Dynamical Systems of Number-Theoretic Origin in the Theory of Aperiodic Order

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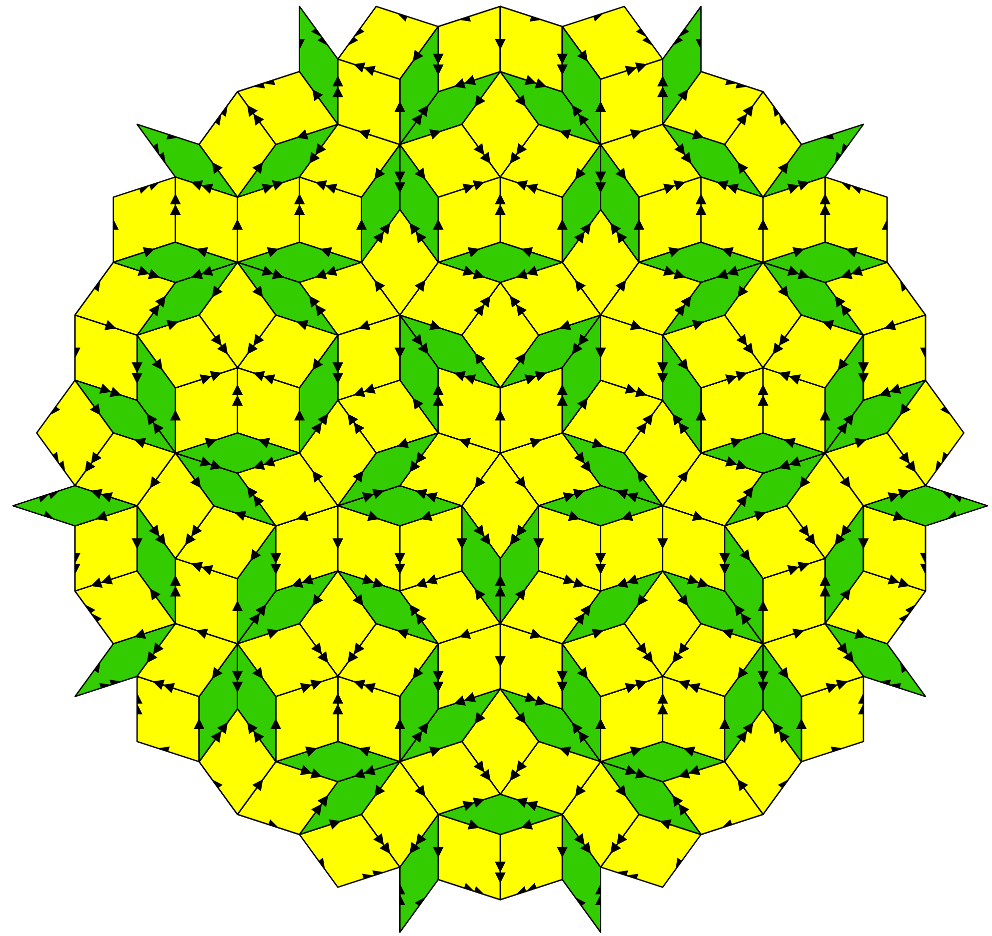
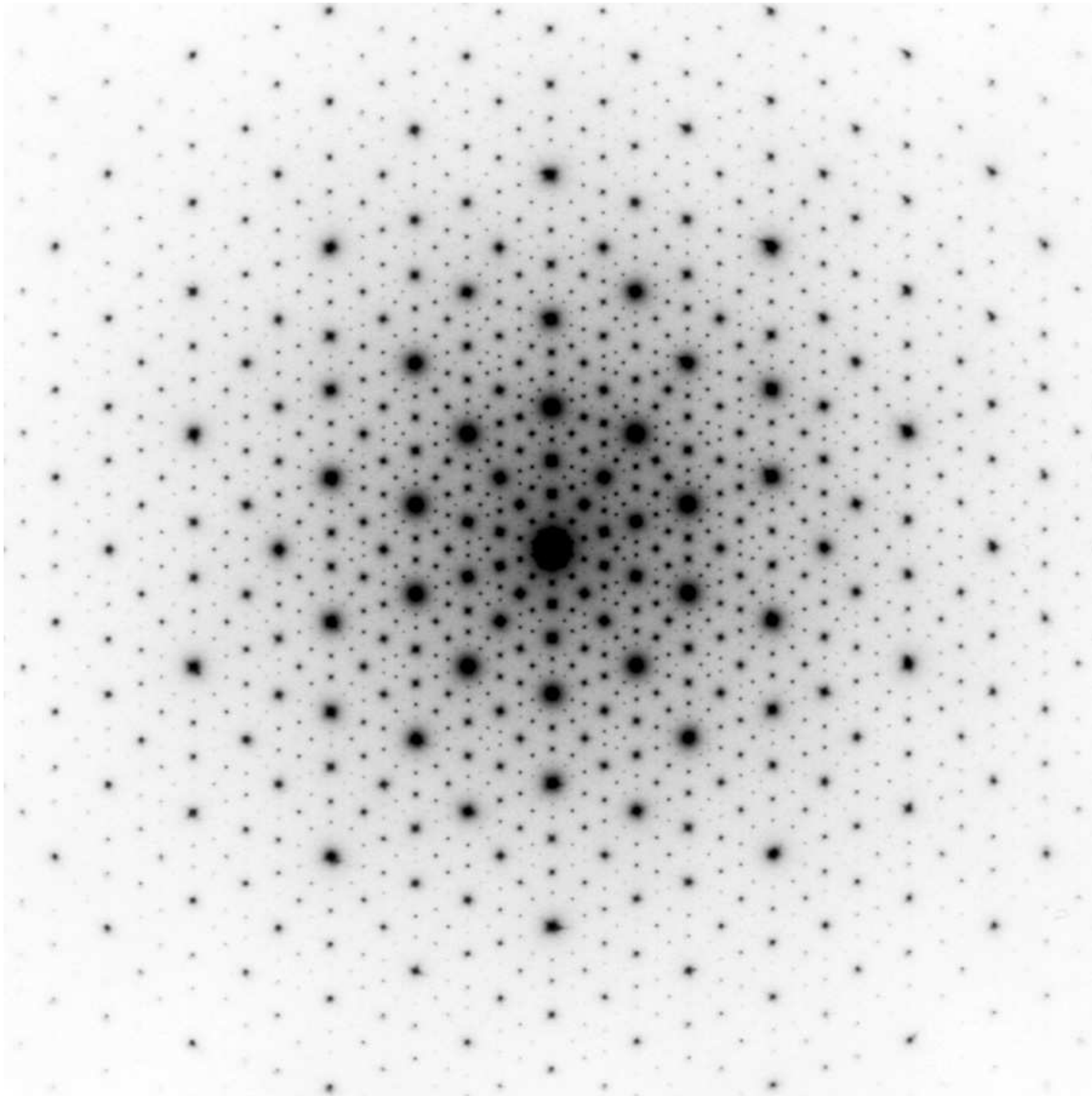
(joint work C. Huck, D. Lenz, R.V. Moody, P. Pleasants† and N. Strungaru)

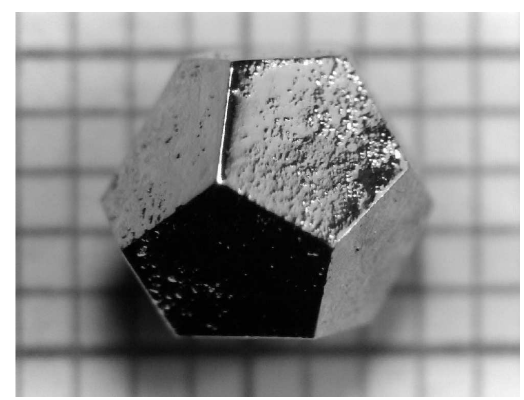
Menu

- Origins in physics
- Spectral notions
- Examples
  - Poisson formula
  - model sets
  - visible points
  - squarefree numbers
  - weak model sets
  - extensions
- Outlook

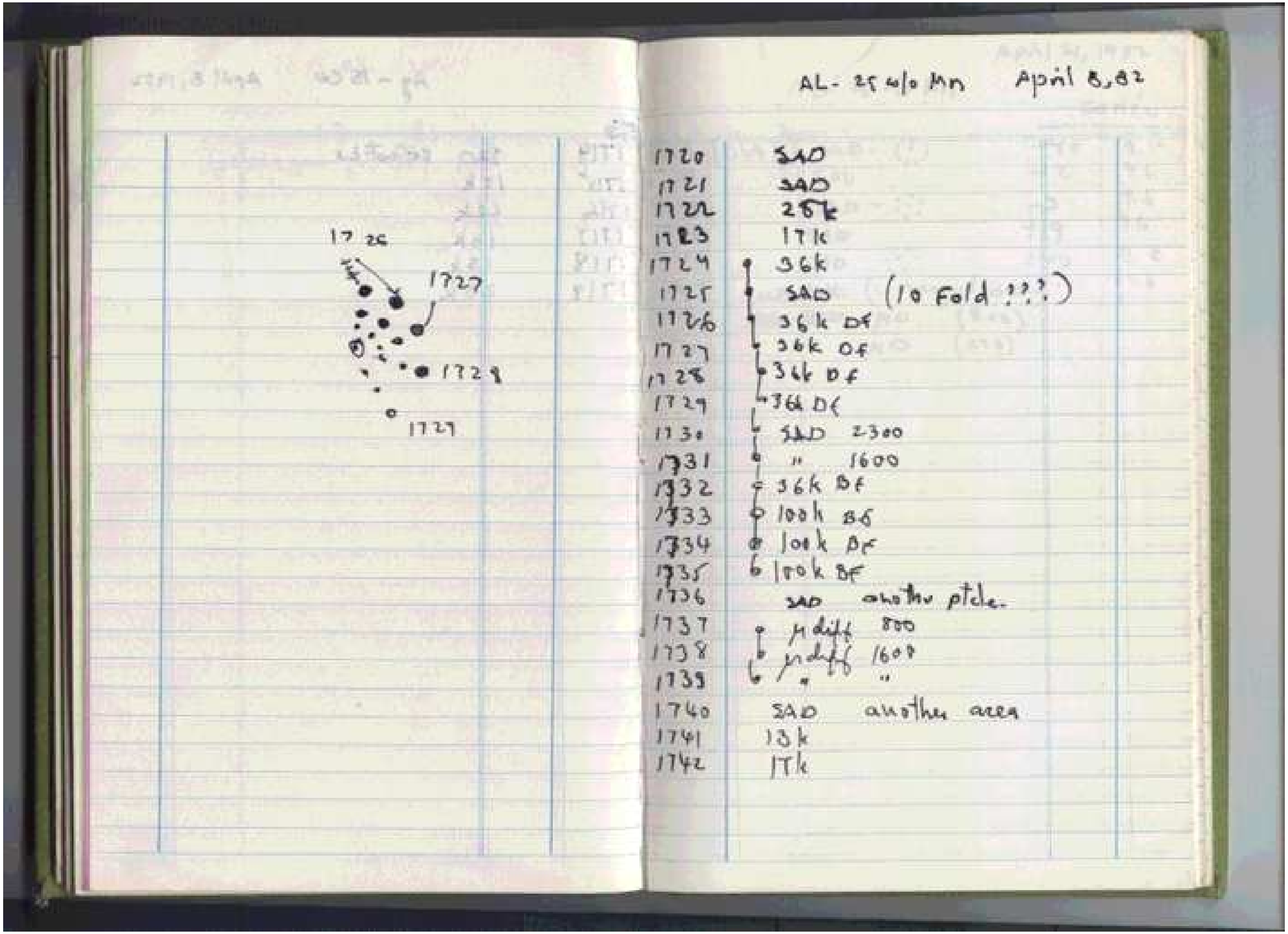


Origins





Origins

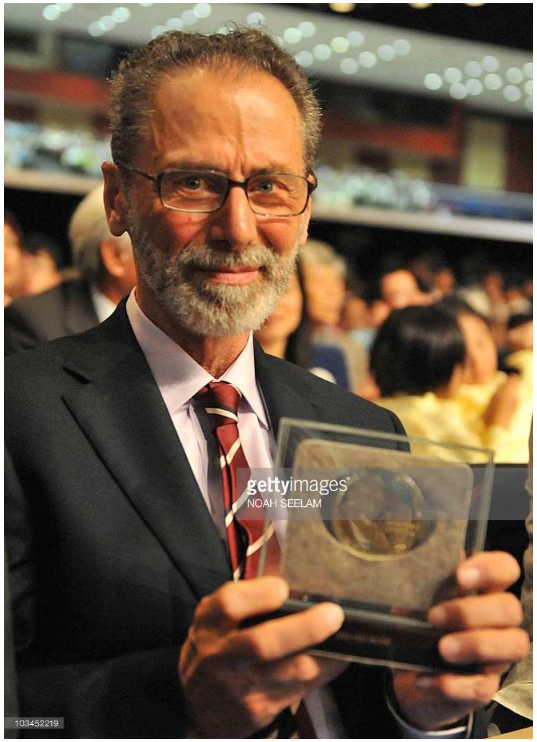


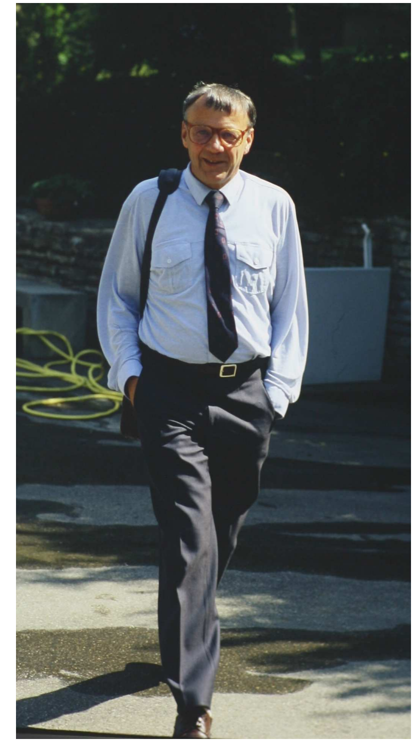
(Photo courtesy of the Ames Laboratory)

Origins

Wolf Prize in Physics 1999
Nobel Prize in Chemistry 2011

Origins



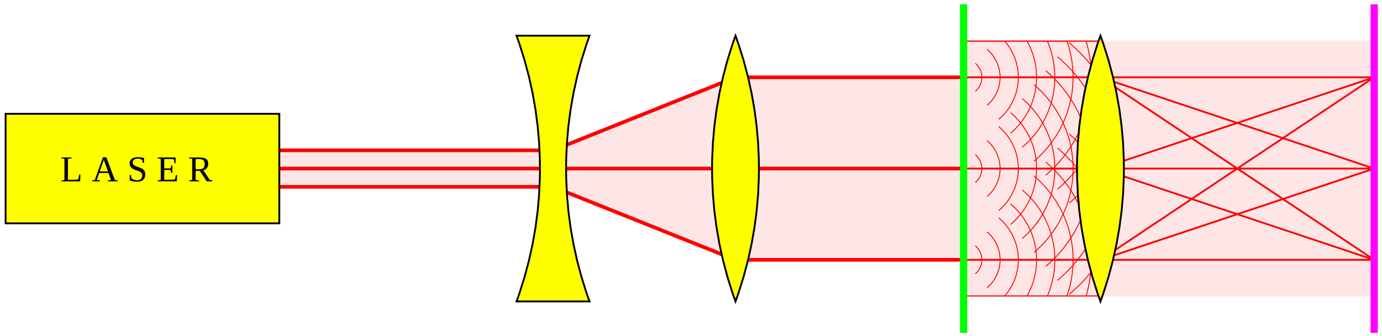


Yves Meyer

Roger Penrose

Peter Kramer

Diffraction theory



Wiener's diagram obstacle $f(x)$, with $\widetilde{f}(x) := \overline{f(-x)}$

$$\begin{array}{ccc} f & \xrightarrow{\;*\;} & f * \widetilde{f} \\ \mathcal{F}\big\downarrow & & \big\downarrow\mathcal{F} \\ \widehat{f} & \xrightarrow{\;|.|^2\;} & |\widehat{f}\,|^2 \end{array}$$

Diffraction theory

Structure translation bounded measure ω

assumed 'self-amenable' (Hof 1995)

Autocorrelation $$\gamma = \gamma_\omega = \omega \circledast \widetilde{\omega} := \lim_{R\to\infty} \frac{\omega|_R * \widetilde{\omega|_R}}{\mathrm{vol}(B_R)}$$

Diffraction $$\boxed{\widehat{\gamma} = \bigl(\widehat{\gamma}\bigr)_{\mathsf{pp}} + \bigl(\widehat{\gamma}\bigr)_{\mathsf{sc}} + \bigl(\widehat{\gamma}\bigr)_{\mathsf{ac}}}$$ (relative to λ)

- pp: Bragg peaks
- ac: diffuse scattering with density
- sc: whatever remains ...

Diffraction theory

Setting $\omega \quad \curvearrowright \quad \gamma = \omega \circledast \widetilde{\omega} \quad \curvearrowleft\!\!\curvearrowright \quad \widehat{\gamma} \quad \not\curvearrowright \quad \omega$

Dirac comb on $\mathbb{Z}$ (similarly for lattices)

$$\omega = \sum_{n\in\mathbb{Z}} w(n)\, \delta_n \qquad \curvearrowright \qquad \gamma = \sum_{m\in\mathbb{Z}} \eta(m)\, \delta_m$$

Autocorrelation coefficients

$$\eta(m) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} w(n)\, \overline{w(n-m)}$$

$\curvearrowright$ Fourier coefficients of spectral measure $\sigma_w = \widehat{\gamma}|_{[0,1)}$

Diffraction versus dynamical spectrum

Dynamical system

$$(\mathbb{X}, \mathbb{Z}, \mu) \quad \text{with} \quad \mathbb{Z} \simeq \{T^n \mid n \in \mathbb{Z}\}$$

$\curvearrowright$ Hilbert space $\mathcal{H} = L^2(\mathbb{X}, \mu)$

$\curvearrowright$ unitary operator on $\mathcal{H}$, $\bigl(U_T f\bigr)(x) := f(Tx)$

$\curvearrowright$ spectrum of U_T (Koopman, von Neumann, Halmos)

Extension analogous definition for other groups, e.g. $\mathbb{R}^d$

Spaces shifts, tilings, Delone sets, measures, ...

(Host 1986, Queffélec 1987, Pytheas Fogg 2002)

(Radin/Wolff 1992, Robinson 1996, Solomyak 1997)

Diffraction versus dynamical spectrum

Theorem Let $(\mathbb{X}, \mathbb{R}^d, \mu)$ be an (ergodic) point set dynamical system with diffraction $\widehat{\gamma}$. Then, $\widehat{\gamma}$ is pure point iff $(\mathbb{X}, \mathbb{R}^d, \mu)$ has pure point dynamical spectrum. The latter then is the group generated by the support of $\widehat{\gamma}$, the so-called Fourier–Bohr spectrum of γ.

(Dworkin 1993, Hof 1995, Schlottmann 2000, Lee/Moody/Solomyak 2002, B/Lenz 2004, Lenz/Strungaru 2009, Lenz/Moody 2012)

Theorem Let $(\mathbb{X}, \mathbb{R}^d, \mu)$ be as above, with finite local complexity. Then, the dynamical spectrum is fully determined by the diffraction measure of the system together with those of (part of) the family of patch-derived factors (patch locator sets).

(Miękisz/van Enter 1992, B/van Enter 2011, B/Lenz/van Enter 2013)

Remark Often, finitely many factors suffice !

Pure point spectra

Point measures $\quad \delta_x\,, \quad \delta_S := \sum_{x\in S} \delta_x$

Poisson's summation formula $\quad \boxed{\widehat{\delta_{\varGamma}} = \mathrm{dens}(\varGamma)\, \delta_{\varGamma^*}}$

for lattice $\varGamma$, dual lattice $\varGamma^*$

Perfect crystals $\quad \omega = \mu * \delta_{\varGamma} \quad$ (μ finite, $\varGamma$ maximal)

$$\Longrightarrow \quad \gamma = \mathrm{dens}(\varGamma)\, (\mu * \widetilde{\mu}) * \delta_{\varGamma}$$

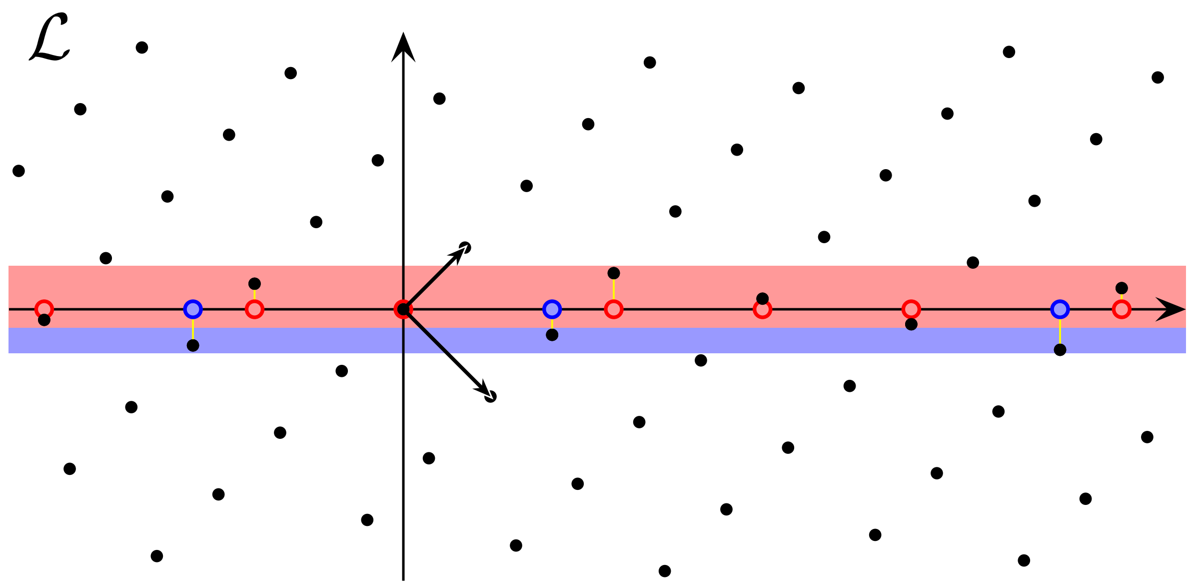
$$\Longrightarrow \quad \boxed{\widehat{\gamma} = \big(\mathrm{dens}(\varGamma)\big)^2 \, \big|\widehat{\mu}\big|^2 \, \delta_{\varGamma^*}} \qquad \text{pure point !!}$$

$\Longrightarrow$ dynamical spectrum $\varGamma^*$, also pure point

Pure point spectra

Silver mean substitution: $a \mapsto aba,\; b \mapsto a \qquad (\lambda_{\mathrm{PF}} = 1 + \sqrt{2})$

Silver mean point set: $\varLambda = \bigl\{ x \in \mathbb{Z}[\sqrt{2}\,] \mid x' \in [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\,] \bigr\}$



Pure point spectra

CPS

$$\begin{array}{ccccc}
\mathbb{R}^d & \xleftarrow{\;\pi\;} & \mathbb{R}^d \times \mathbb{R}^m & \xrightarrow{\;\pi_{\text{int}}\;} & \mathbb{R}^m \\
\cup & & \cup & & \cup \ \text{dense} \\
\pi(\mathcal{L}) & \xleftarrow{\;1-1\;} & \mathcal{L} & \longrightarrow & \pi_{\text{int}}(\mathcal{L}) \\
\| & & & & \| \\
L & & \xrightarrow{\qquad\qquad\star\qquad\qquad} & & L^\star
\end{array}$$

(Meyer 1972)

(Moody 1997)

Model set $\boxed{\varLambda = \{ x \in L \mid x^\star \in W \}}$ (assumed regular)

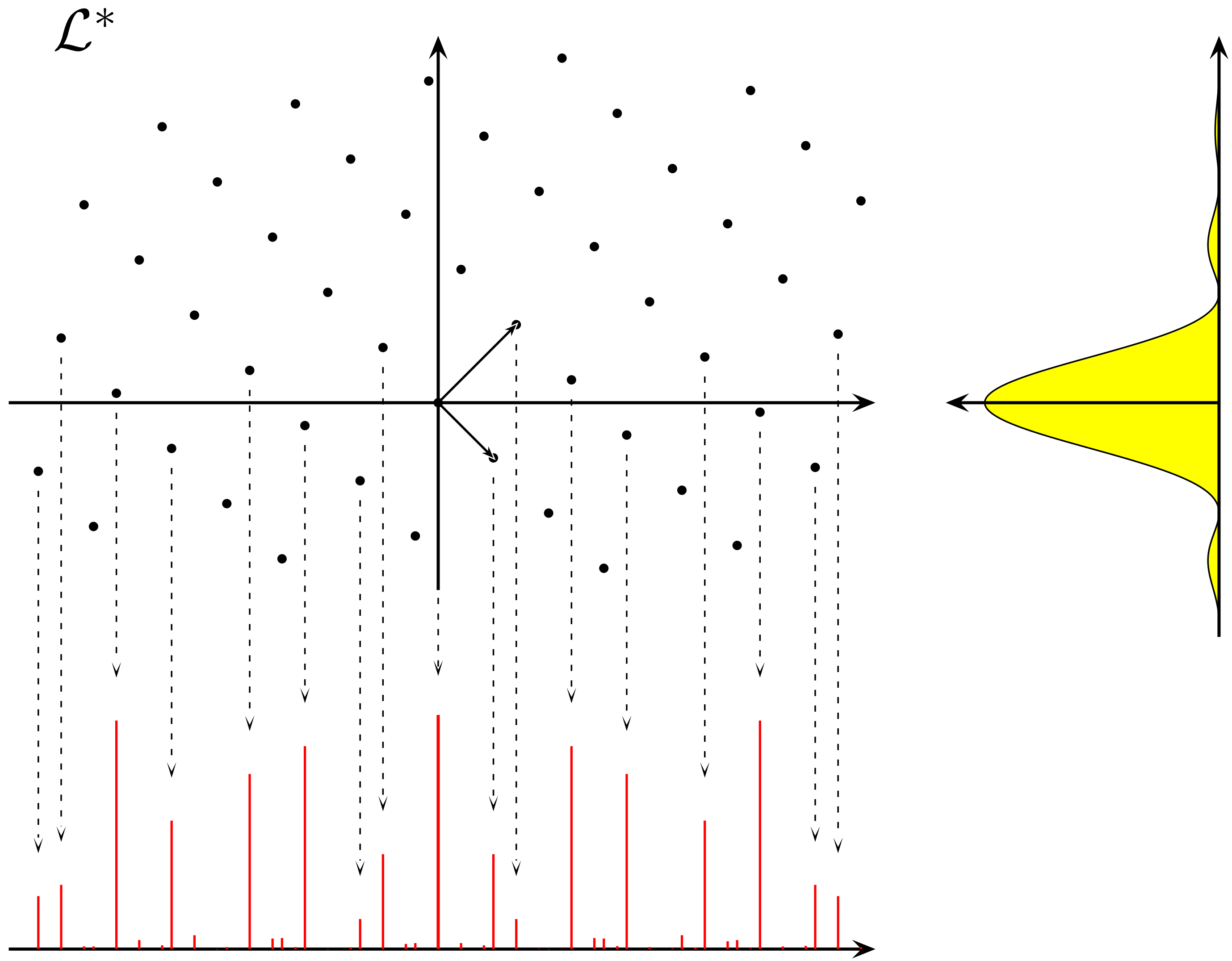
with $W \subset \mathbb{R}^m$ compact, $\lambda(\partial W) = 0$

Diffraction $\boxed{\widehat{\gamma} = \sum_{k \in L^{\circledast}} |A(k)|^2 \, \delta_k}$ pure point !! $(\omega = \delta_\varLambda)$

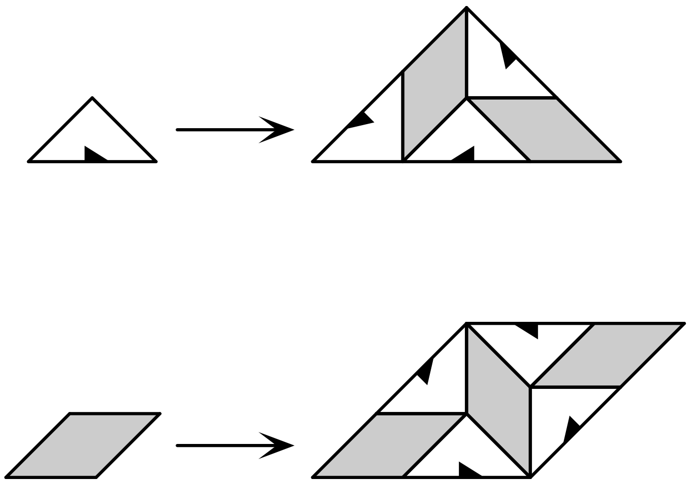
with $L^{\circledast} = \pi(\mathcal{L}^*)$ (Fourier module of $\varLambda$)

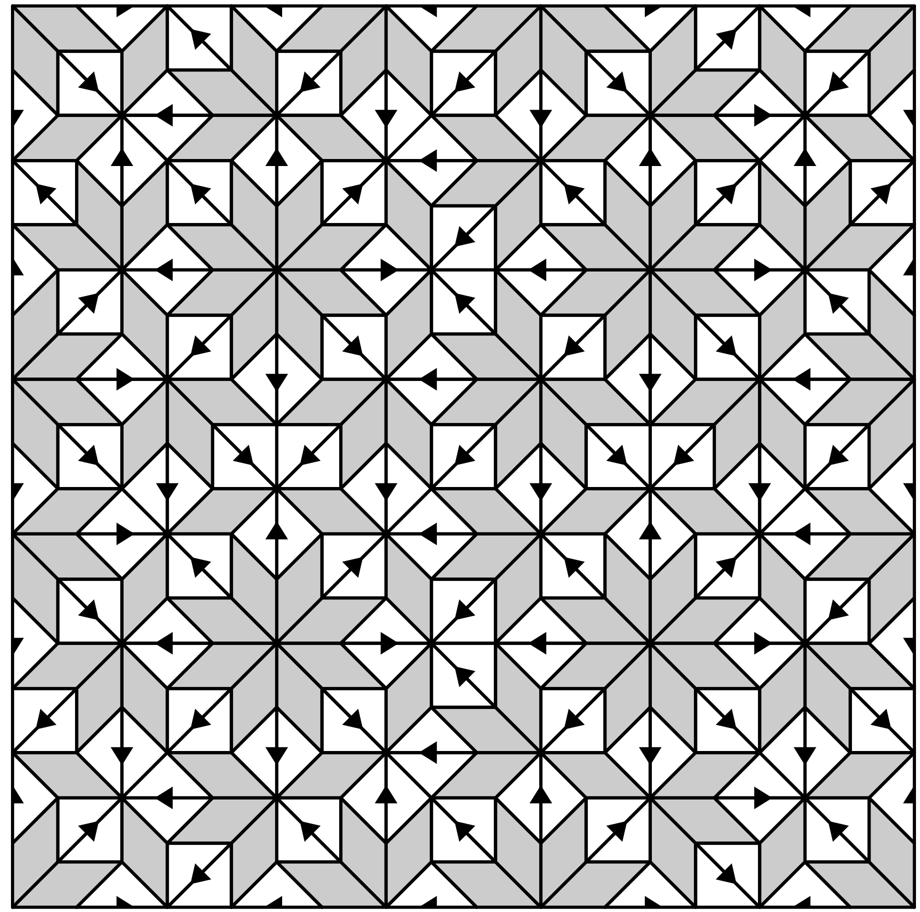
and amplitude $A(k) = \frac{\mathrm{dens}(\varLambda)}{\mathrm{vol}(W)} \, \widehat{1_W}(-k^\star)$

Example: Silver mean diffraction



Example: Ammann–Beenker



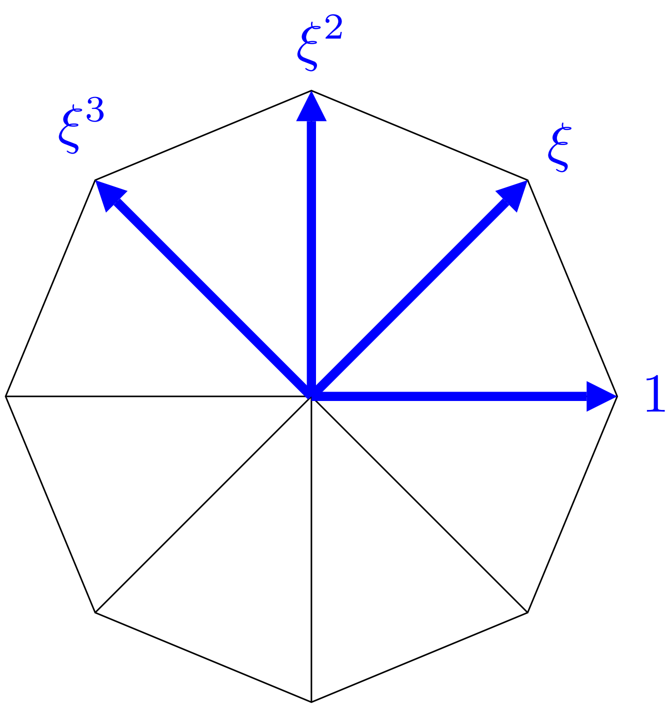


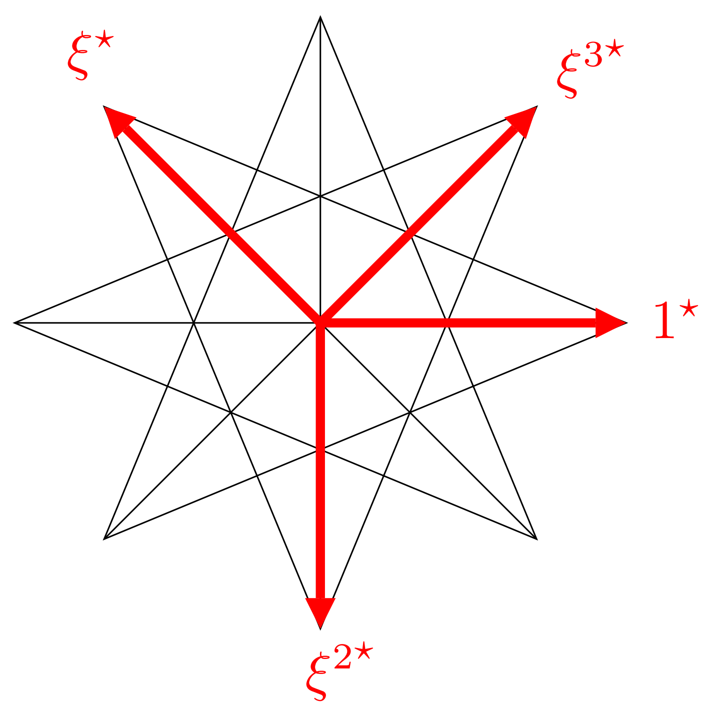
Example: Ammann–Beenker

$L = \mathbb{Z}[\xi]$ $\qquad$ $\mathcal{L} \sim \mathbb{Z}^4 \subset \mathbb{R}^2 \times \mathbb{R}^2$ $\qquad$ O: octagon

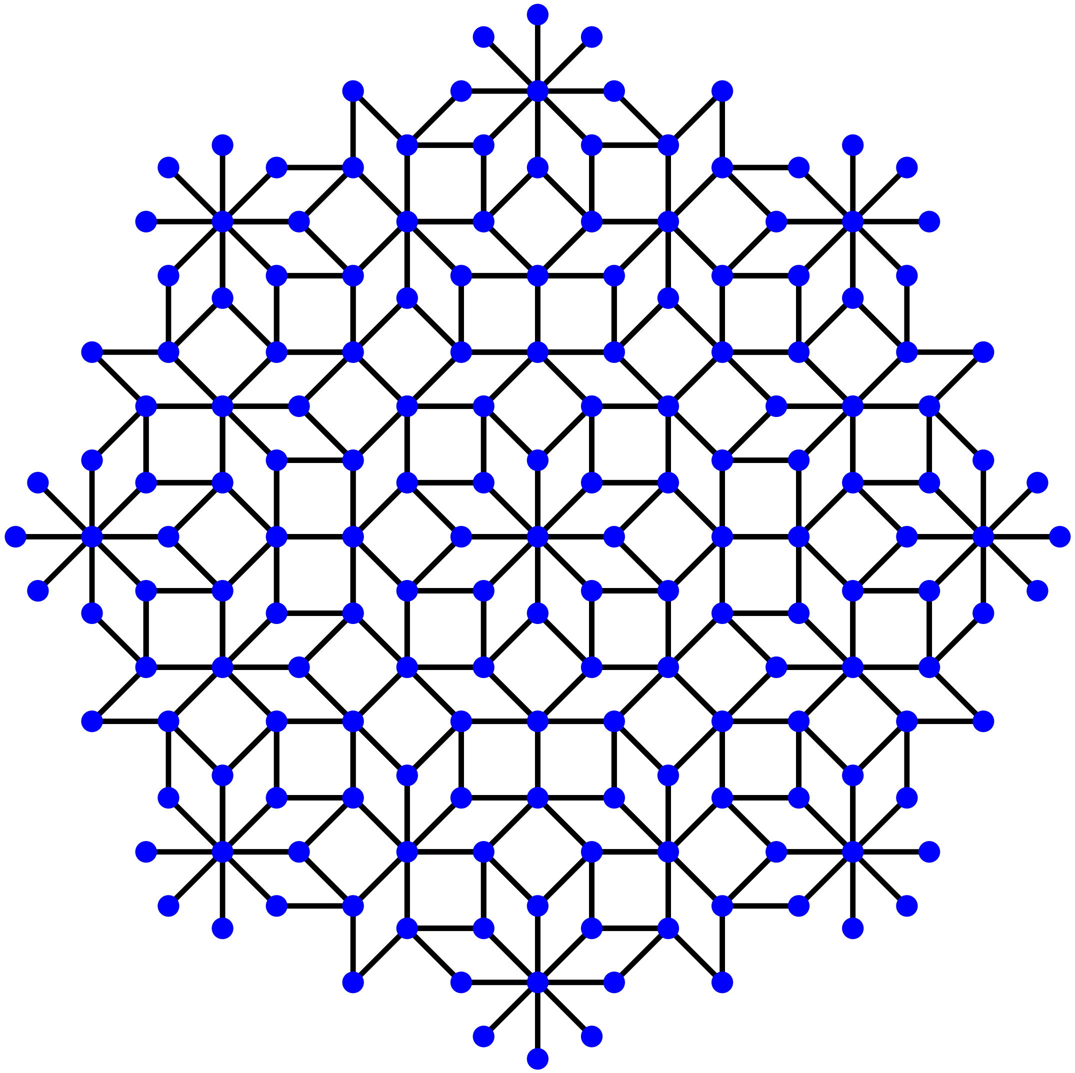
$\xi = \exp(2\pi i/8)$ $\qquad$ $\phi(8) = 4$ $\qquad$ $\star$-map: $\xi \mapsto \xi^3$

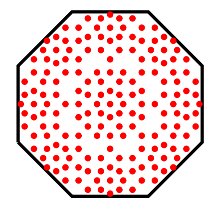
$$\varLambda_{\mathrm{AB}} = \left\{ x \in \mathbb{Z}1 + \mathbb{Z}\xi + \mathbb{Z}\xi^2 + \mathbb{Z}\xi^3 \mid x^\star \in O \right\}$$





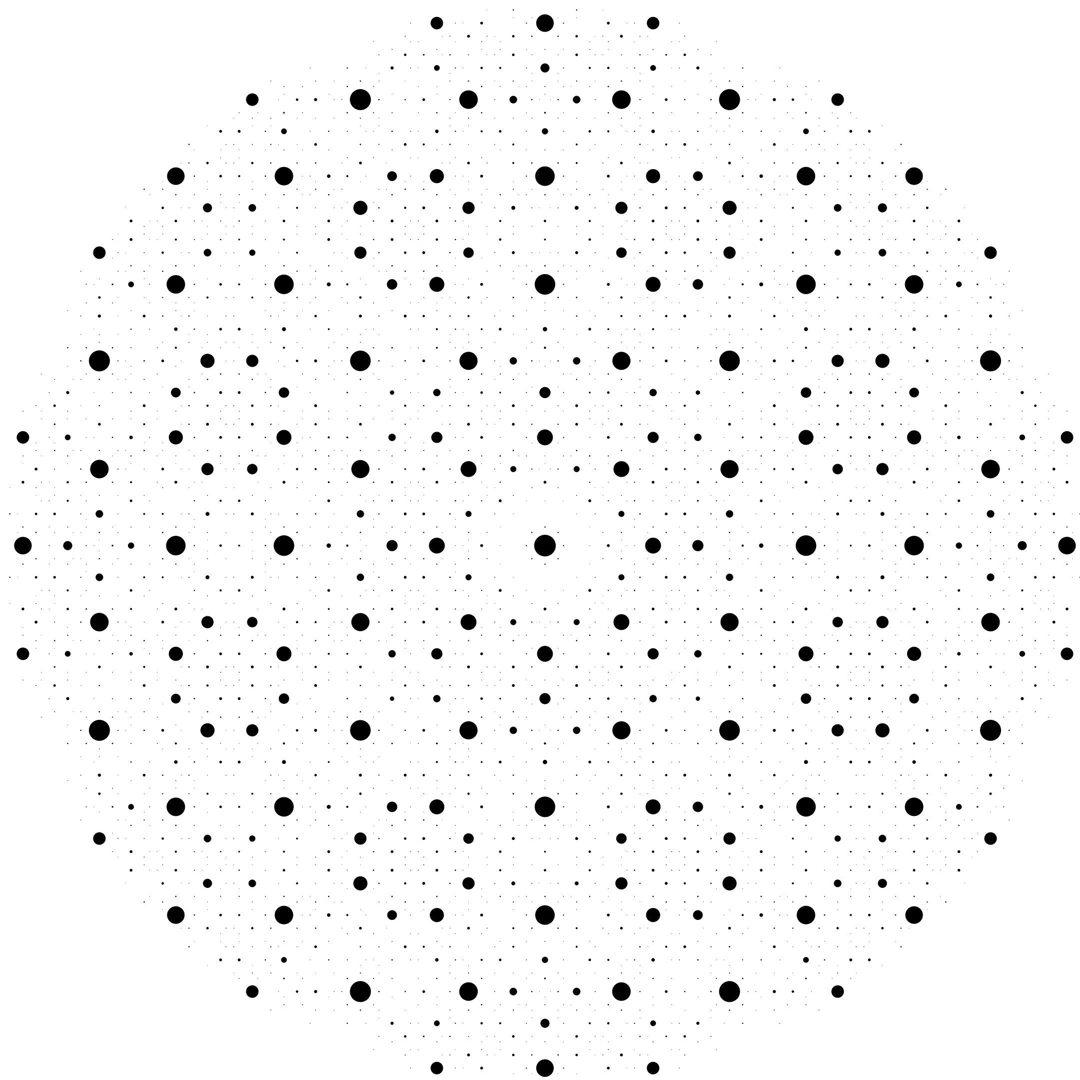
Example: Ammann–Beenker



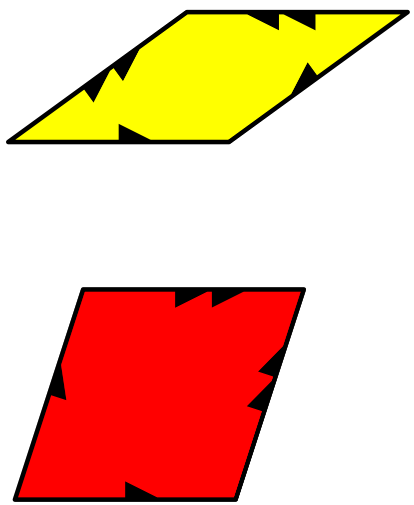


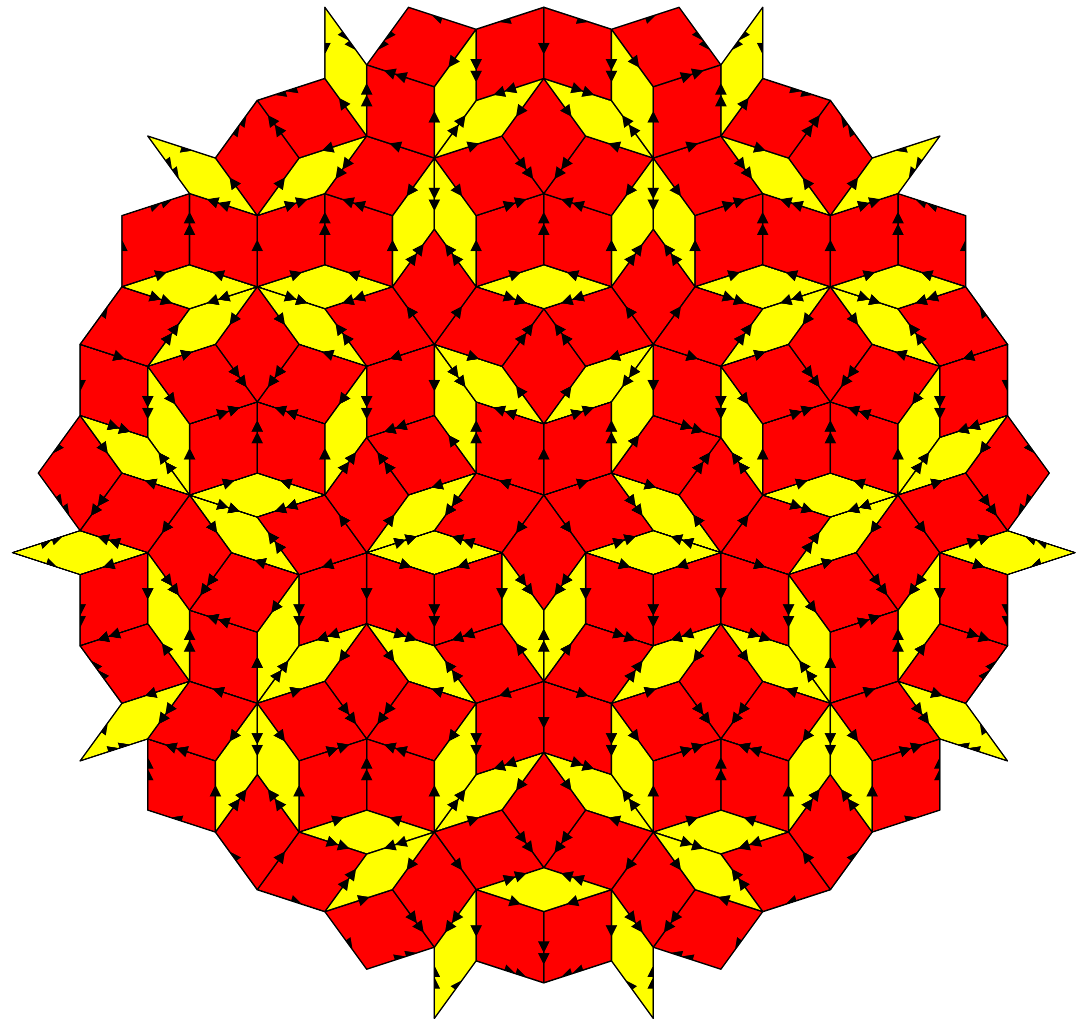
physical space   internal space

Example: Ammann–Beenker



Penrose tiling





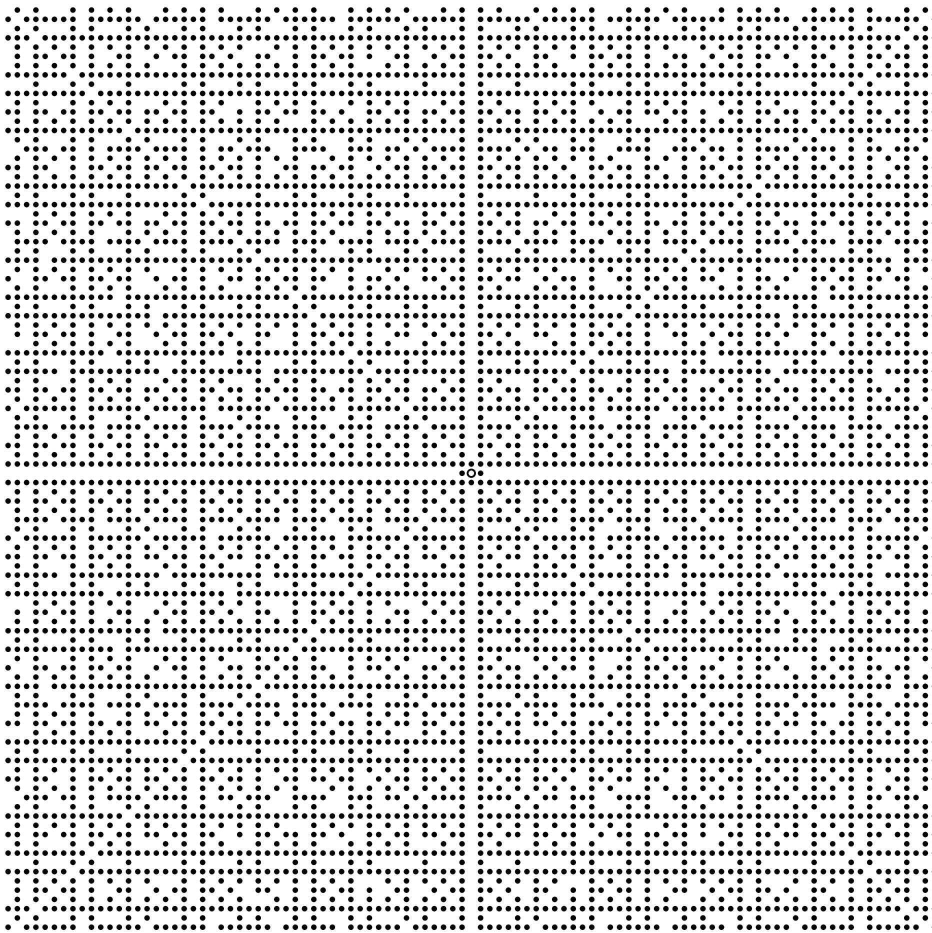
Penrose tiling, ctd.

Kleenex used the Penrose tiling to prevent layers of toilet paper from sticking together. Penrose was not amused.

“*So often we read of very large companies riding rough-shod over small businesses or individuals, but when it comes to the population of Great Britain being invited by a multi-national to wipe their bottoms on what appears to be the work of a Knight of the Realm without his permission, then a last stand must be made.*”

David Bradley, director of Pentaplex (the company that cares for Penrose’s copyrights)

Visible lattice points



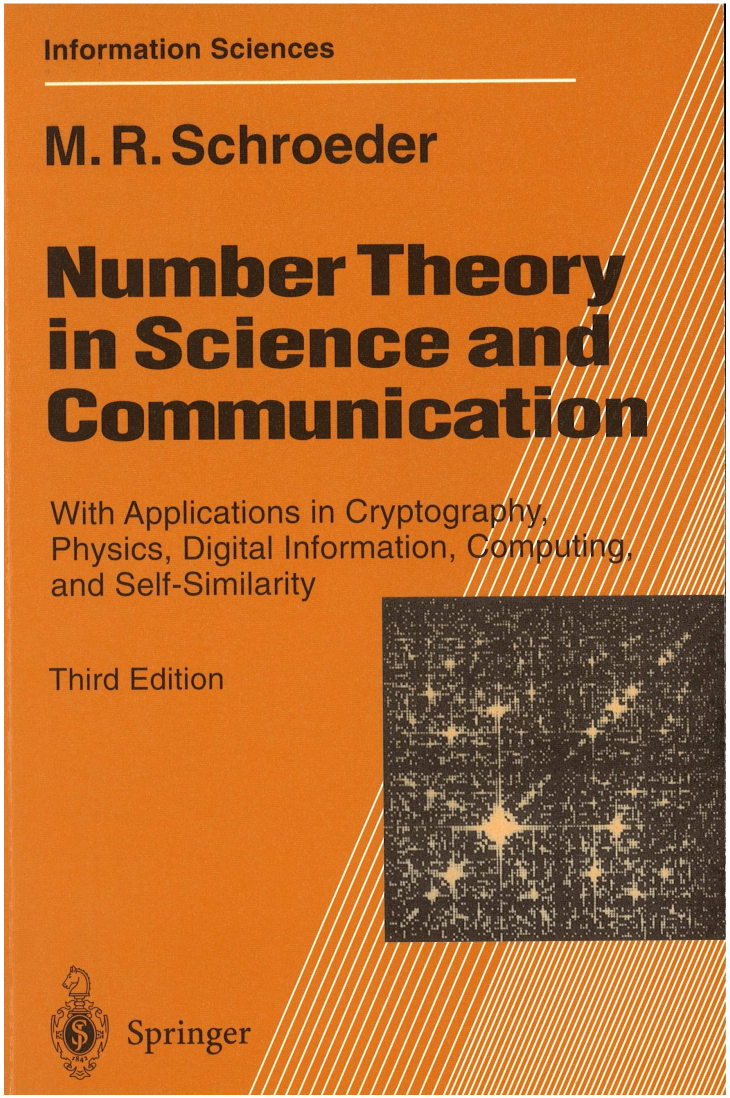
$$V = \{x \in \mathbb{Z}^2 \mid \gcd(x) = 1\}$$

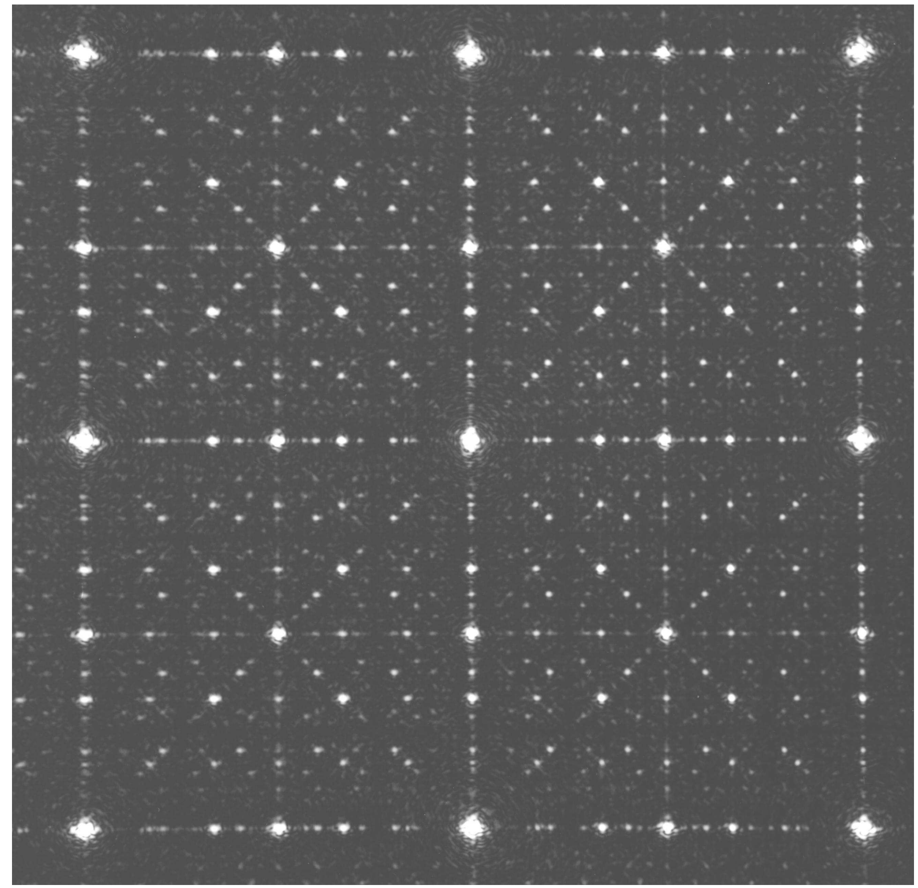
Properties

- $\mathrm{dens}(V) = 6/\pi^2$
- V not Delone
- $V - V = \mathbb{Z}^2$
- pure point diffraction
- weak model set
- $h_{\mathrm{top}}(V) > h_{\mathrm{m}}(V) = 0$

Theorem PP dynamical spectrum, trivial top. point spectrum

Visible lattice points

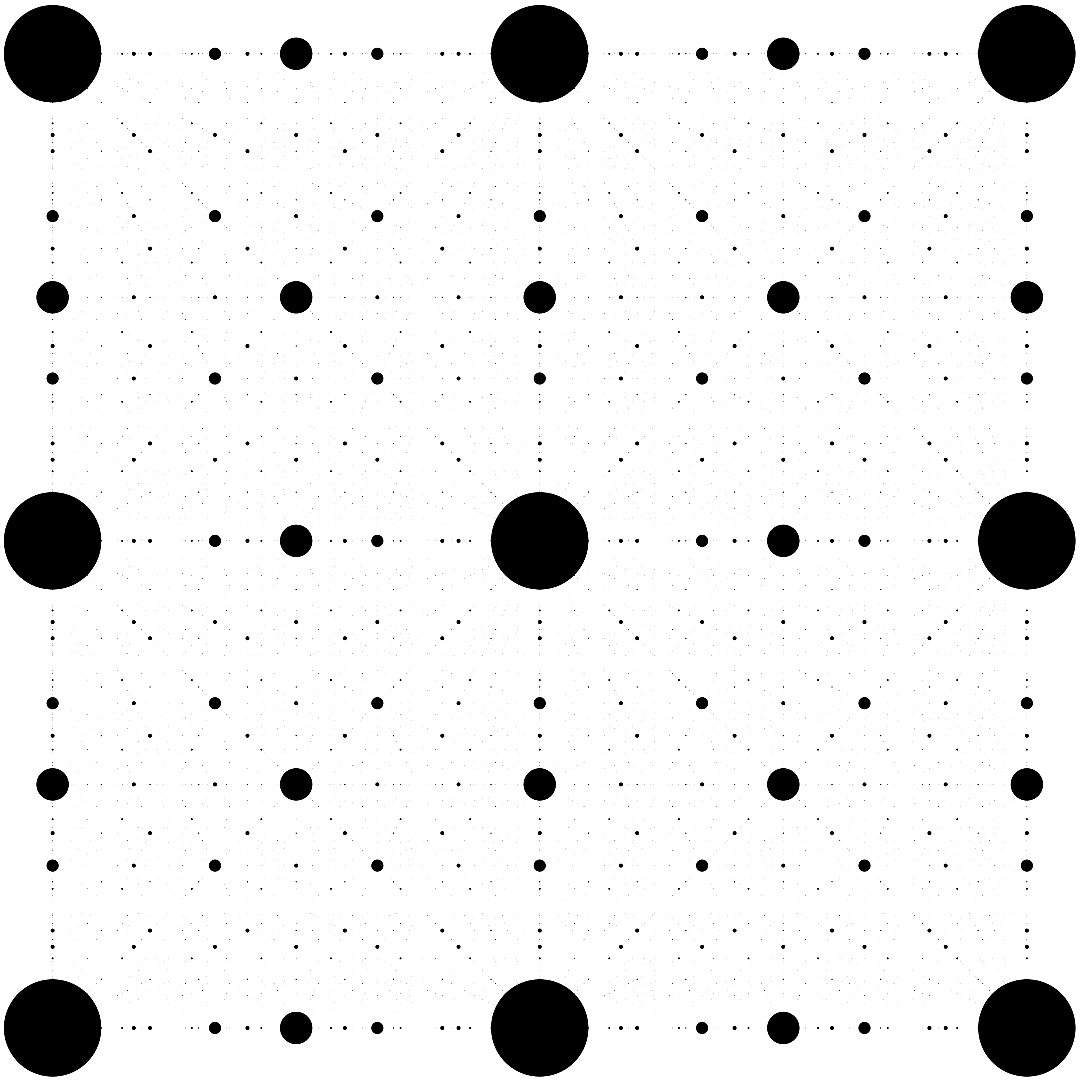




B/Grimm/Warrington 1994

Schroeder 1982, Mosseri 1992, B/Moody/Pleasants 2000, B/Huck 2013

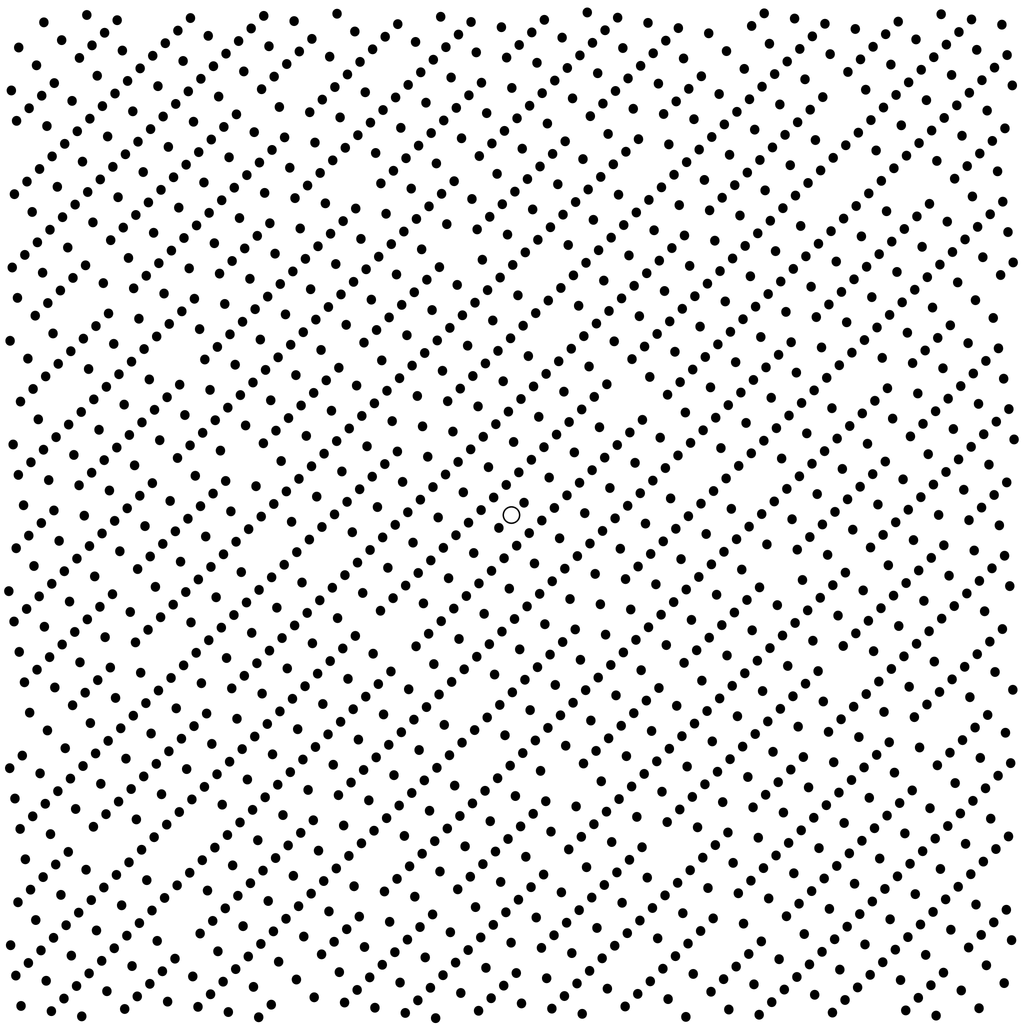
Visible lattice points



Properties

- $\mathbb{Z}^2$-periodic
- D_4-symmetric
- $\mathrm{GL}(2,\mathbb{Z})$-invariant
- support of $\widehat{\gamma}$: $S = \{k \in \mathbb{Q}^2$ with $\mathrm{den}(k)$ square-free$\}$
- intensity for $k \in S$ with $\mathrm{den}(k) = q$ $\left(\frac{6}{\pi^2}\right)^2 \prod_{p|q} \frac{1}{(p^2-1)^2}$

Squarefree integers in $\mathbb{Z}[\sqrt{2}\,]$



$V = \{(x, x') \mid x \text{ sq.-free}\}$

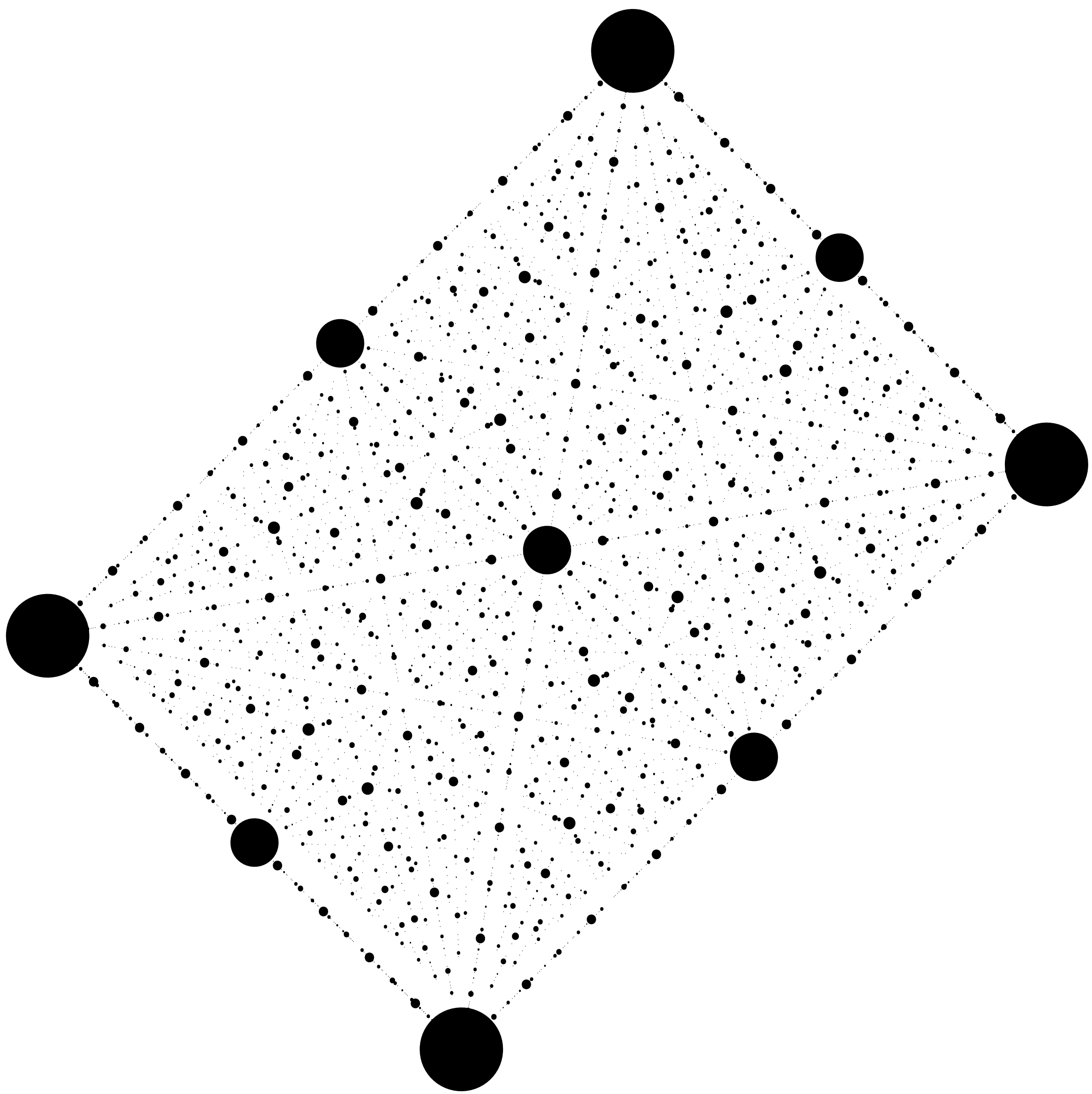
Properties

- $\mathrm{dens}(V) = \frac{24}{\pi^4} = \frac{\mathrm{dens}(\mathcal{L})}{\zeta_K(2)}$
- V not Delone
- $V - V = \langle V \rangle = \mathcal{L}$
- pure point diffraction
- weak model set
- $h_{\mathrm{top}}(V) > h_{\mathrm{m}}(V) = 0$

Theorem PP dynamical spectrum, trivial top. point spectrum

(Cellarosi/Vinogradov 2013, B/Huck 2013)

Squarefree integers in $\mathbb{Z}[\sqrt{2}\,]$



Properties

- $\mathcal{L}^*$-periodic
- $C_2 \times C_2$-symmetric
- $\mathrm{GL}(2,\mathbb{Z})$-invariant
- support of $\widehat{\gamma}$: $S = \{k \in \mathbb{Q}\mathcal{L}^*$ with $\mathrm{den}(k)$ cube-free$\}$
- intensity for $k \in S$ with $\mathrm{den}(k) = q$: $\left(\frac{24}{\pi^4}\right)^2 \prod_{p|q} \frac{1}{(N(p)^2-1)^2}$

Weak model sets (WMS)

CPS

$$\begin{array}{ccccc} G & \xleftarrow{\;\pi\;} & G \times H & \xrightarrow{\;\pi_{\rm int}\;} & H \\ \cup & & \cup & & \cup \ \text{dense} \\ \pi(\mathcal{L}) & \xleftarrow{\;1-1\;} & \mathcal{L} & \longrightarrow & \pi_{\rm int}(\mathcal{L}) \\ \| & & & & \| \\ L & & \xrightarrow{\qquad\qquad \star \qquad\qquad} & & L^{\star} \end{array}$$

G: σ-compact

H: comp. gen.

$\mathcal{A}$: van Hove in G

WMS

$$\boxed{\varLambda = \{ x \in L \mid x^{\star} \in W \}}$$

with $W \subset H$ compact, $\theta_H(W) > 0$

max. density: $\mathrm{dens}(\varLambda) = \mathrm{dens}(\mathcal{L})\, \theta_H(W)$

$$\gamma_{\varLambda} := \lim_{n\to\infty} \frac{\delta_{\varLambda \cap A_n} * \delta_{-(\varLambda \cap A_n)}}{\theta_G(A_n)}$$ (exists !!)

Weak model sets (WMS)

Diffraction $\boxed{\widehat{\gamma} = \sum_{k\in L^0} |A(k)|^2\, \delta_k}$ pure point !! $(\omega = \delta_{\varLambda})$

with $L^0 = \pi(\mathcal{L}^0)$ (annihilator of $\mathcal{L}$ in dual CPS)

amplitude $A(k) = \frac{\mathrm{dens}(\varLambda)}{\theta_H(W)}\, \widehat{1_W}(-k^{\star})$

Hull $\boxed{\mathbb{X}_{\varLambda} = \overline{G + \varLambda}}$

with patch frequency measure μ

μ is ergodic, $\varLambda$ is generic for μ

Theorem $(\mathbb{X}_{\varLambda}, G, \mu)$ has pure point dynamical spectrum: L^0

(Keller/Richard 2015, B/Huck/Strungaru 2015)

Extensions

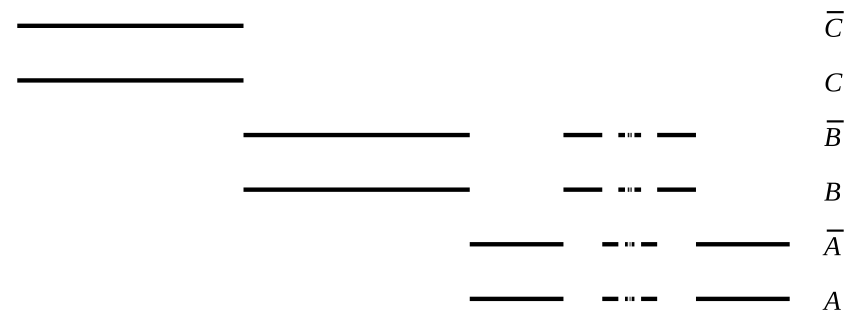
Systems with mixed spectrum (Gähler 2014)

$\sigma: \; A \mapsto BC\bar{B},\, B \mapsto BC\bar{A},\, C \mapsto \bar{A}$

$\curvearrowright$ CPS of silver mean chain

a.e. 2-1 cover of silver mean chain

$\curvearrowright$ mixed spectrum (pp + sc)



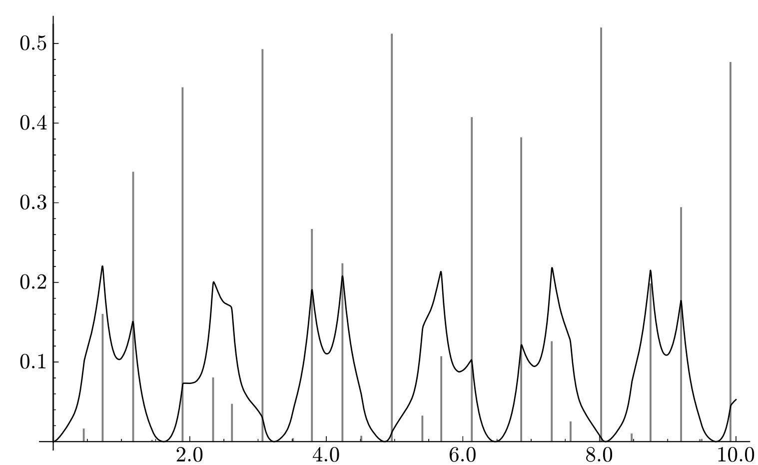
Meyer sets with entropy (Godrèche/Luck 1987, Moll 2014)

$\sigma: \; a \mapsto ab \,/\, ba$ (prob. $p \,/\, q$), $b \mapsto a$

$\curvearrowright$ positive entropy, but still Meyer

stochastic extension of Fibonacci chain

$\curvearrowright$ mixed spectrum (pp + ac)



Outlook

- Diffraction as useful tool
- Diffraction versus dynamical spectrum
- Weak model sets included
- Mixed spectra accessible
- Toeplitz systems as examples
- Generalisations beyond lattice systems
- Extensions to higher dimensions
- Contributions to a classification

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