Average Sample Complexity

Karl Petersen and Benjamin Wilson

University of North Carolina at Chapel Hill
and Stevenson University

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This is a measure of local freedom, taking into account the probabilities of individual configurations.

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Intricacy is low if there is a lot of order ($h = 0$)
or a lot of independence.

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Functioning networks may be most effective when there is a balance between functional segregation and global integration, between freedom of the individual and order of the whole.

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3. $c_{S^c}^n = c_S^n$.

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- Given a system of coefficients, c_S^n, the corresponding *mutual information functional*, $\mathcal{J}^c(X)$ is defined by

$$\begin{aligned}\mathcal{J}^c(X) &= \sum_{S \subset n^*} c_S^n MI(X_S, X_{S^c}) \\ &= \sum_{S \subset n^*} c_S^n [H(X_S) + H(X_{S^c}) - H(X_{S,S^c})].\end{aligned}$$

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1. Exchangeability: invariance by permutations of n;
2. Weak additivity: $\mathcal{I}^c(X, Y) = \mathcal{I}^c(X) + \mathcal{I}^c(Y)$ for any two independent systems $X = \{X_i : i \in n^*\}$ and $Y = \{Y_j : j \in m^*\}$.

Theorem (Buzzi, Zambotti)

Let c_S^n be a system of coefficients and $\mathcal{I}^c$ the associated mutual information functional. $\mathcal{I}^c$ is an intricacy if and only if there exists a symmetric probability measure λ_c on $[0,1]$ such that

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3. For $p = 1/2$, $c_S^n = 2^{-n}$ (uniform).

Definitions using open covers

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Definition (P-W)

Let c_S^n be a system of coefficients. Define the *topological intricacy of* (X, T) *with respect to the open cover* $\mathcal{U}$ to be

$$\mathrm{Int}(X, \mathcal{U}, T) := \lim_{n \to \infty} \frac{1}{n} \sum_{S \subset n^*} c_S^n \log \left(\frac{N(\mathcal{U}_S) N(\mathcal{U}_{S^c})}{N(\mathcal{U}_{n^*})} \right).$$

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There are also Bowen-type definitions using ε-separated or spanning sets, and definitions of average sample pressure.

Theorem

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The proof is based on subadditivity of the sequence

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and Fekete's Subadditive Lemma: for every subadditive sequence a_n, the limit $\lim_{n\to\infty} a_n/n$ exists and is equal to $\inf_n a_n/n$.

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Proposition

For each open cover $\mathcal{U}$,
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The limits in the definitions of $\mathrm{Int}(X, \mathfrak{U}, T)$ *and* $\mathrm{Asc}(X, \mathfrak{U}, T)$ *exist.*

The proof is based on subadditivity of the sequence

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and Fekete's Subadditive Lemma: for every subadditive sequence a_n, the limit $\lim_{n\to\infty} a_n/n$ exists and is equal to $\inf_n a_n/n$.

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For each open cover $\mathfrak{U}$*,*
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In particular, a dynamical system with zero (or relatively low) topological entropy (integrated, ordered) has zero (or relatively low) topological intricacy.

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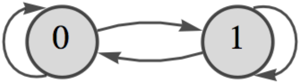
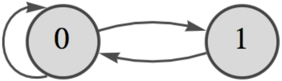
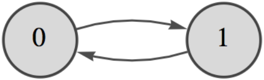
Corollary

For the full r-shift with $c_S^n = 2^{-n}$ for all S,

$$\mathrm{Asc}(\Sigma_r, \mathcal{U}_0, \sigma) = \frac{\log r}{2} \quad \textit{and} \quad \mathrm{Int}(\Sigma_r, \mathcal{U}_0, \sigma) = 0.$$

	Adjacency Graph	Entropy	Asc	Int
Disordered		0.693	0.347	0
		0.481	0.286	0.090
Ordered		0	0	0

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- When we code by k-blocks, $S \subset n^*$ is replaced by $S + k^*$, and the effect on α_{S+k^*} as compared to α_S is similar, since it acts similarly on each of the long subintervals comprising S.

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These are indications that there may be a topological analogue of the following result.

Theorem (Ornstein-Weiss, 2007)

If J is a finitely observable functional defined for ergodic finite-valued processes that is an isomorphism invariant, then J is a continuous function of the measure-theoretic entropy.

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- analogously to the symbolic or topological complexity functions.
- Similarly for the measure-theoretic version: fix a partition α and study the limit, or the function of n.

$$\mathrm{Asc}_\mu(X, T, \alpha) = \lim_{n\to\infty} \frac{1}{n} \sum_{S\subset n^*} c_S^n H_\mu(\alpha_S).$$

So consider Asc for a fixed open cover as a function of n.

$$\mathrm{Asc}(X, \sigma, \mathcal{U}_k, n) = \frac{1}{n} \sum_{S \subset n^*} c_S^n \log N(S).$$

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Example

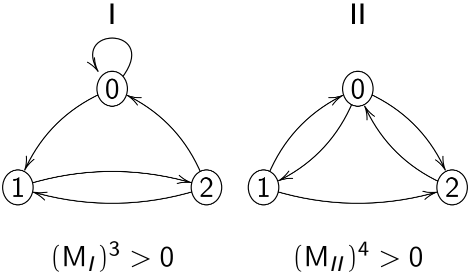
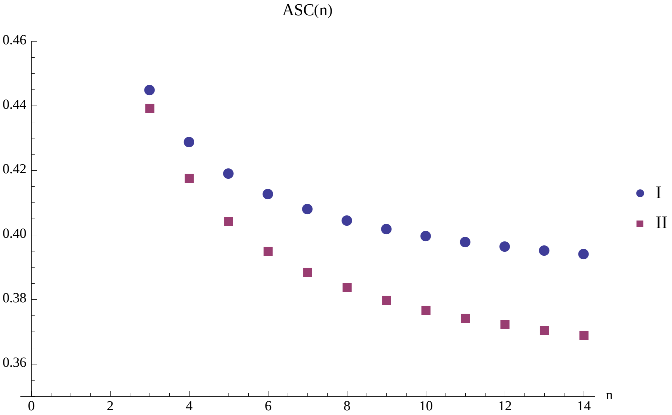
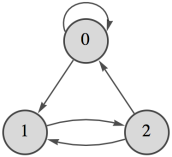
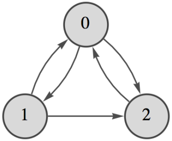


Figure: Graphs of two subshifts with the same complexity function but different average sample complexity functions.

$$\mathrm{Asc}(n) = \frac{1}{n}\frac{1}{2^n}\sum_{S \subset n^*} \log N(S)$$



Adjacency Graph	h_{top}	Asc(10)	Int(10)
	0.481	0.399	0.254
	0.481	0.377	0.208

These SFTs have the same entropy and complexity functions (words of length n) but different Asc and Int functions.

Results in measure-theoretic setting

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- For example $\{0, 2, 3\} \subset 5^* \leftrightarrow 10110$.
- The average entropy, $H_{\mu}(\alpha_S)$, over all $S \subset n^*$, is then an integral and can be interpreted in terms of the entropy of a first-return map to the cylinder $A = [1]$ in a cross (or skew) product of our system X and the full 2-shift, Σ_2.

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Where is the rest of the entropy?

Asc_μ as conditional or fiber entropy

Theorem

$$\mathrm{Asc}_\mu(X, \alpha, T) = \frac{1}{2} \lim_{n\to\infty} \frac{1}{n} H_{\mu\times P_A}\left(\bigvee_{k=0}^{n-1} T_{X\times A}^{-k}\beta \middle| \bigvee_{k=0}^{n-1} T_{X\times A}^{-k}\mathcal{A} \right)$$

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$$= \frac{1}{2} H_{\mu\times P_A}(\beta|\beta^*_{1,\infty} \vee \mathcal{A}^*_{-\infty,\infty})$$

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&= \frac{1}{2} H_{\mu\times P_A}(\beta | \beta_{1,\infty}^{*} \vee \mathcal{A}_{-\infty,\infty}^{*}) \\
&= \frac{1}{2} h_{\mu\times P_A}((\beta, T_{X\times A}, \mu\times P_A) | \mathcal{A}_{-\infty,\infty}^{*})
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Some questions about maximal measures

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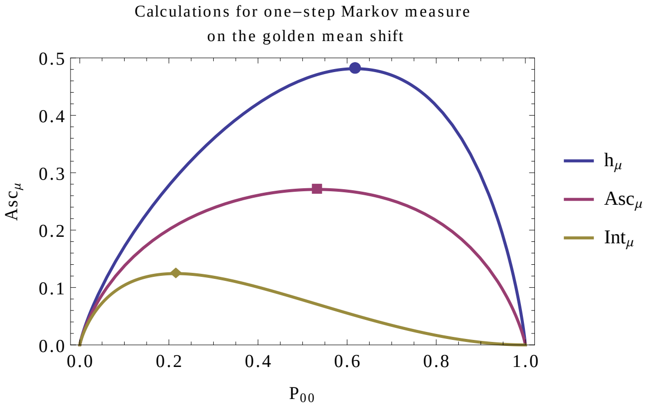

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P_{00}	h_μ	Asc_μ	Int_μ
0.618	**0.481**	0.266	0.051
0.533	0.471	**0.271**	0.071
0.216	0.292	0.208	**0.124**

Table: Calculations for one-step Markov measures on the golden mean shift. Bolded numbers are maxima for given category.

Conj. 2: $\mu_2 \neq \mu_1$

P_{00}	h_μ	Asc_μ	Int_μ
0.618	**0.481**	0.266	0.051
0.533	0.471	**0.271**	0.071
0.216	0.292	0.208	**0.124**

Table: Calculations for one-step Markov measures on the golden mean shift. Bolded numbers are maxima for given category.

P_{000}	P_{100}	h_μ	Asc_μ	Int_μ
0.618	0.618	**0.481**	0.266	0.051
0.483	0.569	0.466	**0.272**	0.078
0	0.275	0.344	0.221	**0.167**

Table: Calculations for two-step Markov measures on the golden mean shift.

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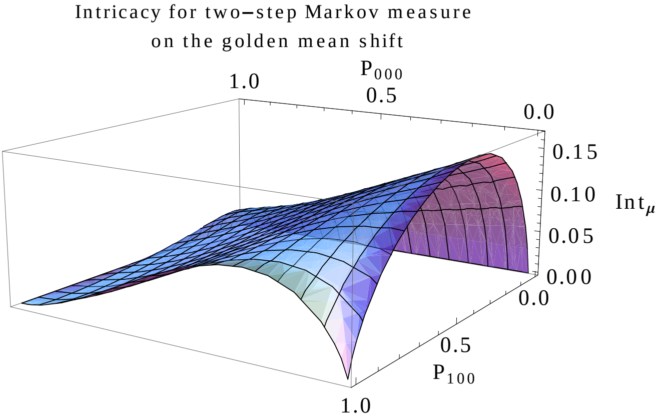

Conj. 5: On the 2-shift there are *two* 1-step Markov measures that maximize $\text{Int}_\mu(X, T, \alpha)$ among all 1-step Markov measures. They are supported on the golden mean SFT and its image under the dualizing map $0 \leftrightarrow 1$.

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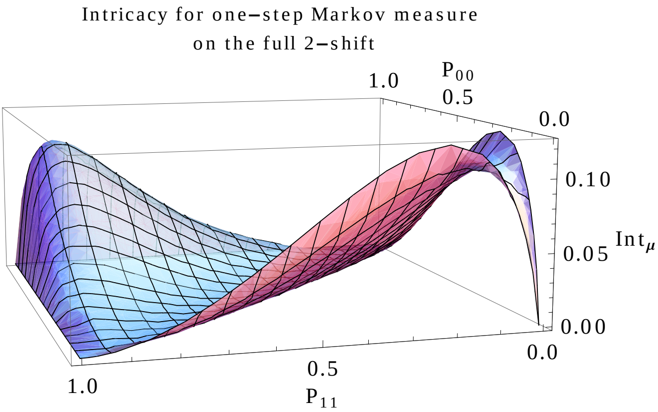

Conj. 6: On the 2-shift there is a 1-step Markov measure that is *fully supported* and is a local maximum point for $\mathrm{Int}_{\mu}(X, T, \alpha)$ among all 1-step Markov measures.

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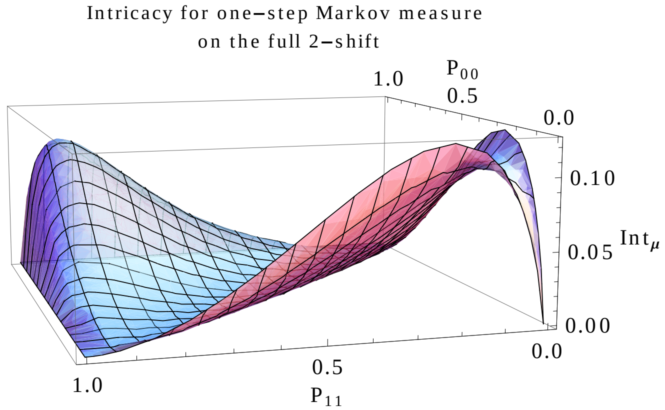

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- First one can consider a function of just a single coordinate that gives the value of each symbol.
- Maximum intricacy may be useful for finding areas of high information activity, such as working regions in a brain (Edelman-Sporns-Tononi) or coding regions in genetic material (Koslicki-Thompson).

Thank you!

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