EXERCISES: CHILE LECTURES

1) Prove that for each dimension, d, there is a constant C_d such that for each $A \in M_{d\times d}(\mathbb{R})$,

$$(1/C_d)\max_{i,j}|A_{ij}| \le s_1(A) \le C_d \max_{i,j}|A_{ij}|.$$

2) Prove that for each dimension, d, there is a constant C_d such that for each $A \in M_{d\times d}(\mathbb{R})$,

$$(1/C_d)\max_{i,j,k,l}|A_{ij}A_{kl} - A_{ik}A_{jl}| \le s_2(A) \le C_d \max_{i,j,k,l}|A_{ij}A_{kl} - A_{ik}A_{jl}|.$$

3) Prove that for each dimension, d, there is a constant C_d such that for each $A \in M_{d\times d}(\mathbb{R})$, $\alpha(A)/C_d \le s_d(A) \le \alpha(A)$, where $\alpha(A) = \min_i d(A_i, \mathrm{lin}\{A_j : j \neq i\})$ and A_i denotes the ith row of A.

4) Let σ be a measure-preserving transformation and let $f \in L^1(\Omega, \mathbb{P})$. Prove that for $\mathbb{P}$-a.e. ω, $f(\sigma^n\omega)/n \to 0$. (Hints: the easiest proof that I know goes via the Birkhoff ergodic theorem; for a more elementary proof, try a proof based on Borel–Cantelli)

5) Let σ be an invertible ergodic measure-preserving transformation and let (f_n) be a sub-additive sequence of functions.

Prove that for $\mathbb{P}$-a.e. ω, $(1/n)f_n(\sigma^{-n}\omega)$ is a convergent sequence, and that the limit is $\lim_n (1/n)\int f_n\,d\mathbb{P}$.

Deduce that the dual cocycle has the same exponents as the primal cocycle.

6) Consider the three matrices $H = \begin{pmatrix} 4 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$ and $R_\pm = \mathsf{Rot}_{\pm\pi/3}$. Let μ be any ergodic invariant measure on the symbols $\{H, R_\pm\}$ with $\mu([H]) > 0$. Prove that the Lyapunov exponents of the cocycles are non-zero.

[Hint: consider splitting the directions (= the Grassmannian) into suitable cones.]