A probability concentration phenomenon in system reliability under dependent failures

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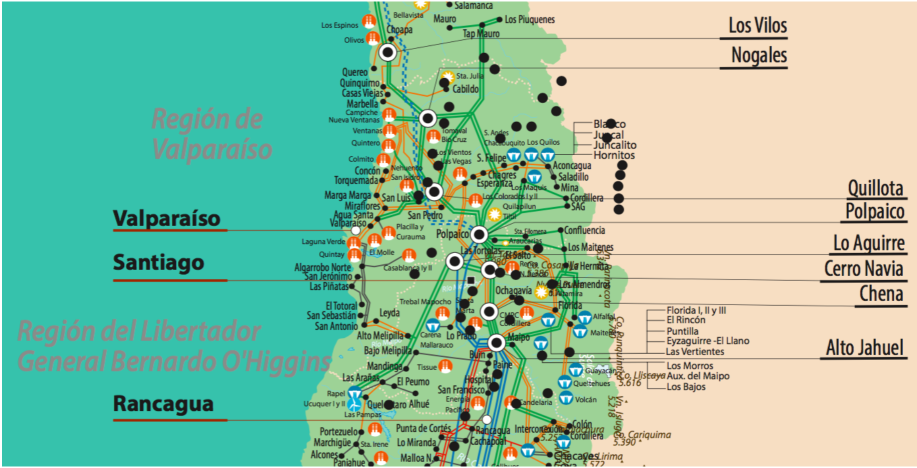


Figure: Power network, central part of Chile (“Sistema Interconectado Central”)

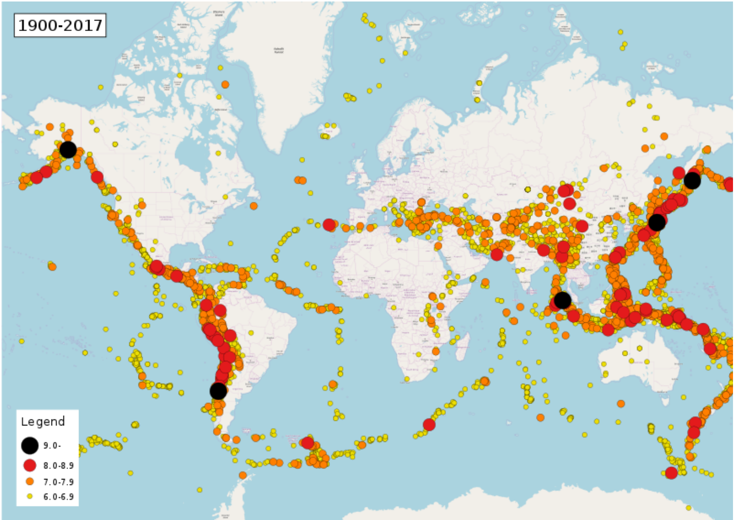


Figure: Earthquakes of magnitude ≥ 6.0 in span 1900-2017

Largest earthquakes by magnitude [edit]

Listed below are all the 36 known earthquakes with an estimated magnitude of 8.5 or hig
enough data, this gives a rough estimate of its frequency per century. (The timeframe does not include outlying events like the earlier 1361 Shōhei earthquake and 869 Sanriku earthquake, both estimated to have magnitude ≥ 8.5.)

Prior to the development and deployment of seismographs – starting around 1900 – magnitudes can only be estimated, based on historical reports of the extent and severity of damage.[2]

Rank	Date	Location	Event	Magnitude
1	May 22, 1960	Valdivia, Chile	1960 Valdivia earthquake	9.4–9.6
2	March 27, 1964	Prince William Sound, Alaska, United States	1964 Alaska earthquake	9.2
3	December 26, 2004	Indian Ocean, Sumatra, Indonesia	2004 Indian Ocean earthquake	9.1–9.3
4	March 11, 2011	Pacific Ocean, Tōhoku region, Japan	2011 Tōhoku earthquake	9.1[3]
5	November 4, 1952	Kamchatka, Russian SFSR, Soviet Union	1952 Kamchatka earthquakes	9.0[4]
6	August 13, 1868	Arica, Chile (then Peru)	1868 Arica earthquake	8.5–9.0 (est.)
7	January 26, 1700	Pacific Ocean, USA and Canada (then claimed by the Spanish Empire and the British Empire)	1700 Cascadia earthquake	8.7–9.2 (est.)
8	April 2, 1762	Chittagong, Bangladesh (then Kingdom of Mrauk U)	1762 Arakan earthquake	8.8 (est.)
9	November 25, 1833	Sumatra, Indonesia (then part of the Dutch East Indies)	1833 Sumatra earthquake	8.8 (est.)
10	January 31, 1906	Ecuador – Colombia	1906 Ecuador–Colombia earthquake	8.8[5]
11	February 27, 2010	Offshore Maule, Chile	2010 Chile earthquake	8.8[5]
12	August 15, 1950	Assam, India – Tibet, China	1950 Assam–Tibet earthquake	8.7
13	October 28, 1707	Pacific Ocean, Shikoku region, Japan	1707 Hōei earthquake	8.7–9.3 (est.)
14	July 8, 1730	Valparaiso, Chile (then part of the Spanish Empire)	1730 Valparaiso earthquake	8.7 (est.)[6]
15	November 1, 1755	Atlantic Ocean, Lisbon, Portugal	1755 Lisbon earthquake	8.5–9.0
16	February 4, 1965	Rat Islands, Alaska, United States	1965 Rat Islands earthquake	8.7
17	October 28, 1746	Lima, Peru (then part of the Spanish Empire)	1746 Lima–Callao earthquake	8.6 (est.)
18	March 28, 1787	Oaxaca, Mexico (then part of the Spanish Empire)	1787 Mexico earthquake	8.6 (est.)
19	March 9, 1957	Andreanof Islands, Alaska, United States	1957 Andreanof Islands earthquake	8.6[5]
20	March 28, 2005	Sumatra, Indonesia	2005 Nias–Simeulue earthquake	8.6[5]
21	April 11, 2012	Indian Ocean, Sumatra, Indonesia	2012 Aceh earthquake	8.6
22	December 16, 1575	Valdivia, Chile (then part of the Spanish Empire)	1575 Valdivia earthquake	8.5 (est.)
23	November 24, 1604	Arica, Chile (then part of the Spanish Empire)	1604 Arica earthquake	8.5 (est.)
24	May 13, 1647	Santiago, Chile (then part of the Spanish Empire)	1647 Santiago earthquake	8.5 (est.)
25	May 24, 1751	Concepción, Chile (then part of the Spanish Empire)	1751 Concepción earthquake	8.5 (est.)
26	November 19, 1822	Valparaíso, Chile	1822 Valparaíso earthquake	8.5 (est.)
27	February 20, 1835	Concepción, Chile	1835 Concepción earthquake	8.5 (est.)
28	February 16, 1861	Sumatra, Indonesia	1861 Sumatra earthquake	8.5
29	May 9, 1877	Iquique, Chile (then Peru)	1877 Iquique earthquake	8.5 (est.)
30	November 10, 1922	Atacama Region, Chile Catamarca Province, Argentina	1922 Vallenar earthquake	8.5[7]
31	February 1, 1938	Banda Sea, Indonesia (then part of the Dutch East Indies)	1938 Banda Sea earthquake	8.5[5]
32	October 13, 1963	Kuril Islands, Russia (USSR)	1963 Kuril Islands earthquake	8.5[5]
33	September 12, 2007	Sumatra, Indonesia	2007 Sumatra earthquakes	8.5[5]
34	October 20, 1687	Lima, Peru (then part of the Spanish Empire)	1687 Peru earthquake	8.5 (est.)
35	October 17, 1737	Kamchatka, Russia	1737 Kamchatka earthquakes	8.5 (est.)
36	June 15, 1896	Pacific Ocean, Tōhoku region, Japan	1896 Sanriku earthquake	8.5 (est.)

Figure: All registered earthquakes of magnitude ≥ 8.5

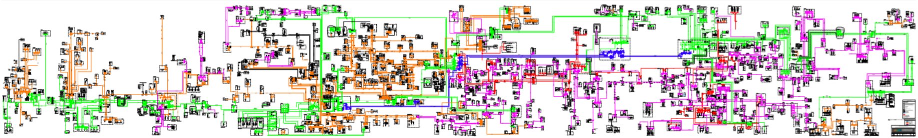


Figure: Power network, central part of Chile, zoom 5%

Introduction

Reliability/Resiliency of a system

Ability **of the system** to keep on working despite failure **of its components**

In energy & telecomm systems:

- *IEEE Reliability Society*, *IEEE Transactions in reliability* (est. 1952, IF 2.79)
- Repairable vs. unrepairable components
- Focus: k-out-of-n reliability, mostly connectivity, exogenous shocks

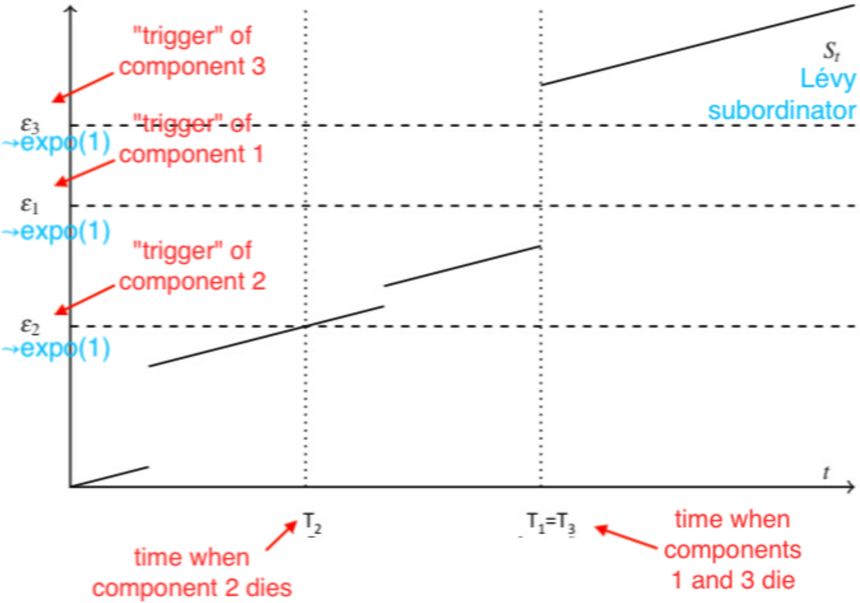
Our work:

"system lifetime due to occurrence of exogenous shocks"

when

- shocks can hit multiple components at the same time (i.e. non-iid shocks)
- components are unrepairable
- system lifetime := time when last* component fails

Example: system with 3 components



A model for dependent lifetimes of components

We consider **Lévy-frailty Marshall-Olkin** distribution (Mai & Scherer 2013):

- n = number of components
- For each component i there is a "trigger" $\varepsilon_i \rightsquigarrow expo(1)$ *all triggers ε_i are iid
- There is a Lévy subordinator $(S_t : t \geq 0)$ *S is independent of the triggers (ε_i)
- Component i "dies" first time S up-crosses ε_i: $\min\{t \geq 0 : S_t \geq \varepsilon_i\}$

We obtain: a multivariate distribution of lifetimes of the components

Note on **Marshall-Olkin** distribution (1967):

- Model for simultaneous failures of components
- Generalizes memoryless property to $\mathbb{R}^n$
- Easy to simulate, **Popular!**
- ☹ n components $\Rightarrow 2^n - 1$ parameters!

Lévy-frailty Marshall-Olkin:

- ☺ alleviates this parametrical complexity
- ☺ "almost iid" model

Main result

Consider a system where

- n := number of components
- lifetime of components follows a *Lévy-frailty Marshall-Olkin* model
- system lifetime is T_{sys} := time when last component fails

Consider the hypotheses

$$(\mathbf{A}_\alpha) \quad \mathbb{P}(S_1 > t) \underset{t\to\infty}{\approx} \frac{const.}{t^\alpha} \qquad \text{and} \qquad (\mathbf{B}) \quad 0 < Var(S_1) < \infty$$

Theorem

If $(\mathbf{A}_\alpha)$ *holds for* $\alpha \in (1,2)$ *or* $(\mathbf{B})$ *holds (put* $\alpha := 2$*), then*

$$\frac{T_{sys} - \log n/\mathbb{E}S_1}{(\log n)^{1/\alpha}/\mathbb{E}S_1} \xrightarrow[n\to\infty]{distribution} \sigma \cdot \underbrace{Stable_\alpha(1,-1,0)}_{=Normal(0,2) \text{ if } \alpha=2},$$

where σ *= constant related to the tail of* S_1*.*

∗ Also holds when T_{sys} := time of k^{th} failure (out of a total of n), "$k \approx n$"

Perspectives & applications I

Perspective 1: an asymptotic analysis result

On $\mathbb{P}(T_{sys} > t) = \mathbb{P}(n^{\text{th}}$ failure hasn't occured by $t)$: $\xrightarrow[n\to\infty]{\text{fixed } t} 1, \quad \xrightarrow[t\to\infty]{\text{fixed } n} 0.$

Question: is there a non-trivial "regime" as t and n grow together?

Answer: our result says YES, plug in $t = t_n := [\log n + s(\log n)^{1/\alpha}]/\mathbb{E}S_1$
to obtain $\mathbb{P}(T_{sys} > t_n) \xrightarrow[n\to\infty]{} \mathbb{P}(\sigma \cdot Stable_\alpha(1, -1, 0) > s)$

Perspective 2: a Central Limit Theorem–type result

$$\frac{\sum_{i=1}^n \xi_i - n \cdot \mathbb{E}\xi}{n^{1/2}} \xrightarrow[n\to\infty]{distr.} \sigma \cdot N(0,1) \quad \text{vs.} \quad \frac{T_{sys} - \log n/\mathbb{E}S_1}{(\log n)^{1/\alpha}/\mathbb{E}S_1} \xrightarrow[n\to\infty]{distr.} \sigma \cdot S_\alpha(1,-1,0)$$

where $\xi_i \rightsquigarrow$ iid, $Var(\xi_1) < \infty$

under *Lévy-frailty M-O* model

Application 1: confidence bounds

Assume n components and (A_α) holds, $\alpha \in (1,2)$. With confidence $1 - \epsilon$,

$$T_{sys} \in \left[\frac{\log n}{\mathbb{E}S_1} - \frac{\sigma(\log n)^{1/\alpha}}{\mathbb{E}S_1} z^{\epsilon/2}_{Stable_\alpha(1,-1,0)}, \frac{\log n}{\mathbb{E}S_1} + \frac{\sigma(\log n)^{1/\alpha}}{\mathbb{E}S_1} z^{1-\epsilon/2}_{Stable_\alpha(1,-1,0)} \right].$$

Perspectives & applications II

Application 1: confidence bounds (continued)

If the model is erroneously assumed to be iid, with confidence $1-\epsilon$,

$$T_{sys} \in \left[\frac{\log n}{\mathbb{E}S_1} - \frac{z^{\epsilon/2}_{\text{Gumbel}(1,0)}}{\mathbb{E}S_1}, \frac{\log n}{\mathbb{E}S_1} + \frac{z^{1-\epsilon/2}_{\text{Gumbel}(1,0)}}{\mathbb{E}S_1} \right].$$

Perspective 3: a result on Extreme Value Theory

$$T_{k:n} := \text{time of } k^{\text{th}} \text{ failure (out of } n \text{ component failure times)}$$

Our result holds for T_{sys} := any of the *upper order statistics* $T_{k:n},\ T_{(k+1):n},\ \ldots,\ T_{n:n}$, for any $k = k(n) \nearrow \infty$ satisfying: $\log(n-k) = o\left((\log n)^{1/\alpha}\right)$.

Application 2: reliability in networks

Samaniego signature (Marichal et al. 2011): for "most" system failure times

$$\mathbb{P}\left(\begin{array}{c}\text{system is alive}\\ \text{at time } t\end{array}\right) = \sum_{k=1}^{n} \mathbb{P}\left(\begin{array}{c}\text{system fails because}\\ \text{of } k^{th} \text{ failure}\end{array}\right) \mathbb{P}\left(\begin{array}{c}k^{th} \text{ failure occurs}\\ \text{at time} \geq t\end{array}\right)$$

Summary & main contributions

- We derive asymptotical results for the last-failure times of a *Lévy-frailty Marshall-Olkin* model for failure times

- Theoretical contributions:
  - Asymptotic result, very useful for approximations
  - New result on probability concentration of upper order statistics
  - New result in Extreme Value Theory

- Engineering/applications contributions:
  - We give confidence intervals for system failure times
  - Our result has potential to tackle more general system failure times through *Samaniego signature* result

- All in all, contribution to systemic risk, reliability, applied probability & multivariate statistics

Thanks!