Automorphism groups of subshifts via group extensions

(joint work with Ville Salo)

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Basic notions from symbolic dynamics

$\mathcal{A}$ some finite (discrete) **alphabet**

G a countable, discrete and **finitely generated (f.g.) group**

$\sigma: \; G \times \mathcal{A}^G \to \mathcal{A}^G$ (left) **shift action** of G on the full shift $\mathcal{A}^G$ (homeomorphisms)

$(g, x) \mapsto \sigma_g(x)$ where $\forall h \in G$: $\big(\sigma_g(x)\big)_h := x_{g^{-1}h}$

G-**subshift**: $X \subseteq \mathcal{A}^G$ shift invariant, closed subset

given by a family of forbidden patterns $\mathcal{F} \subseteq \bigcup_{F \subsetneq G \text{ finite}} \mathcal{A}^F$ on finite shapes such that

$$\mathsf{X}_{\mathcal{F}} := \left\{ x \in \mathcal{A}^G \;\middle|\; \forall\, F \subsetneq G \text{ finite}: \; x|_F \notin \mathcal{F} \right\}$$

X is a G-**SFT** $:\Longleftrightarrow \exists\, \mathcal{F} \subseteq \bigcup_{F \subsetneq G \text{ finite}} \mathcal{A}^F$ with $|\mathcal{F}| < \infty$ and $X = \mathsf{X}_{\mathcal{F}}$ (local rules)

X is a G-**sofic shift** $:\Longleftrightarrow X$ is a (subshift) factor of some G-SFT

The **automorphism group** of X:

$$\mathrm{Aut}(X) := \{\phi: \; X \to X \text{ homeomorphism} \mid \forall\, g \in G: \; \sigma_g \circ \phi = \phi \circ \sigma_g\}$$

Questions about $\mathrm{Aut}(X)$

$$\mathrm{Aut}(X) := \{\phi : X \to X \text{ homeomorphism} \mid \forall\, g \in G : \ \sigma_g \circ \phi = \phi \circ \sigma_g\}$$

- Describe $\mathrm{Aut}(X)$ **explicitly** or even construct some subshift X which has a given group as $\mathrm{Aut}(X)$ (**realization problem**).
- Study the algebraic structure, i.e. **subgroups** and **algebraic properties** (torsion, simple, **residually finite**) of $\mathrm{Aut}(X)$.
- Decide when $\mathrm{Aut}(X)$ is “small” (**finitely generated**, **amenable**, **virtually-$\mathbb{Z}$** etc.) vs. “big” and “complicated”.
- **Classify** subshifts using $\mathrm{Aut}(X)$ (Fact: two subshifts X, Y are **non-conjugate** whenever $\mathrm{Aut}(X)$ and $\mathrm{Aut}(Y)$ are **non-isomorphic** groups).

We will see **distinct classes** of subshifts which have **very different** automorphism groups.

Basic notions from group theory

Given groups G, H, K we say G is a **group extension of** H **by** K iff

$$\exists \text{ short exact sequence } \quad \mathbf{1} \to K \xrightarrow{\imath} G \xrightarrow{\phi} H \to \mathbf{1}$$

i.e. $\imath: K \hookrightarrow G$ (inclusion) and $\phi: G \twoheadrightarrow H$ (surjection) are homomorphisms, $\imath(K) = \ker(\phi)$.

If there exists a **section** $\tau: H \to G$ (homomorphism) such that $\phi \circ \tau = \mathrm{Id}_H$, we say the extension is **(right) split**.

Lemma: Let G, H, K be groups.

(1) If $\mathbf{1} \to K \xrightarrow{\imath} G \xrightarrow{\phi} H \to \mathbf{1}$ is **right split** then G is (isomorphic to) a **semi-direct product** $K \rtimes_\psi H$.

(2) If furthermore $\tau(H) \trianglelefteq G$ then $G \cong K \times H$ is a **direct product**.

Basic ways of forming new groups from G and H:

$G \times H = \{(g, h) \mid g \in G,\ h \in H\}$ with product $(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2)$.

$G \rtimes_\psi H = \{(g, h) \mid g \in G,\ h \in H\}$ with product $(g_1, h_1) \cdot (g_2, h_2) = (g_1\, \psi_{h_1}(g_2), h_1 h_2)$,
where H acts on G (from the left) by group automorphisms $\psi: H \times G \to G$ such that $g \mapsto \psi_h(g) = \psi(h, g)$.

The smallest case – A classification of $\mathrm{Aut}(X)$ for finite subshifts

Observation: $\mathrm{Aut}(X)$ is **finite** if and only if X is **finite**. (otherwise $\mathrm{Aut}(X)$ is countably infinite)

Observation: Every automorphism **maps a periodic point** into a **periodic point** of the **same period**. (induces a permutation on $\mathrm{Per}_n(X) = \{x \in X \mid \sigma^n(x) = x\}$)

We obtain a **complete description** of automorphism groups for **finite** $\mathbb{Z}$-subshifts:

Corollary: Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a finite $\mathbb{Z}$-subshift. Then the shift σ is (conjugate to) a permutation $\pi \in S_{|X|}$. Let $\pi = \prod_{i=1}^{I} \prod_{j=1}^{J_i} \pi_{i,j}$ be the **cycle decomposition** of π where each $\pi_{i,j}$ is an i-cycle. Then

$$\mathrm{Aut}(X) \cong \prod_{i=1}^{I} (\mathbb{Z}/i\mathbb{Z})^{J_i} \rtimes S_{J_i} \ .$$

(similar result for finite G-subshifts, however cyclic groups $\mathbb{Z}/i\mathbb{Z}$ would have to be replaced by more general finite groups)

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Proposition: (1) Let $X_1 \cong X_2 \cong \ldots \cong X_I$ be a family of **disjoint transitive** G-subshifts all **conjugate** one to another. Then

$$\mathrm{Aut}\big(\bigsqcup_{i=1}^{I} X_i\big) \cong \mathrm{Aut}(X_1)^I \rtimes S_I \ ,$$

where the symmetric group S_I acts on $\bigsqcup_{i=1}^{I} X_i$ by permuting its components.

(2) Let $\left(X_{i,j}\right)_{i,j}$ be a family of **disjoint transitive** G-subshifts where $1 \leqslant i \leqslant I$ and $1 \leqslant j \leqslant J_i$, and suppose that $X_{i,j} \cong X_{i',j'} \iff i = i'$. Then

$$\mathrm{Aut}\big(\bigsqcup_{i,j} X_{i,j}\big) \cong \prod_{i=1}^{I} \mathrm{Aut}(X_{i,1})^{J_i} \rtimes S_{J_i} \ .$$

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The other extreme – Known results about $\mathrm{Aut}(X)$ for (mixing) $\mathbb{Z}$-SFTs X

Principal tool in all proofs: Automorphisms given by (local) **Marker constructions**

- Non-trivial, mixing $\mathbb{Z}$-SFTs have **positive entropy** (large flexibility for constructions)
- $\mathrm{Aut}(X)$ is countable, non-abelian, with **center isomorphic to** $\mathbb{Z}$ (powers of the shift map)
- $\mathrm{Aut}(X)$ is **huge**. In particular contains isomorphic copies of every finite group, $\mathbb{Z}$, the free group on countably many generators and countable direct sums of such groups
- $\mathrm{Aut}(X)$ is **not finitely generated** and **not amenable**. (exponential growth in f.g. subgroups)
- $\mathrm{Aut}(X)$ is **residually finite** (use denseness of periodic points and periodic point representation)
- $\mathrm{Aut}(X)$ has **decidable word problem**

(BLR 1988, KR 1990, etc.)

Qualitatively all Aut-groups of mixing $\mathbb{Z}$-SFTs look (more or less) the same

Those results extend to the setting of **mixing $\mathbb{Z}$-sofic shifts**

A general philosophy for studying $\mathrm{Aut}(X)$

Automorphisms do preserve both the **topological** as well as the **dynamical** structure

(e.g. orbits go to orbits, periodic points to periodic points, isolated points to isolated points etc.)

Metatheorem: X some G-subshift, $Y \subsetneq X$ a subset **defined uniquely** by "good" topological and dynamical properties, then every automorphism $\phi \in \mathrm{Aut}(X)$ preserves Y.

In particular the **restriction homomorphism** $\phi : \mathrm{Aut}(X) \to \mathrm{Aut}(Y)$ is well-defined with $\phi|_Y \in \mathrm{Aut}(Y)$.

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This works in particular for $Y = \mathrm{Fix}_H(X) = \{x \in X \mid \mathrm{Stab}(x) = H\}$ (set of H-periodic points)

$Y = \mathcal{T}(X)$ (set of transitive points)

$Y = \Omega(X)$ (set of non-wandering points)

$Y = X'$ (Cantor-Bendixson derivative)

$Y =$ set of points $x \in X$ whose language $\mathcal{L}(x)$ is a maximal (resp. minimal) element of the subpattern poset (ordered by inclusion) of X (e.g. transitive points or fixed points)

(obviously also works for the complement of each Y)

Cantor-Bendixson rank of a subshift

X a G-subshift

Define the **Cantor-Bendixson derivative** of X

$$X' := X \setminus \text{ isolated points} = \{x \in X \mid \forall\, x \in \mathcal{U} \subseteq X \text{ clopen: } |\mathcal{U}| \geqslant 2\}$$

$X^0 := X$, $X^{\alpha+1} := (X^\alpha)'$, $X^\lambda := \bigcap_{\alpha<\lambda} X^\alpha$ (transfinite induction for ordinals α and limit ordinals λ)

The **Cantor-Bendixson rank** of X is the smallest ordinal α such that $X^{\alpha+1} = X^\alpha$

Example: $\mathcal{A} = \{0,1\}$, $k \in \mathbb{N}_0$

$X_{\leqslant k} := \{x \in \mathcal{A}^G \mid \#_1(x) \leqslant k\}$ (subshift whose points contain at most k copies of symbol 1)

$G = \mathbb{Z}$: $X_{\leqslant 0} = \{0^\infty\}$, $X_{\leqslant 1} = \{0^\infty\} \cup \mathrm{Orb}\{0^\infty.1\,0^\infty\}$ (sunny side up shift), etc.

$\forall\, k \in \mathbb{N}$: $(X_{\leqslant k})' = X_{\leqslant k-1}$ and $(X_{\leqslant 0})' = \varnothing$ hence $X_{\leqslant k}$ has C.B.-rank $k+1$

Cantor-Bendixson derivatives and a first result for countable $\mathbb{Z}$-sofics

Theorem: Let $X \subseteq \mathcal{A}^G$ be any G-subshift, and let $Y = X'$ denote its **Cantor-Bendixson derivative**. Then the sequence

$$\mathbf{1} \to K \to \mathrm{Aut}(X) \to H \to \mathbf{1}$$

is **exact**, where $H = \{\phi|_Y \mid \phi \in \mathrm{Aut}(X)\} \leqslant \mathrm{Aut}(Y)$ and K is a direct union of subgroups of G^*-by-finite groups.

(Given a (f.g.) group G, let G^* denote the family of groups obtained from G by taking finite direct products.)

Remark: The image H of the restriction homomorphism can be **very small** inside $\mathrm{Aut}(Y)$. We do not know if the above group extension always splits (it does in all our explicit examples).

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Theorem: Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a **countable $\mathbb{Z}$-sofic** shift. Then $\mathrm{Aut}(X)$ has **decidable torsion problem**, i.e. for each $f \in \mathrm{Aut}(X)$ it is decidable whether or not f generates a finite group. (already in Salo & Törmä, 2012)

(Isolated points in countable $\mathbb{Z}$-sofic are eventually periodic and X' can be computed from X using the graph presentation of X.)

Some explicit computations of $\mathrm{Aut}(X)$ Assume $G = \mathbb{Z}$ for simplicity

$\mathrm{Aut}(X_{\leqslant 1}) = \langle \sigma \rangle \simeq \mathbb{Z}$ (unitary point is fixed by all automorphisms, only shifts on isolated points)

$\mathrm{Aut}(X_{\leqslant 2}) \simeq \mathbb{Z} \times (\mathbb{Z}^\infty \rtimes S_\infty)$ (theorem applies, explain structure, not finitely generated) . . .

In principle able to determine $\mathrm{Aut}(X_{\leqslant k})$ explicitly for all $k \in \mathbb{N}$, but expressions get more and more complicated.

Proposition: The infinite periodic subshift has $\mathrm{Aut}\big(\overline{\bigcup_{n\in\mathbb{N}} \mathrm{Orb}\{(1\,0^n)^\infty\}}\big) \simeq \mathbb{Z} \times \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_n$

Idea of proof: $\big(\overline{\bigcup_{n\in\mathbb{N}} \mathrm{Orb}\{(1\,0^n)^\infty\}}\big)' = X_{\leqslant 1}$ and in this particular case our exact sequence splits with normal image, which gives $\mathbb{Z}$ in the direct product.

So we only have to determine the kernel of the restriction map.

Every automorphism preserves period, thus maps periodic orbits into themselves and (being a sliding-block-code) also has to fix points with very large periods.

More results about countable $\mathbb{Z}$-sofics

X **uncountable, transitive** $\mathbb{Z}$-sofic shift (dense periodic points) $\Longrightarrow \operatorname{Aut}(X)$ is **residually finite**
($\mathbb{Z}$-subshifts with dense periodic points always have residually finite automorphism groups)

Proposition: Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a **countable** $\mathbb{Z}$-sofic shift. (zero entropy, no Marker automorphisms)

- If C.B.-rank of X is 1 (i.e. $|X| < \infty$), then $\operatorname{Aut}(X)$ is **finite**.
- If C.B.-rank of X is 2 (i.e. orbit closure of finitely many eventually periodic points), then $\operatorname{Aut}(X) \leqslant \mathbb{Z}^n \rtimes G$ for some $G \leqslant S_n$.
- If C.B.-rank of X is at least 3, then $\mathbb{Z}^\infty \rtimes S_\infty \leqslant \operatorname{Aut}(X)$ and in particular $\operatorname{Aut}(X)$ is **not residually finite**.

(Explain $X_{\leqslant k}$ examples, mention complexity function)

The last item is a consequence of the following more general theorem:

Theorem: Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a countable $\mathbb{Z}$-subshift containing an **infinite set of isolated points** in distinct orbits with the **same eventually periodic tails**. Then $\mathbb{Z}^\infty \rtimes S_\infty \leqslant \operatorname{Aut}(X)$.

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Comparison between $\mathrm{Aut}(X_{\leqslant 2})$ and $\mathrm{Aut}\big((X_{\leqslant 1})^2\big)$

Taking Cartesian products of countable subshifts (even countable $\mathbb{Z}$-sofics) makes the Aut-group much **more complicated** and in particular **destroys many algebraic properties**.

$\mathrm{Aut}(X_{\leqslant 2}) \cong \mathbb{Z} \times (\mathbb{Z}^\infty \rtimes S_\infty)$	$\not\cong$	$\mathrm{Aut}\big((X_{\leqslant 1})^2\big) \cong (\mathbb{Z}^\infty \rtimes S_\infty) \rtimes (\mathbb{Z}^\infty \rtimes S_2)$
amenable		amenable
not residually finite		not residually finite
not finitely generated (S_∞ + lemma)		**finitely generated** (3 generators)
locally virtually abelian		not locally virtually solvable
all f.g. subgroups have **polynomial growth**		**exponential growth** (free monoid on 2 generators)

$\mathrm{Aut}\big((X_{\leqslant 1})^2\big)$ already contains isomorphic copies of **all finite** and **all finitely generated abelian** groups. (huge automorphism group despite being zero entropy, countable $\mathbb{Z}$-sofic)

Remark: There is a weak partial result about when the automorphism group of a Cartesian product of G-subshifts X_1, X_2 is a (semi-direct) product of $\mathrm{Aut}(X_1)$ and $\mathrm{Aut}(X_2)$. (uses strong independence assumptions)

Structure of $\mathrm{Aut}(X)$ for countable G-subshifts

Iterated application of the theorem on the Cantor-Bendixson derivative gives very strong structural results about $\mathrm{Aut}(X)$.

Definition: Let $\mathcal{G}$ be a class of (abstract) groups such that

- $\mathcal{G}$ contains all **finite groups**,
- $\mathcal{G}$ is closed under taking **subgroups**,
- $\mathcal{G}$ is closed under forming **directed unions**, and
- $\mathcal{G}$ is closed under forming **group extensions**.

Then we say $\mathcal{G}$ is an **elementary class**. The **elementary closure** of a group (or family of groups) G is the smallest class of groups which is elementary and contains G. Groups in the elementary closure of G are called G-**elementary**.

Remark: Definition contains "natural" group operations and almost coincides with definition of **elementary amenable (EA)** (abelian groups not included).

Theorem: Let $X \subseteq \mathcal{A}^G$ be a countable G-subshift. Then $\mathrm{Aut}(X)$ is in the **elementary closure of** G.

Consequences on $\mathrm{Aut}(X)$ for countable G-subshifts

Theorem: Let $X \subseteq \mathcal{A}^G$ be a countable G-subshift. Then $\mathrm{Aut}(X)$ is in the **elementary closure of** G.

Proof ingredients: C-B-derivative Theorem implies $\ker(\phi)$ is G-elementary.

$\phi_1 : G_0 \to G_1$, $\phi_2 : G_1 \to G_2$ homomorphisms, then $\ker(\phi_2 \circ \phi_1)$ is a group extension of a subgroup of $\ker(\phi_2)$ by a subgroup of $\ker(\phi_1)$.

Transfinite induction on C-B-rank of subshift, the kernel of ϕ stays in elementary closure.

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Transfinite induction on C-B-rank of subshift, the kernel of ϕ stays in elementary closure.

Corollary: If G is **amenable (resp. elementary amenable, torsion, locally finite)**, and $X \subseteq \mathcal{A}^G$ is a countable G-subshift, then $\mathrm{Aut}(X)$ is **amenable (resp. elementary amenable, torsion or locally finite)**.

Corollary: In particular when X is a **countable $\mathbb{Z}$-subshift**, $\mathrm{Aut}(X)$ is **elementary amenable**. (For uncountable $\mathbb{Z}$-sofics (positive entropy) $\mathrm{Aut}(X)$ is **not amenable**)

Corollary: Let $X \subseteq \mathcal{A}^G$ be a G-subshift. Then there exists a **perfect** G-subshift $Y \subseteq X$ such that $\mathrm{Aut}(X)$ is a **group extension** of $\mathrm{Aut}(Y)$ by a G-**elementary group**.

What remains of Ryan's theorem?

Useful (only tool) for proving non-isomorphism of automorphism groups for mixing $\mathbb{Z}$-SFTs

Proposition: Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a **countable $\mathbb{Z}$-sofic**, then

$$\mathcal{Z}\big(\mathrm{Aut}(X)\big) \cong \mathbb{Z}^k \times {\mathbb{Z}_2}^l \qquad \text{for some } k \in \mathbb{N} \text{ and } l \in \{0,1\}$$

However, there exists $X \subseteq \mathcal{A}^{\mathbb{F}_2}$ **countable sofic shift on the free group** on 2 generators such that:

- X is faithful (i.e. $\bigcap_{x \in X} \mathrm{Stab}(x) = \mathbf{1}$)
- C-B-rank of X is 2
- $\mathrm{Aut}(X) = \mathbf{1}$