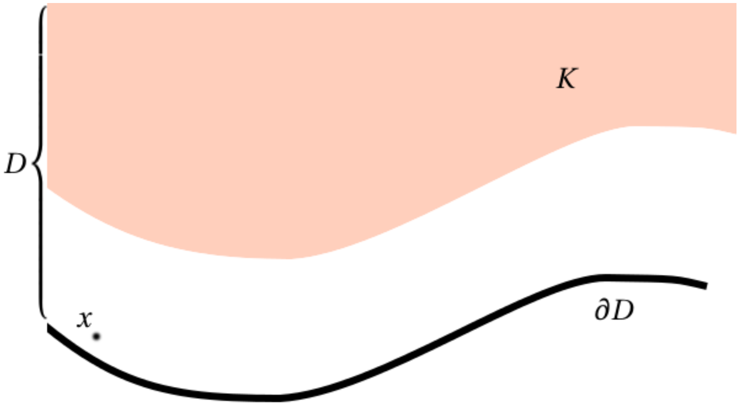
If D is bounded with $C^{1,1}$ boundary and if σ and b are C^1, then A1 and A2 holds true.

A1 & A2 in the multi-dimensional diffusion case

How do we prove A1 and A2 in this situation ?



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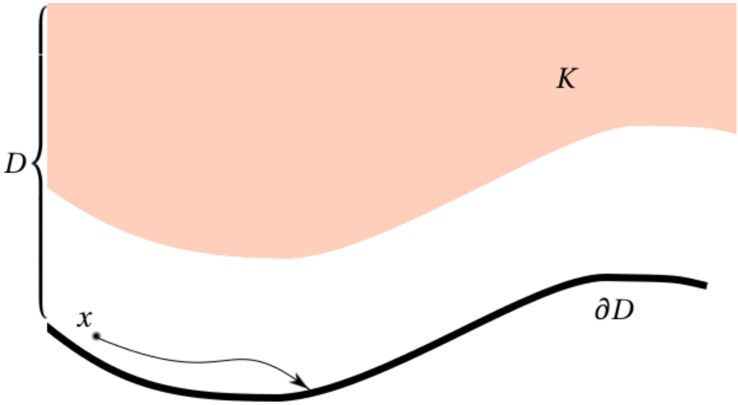

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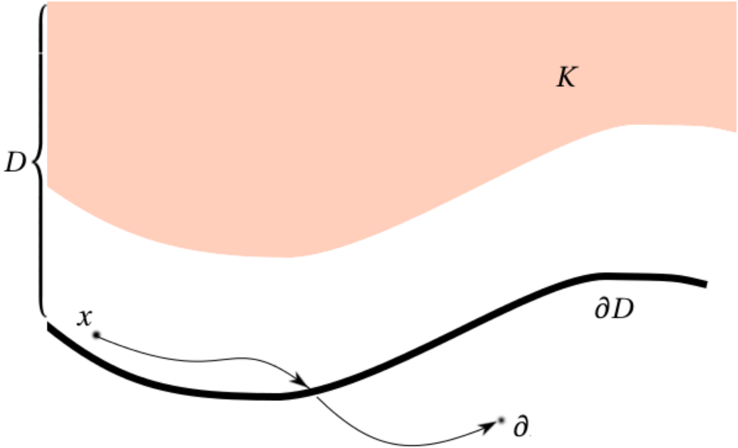

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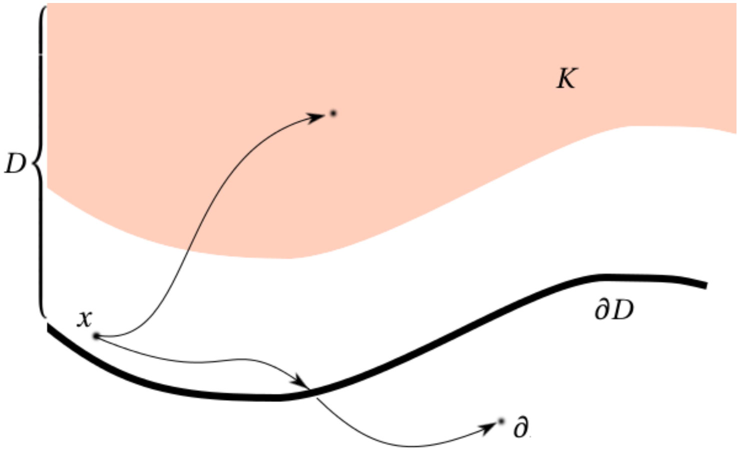

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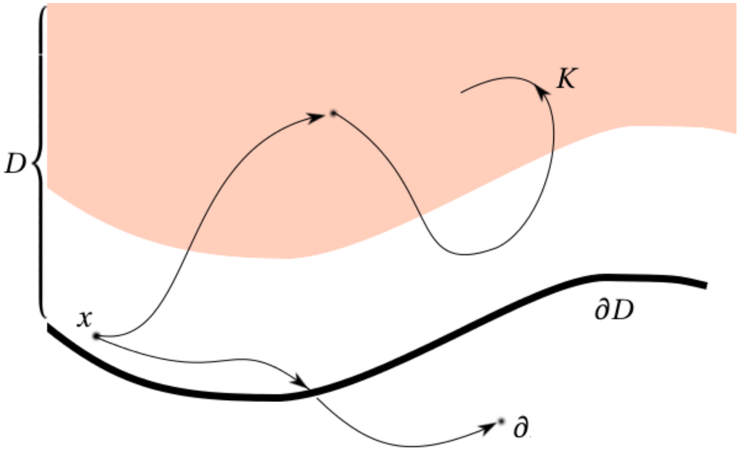

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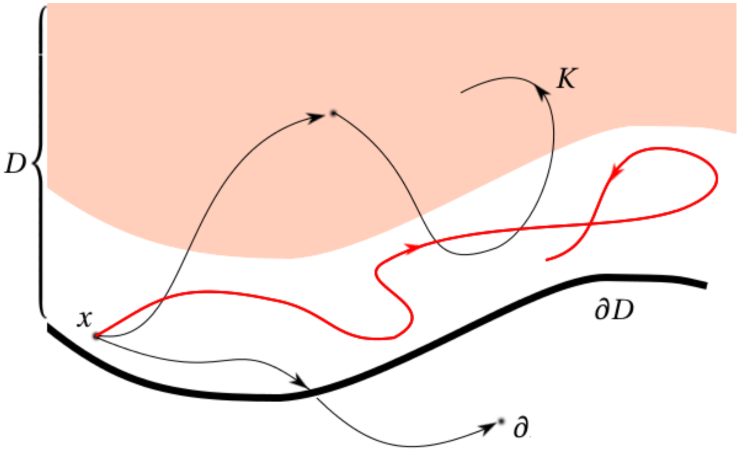

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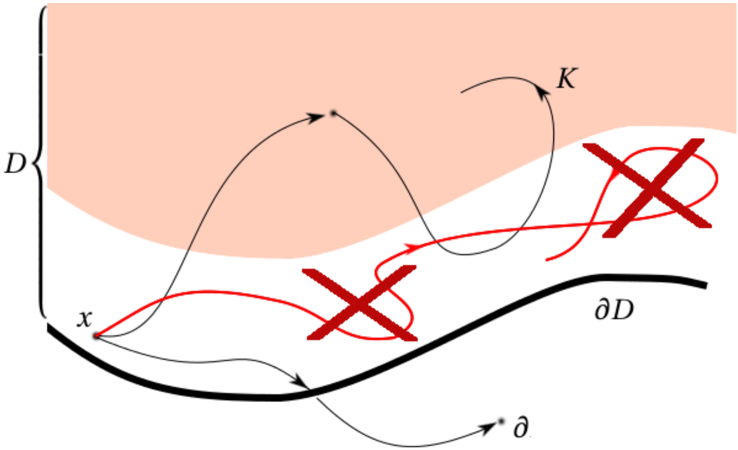

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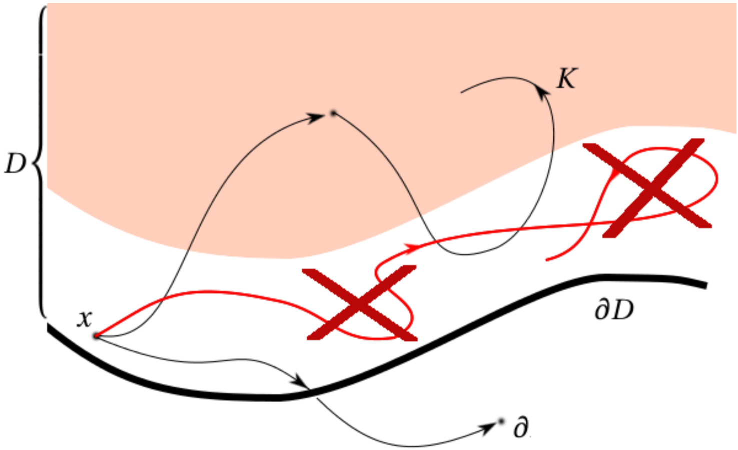


Necessary condition : $\mathbb{P}_x(\tau_K < 1) \geq c\,\mathbb{P}_x(1 < \tau_\partial)$ (difficult).

A1 & A2 in the multi-dimensional diffusion case

Ideas of the proof

- If the boundary of D is C^2, then simply apply the Itô's formula to $d(X_t, \partial D)$. This and Lipschitz regularity of the coefficients allow to conclude (rather indirectly).

A1 & A2 in the multi-dimensional diffusion case

Ideas of the proof

- If the boundary of D is C^2, then simply apply the Itô's formula to $d(X_t, \partial D)$. This and Lipschitz regularity of the coefficients allow to conclude (rather indirectly).

- Otherwise, you can use so-called two sided estimates : there exists a probability measure ν on D such that

$$f(x)\,\nu(\cdot) \leq \mathbb{P}_x(X_1 \in \cdot) \leq c\,f(x)\,\nu(\cdot),\ \forall x \in D.$$

  These are known to hold for many processes, with sufficient regularity of the diffusion coefficient (typically C^1) and of the boundary of ∂D (typically $C^{1,1}$).

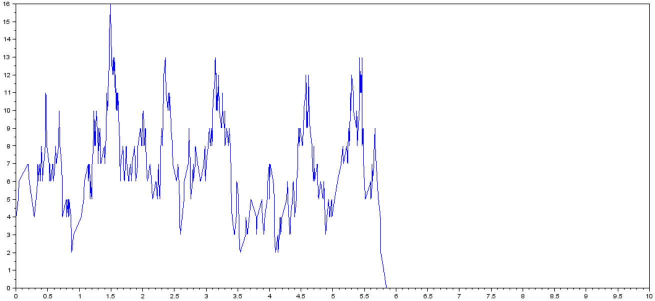
  → simplify Knobloch Partzsch 2010,

  → do not use Birkhoff 1957

5. Other benefits coming with A1 and A2

Q-processus

We are looking now at the behavior of the process *a long time before extinction*



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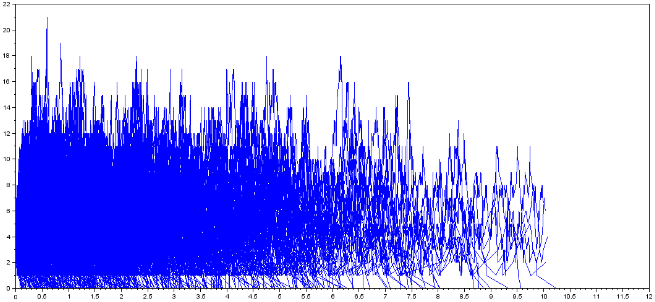

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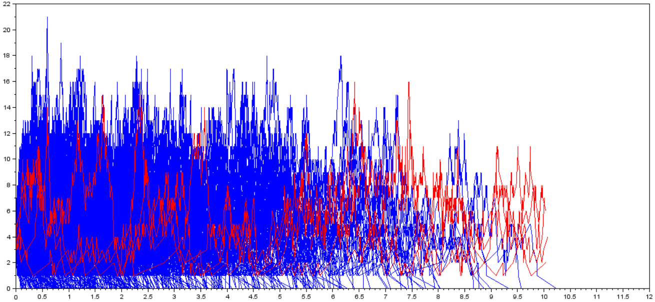

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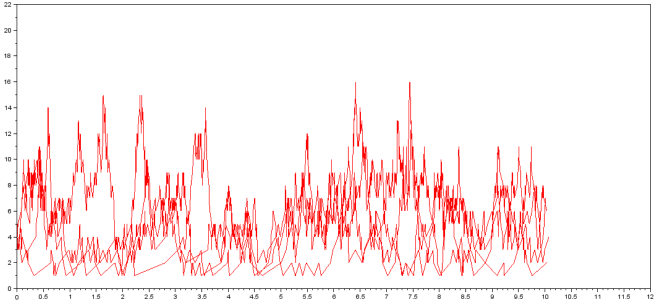

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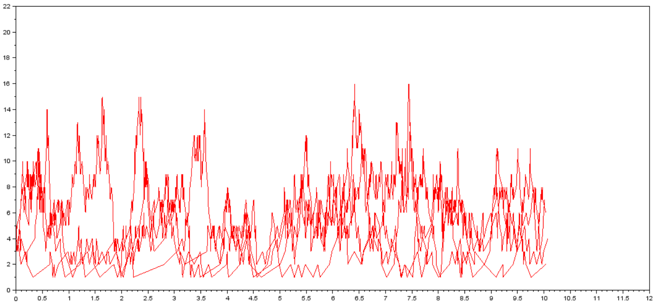

Question : $\mathbb{P}_x\left(X_t \in \cdot \middle| T < \tau_\partial\right)$ when $T \to +\infty$?

Definition of the Q-process

When it exists, the Q-process Y is the process with law $\mathbb{Q}$ defined, for all $x \in E$ and $t \geq 0$, by

$$\mathbb{Q}_x\left((Y_u)_{u\in[0,t]} \in \cdot\right) = \lim_{T\to\infty} \mathbb{P}_x\left((X_u)_{u\in[0,t]} \in \cdot \mid T < \tau_\partial\right).$$

This is a Markov process without absorption.

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Theorem (Champagnat, V. 2016)

Assume that X satisfies A1 and A2. Then the Q-process Y is well defined and is exponentially ergodic :

$$\|\mathbb{Q}_\mu(Y_t \in \cdot) - \beta(\cdot)\|_{TV} \leq 2(1 - c_1 c_2)^{\lfloor t/t_0 \rfloor} \|\mu - \beta\|_{TV},$$

where $\beta(f) := \alpha(\eta f)$, $\forall f$ and some function η.

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This is a penalization problem (Roynette, Valois and Yor 2005).

Quantification of the convergence toward the Q-process

Theorem (Champagnat, V. 2016)

Assume that X satisfies A1 and A2. Then there exists a positive constant C such that, for all $x \in E$, all $0 \leq t \leq T$,

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Reciprocally, if the Q-process is well defined, exponentially ergodic and if

$$\sup_{x\in E} \left\| \mathbb{Q}_x((Y_u)_{u\in[0,t]} \in \cdot) - \mathbb{P}_x((X_u)_{u\in[0,t]} \in \cdot \mid T < \tau_\partial) \right\|_{TV} \xrightarrow[T\to+\infty]{} 0,$$

then A1 and A2 holds true.

Conditional ergodic theorems

Corollary

Assume that X satisfies A1 and A2. Then there exists a positive constant C such that, for all $T > 0$ and all bounded measurable functions $f : E \to \mathbb{R}$,

$$\left| \mathbb{E}_x \left(\frac{1}{T} \int_0^T f(X_t)\, dt \mid T < \tau_\partial \right) - \beta(f) \right| \leq \frac{C\, \|f\|_\infty}{T},$$

where β is the unique stationary distribution of the Q-process.

THE END