Principal–Agent problems and applications to electricity demand management

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Outline

Motivation

B. Salanié, The economics of contracts

Customers know more about their tastes than firms, firms know more about their costs than the government and all agents take actions that are at least partly unobservable.

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- Vast economic literature revisiting general equilibrium theory by incorporating incentives and asymmetry of information.
- Moral hazard: situation where an Agent can benefit from an action (unobservable), whose cost is incurred by others $\longrightarrow$ a type of externality
- How can one design "optimal" contracts?

Modelisation

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- Agent can accept or refuse it, but is then fully committed.
- When he accepts, Agent has to perform a costly task (examples to come).

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- Choice of Agent impacts the **distribution** of an output process X

$$X_t = X_0 + \int_0^t \lambda_s(X, \nu_s) ds + \int_0^t \sigma_s(X, \nu_s) dW_s^\nu, \ t \in [0, T],$$

where W^ν is a $\mathbb{P}^\nu-$Brownian motion.

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where W^{ν} is a $\mathbb{P}^{\nu}-$Brownian motion.

- Profit of Principal depends on X, which he observes. But ν is inaccessible! $\Longrightarrow$ hidden action $\Longrightarrow$ asymmetry of information.

Examples (from finished or on-going works)

• Optimal remuneration of managers (joint with Élie, Mastrolia, Réveillac and Villeneuve)

$$\begin{cases} X \longrightarrow \text{value of the firm.} \\ \nu \longrightarrow \text{work of the manager.} \end{cases} \quad X_t = X_0 + \int_0^t \nu_s ds + \sigma W_s^{\nu}.$$

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• Contract $\longrightarrow$ salary of the manager.

Examples (from finished or on-going works)

- Delegated portfolio management (joint with Cvitanić and Touzi).

$$\begin{cases} X \longrightarrow \text{value of the portfolio.} \\ \nu \longrightarrow \text{investment choices.} \end{cases} \quad X_t = X_0 + \int_0^t \nu_s \cdot (b_s ds + \sigma_s dW_s^\nu).$$

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- Contract $\longrightarrow$ remuneration of the fund manager.

Examples (from finished or on-going works)

- Electricity demand response management (joint with Aïd and Touzi).

$$\begin{cases} X \longrightarrow \text{deviation from baseline consumption.} \\ \nu = (\alpha, \beta) \longrightarrow \text{effort on the mean and the volatility of consumption.} \end{cases}$$

$$X_t = X_0 - \int_0^t \alpha_s \cdot \mathbf{1} ds + \int_0^t \mathrm{diag}(\sigma)\sqrt{\beta_s} \cdot dW_s^\nu.$$

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- Contract $\longrightarrow$ compensation of the consumer.

Examples (from finished or on-going works)

• Optimal securitization of mortgage loans (joint with Pagès, Hernández Santibáñez and Zhou).

$$\begin{cases} X \longrightarrow \text{number of observed defaults of the } I \text{ loans.} \\ \nu \longrightarrow \text{bank monitoring actions (number of loans non-monitored).} \end{cases}$$

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- Contract $\longrightarrow$ remuneration of the bank, and liquidation procedure.

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with cost

$$c(a,b) := \frac{1}{2}\left(\sum_{i=1}^{N} \frac{a_i^2}{\mu_i} + \sum_{j=1}^{d} \frac{\sigma_j^2 b_j^{-\eta_j}}{\lambda_j \eta_j}\right),$$

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- Dependence of ξ on the whole trajectory of X is **crucial**.
- Agent faces a non–Markovian stochastic control problem.

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Principal looks for a Stackelberg equilibrium and proceeds in 2 steps.

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- h: cost of variability.
- Ξ_R: contracts such that $V^A(\xi) \geq R$ (participation constraint).

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- Agent's problem: non–Markovian control $\longrightarrow$ (2)BSDEs.
- Principal's problem: non–standard optimization over random variables $\longrightarrow$ necessary reinterpretation to pave the way to explicit solutions and, at least, to **efficient numerical methods!**

Outline

Intuition and verification

- Dynamic Agent's problem writes

$$V_t^A(\xi) := \sup_{\nu} \mathbb{E}^{\mathbb{P}^\nu}\left[\xi(X) + \int_t^T \big(f(X_s) - c(\nu_s)\big)ds \Big| \mathcal{F}_t\right],\ t \in [0, T].$$

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- Hamiltonian naturally associated to this control problem is

$$H(t,x,z,\gamma) := \sup_{v} h(t,x,z,\gamma,a,b),$$

$$h(t,x,z,\gamma,a,b) := a \cdot \mathbf{1}z + \frac{1}{2}\left|\mathrm{diag}(\sigma)\sqrt{b}\right|^2\gamma + f(x) - c_t(x,a,b).$$

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- Optima

$$\widehat{a}(z) := \mu z^-, \ \widehat{b}_j(\gamma) := 1 \wedge (\gamma^- \lambda_j)^{-\frac{1}{1+\eta_j}}.$$

Intuition and verification

- If $V_t^A(\xi) =: v^A(t,X)$ is a "smooth" function of (t,X), where smooth means that an Itō formula is valid

$$dv^A(t,X) = \partial_t v^A(t,X)dt + Z_t \cdot dX_t + \frac{1}{2}\Gamma_t d\langle X\rangle_t,$$

with the correspondence

$$Z_t \longrightarrow "\partial_x v^A(t,X)", \quad \Gamma_t \longrightarrow "\partial_{xx} v^A(t,X)".$$

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- Z and Γ only supposed integrable $\longrightarrow$ Sobolev type regularity.
- Dynamic programming and martingale optimality principle $\Longrightarrow$

$$\partial_t v^A(t,X) + H(t,X,Z_t,\Gamma_t) = 0.$$

Intuition and verification

- To sum up

$$V_0^A(\xi) = \xi + \int_0^T H(t, X, Z_t, \Gamma_t)dt - \frac{1}{2}\int_0^T \Gamma_t d\langle X\rangle_t - \int_0^T Z_t \cdot dX_t.$$

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- It is a non–linear representation theorem for ξ! In addition, optimal effort of Agent is given by

$$\widehat{\nu}_t := \big(\widehat{a}(Z_t), \widehat{b}_j(\Gamma_t)\big).$$

Intuition and verification

The converse statement is true! It is a simple verification theorem

Theorem

If, for an "admissible" pair (Z,Γ) and $V_0 \in \mathbb{R}$, the contract $\xi_T^{V_0,Z,\Gamma}$, with

$$\xi_t^{V_0,Z,\Gamma} := V_0 - \int_0^t H(s,X,Z_s,\Gamma_s)ds + \frac{1}{2}\int_0^t \Gamma_s d\langle X\rangle_s - \int_0^t Z_s \cdot dX_s,$$

is offered to Agent, then his utility is V_0 and his optimal effort

$$\widehat{\nu}_t = \left(\widehat{a}(Z_t), \widehat{b}_j(\Gamma_t)\right).$$

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However, *quid* of the existence of Z and Γ for general contracts?

2BSDEs

- Existence of Z is classical (standard representation theorem), but less clear for Γ. Introduce

$$F(t,x,z,b) := \sup_{\gamma} \left\{ \frac{1}{2}\gamma \left|\mathrm{diag}(\sigma)\sqrt{b}\right|^2 - H(t,x,z,\gamma) \right\}.$$

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$$F(t,x,z,b) := \sup_{\gamma}\left\{\frac{1}{2}\gamma\left|\mathrm{diag}(\sigma)\sqrt{b}\right|^2 - H(t,x,z,\gamma)\right\}.$$

- We have for any control β

$$V_0^A(\xi) = \xi - \int_0^T F(t,X,Z_t,\beta_t)dt - \int_0^T Z_t\cdot dX_t + K_T^{\beta},$$

where K^{β} is non–decreasing with

$$K_T^{\beta} := \int_0^T F(t,X,Z_t,\beta_t)dt - \frac{1}{2}\Gamma_t d\langle X\rangle_t + H(t,X,Z_t,\Gamma_t)dt.$$

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2BSDEs

- Relaxation of existence of $\Gamma \iff K^{\beta}$ not necessarily absolutely continuous.
- The triplet $(V^A(\xi), Z, (K^{\beta})_{\beta})$ solves then exactly a 2BSDE with terminal condition ξ and generator F!
- Study of wellposedness of these equations and their generalizations with Élie, Kazi-Tani, Piozin, Mastrolia, Matoussi, Sabbagh, Soner, Tan, Touzi, Zhang and Zhou.

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2BSDEs

- 2BSDEs allow for a general probabilistic representation of the value function of Agent, but do not give in general access to optimal effort!
- Nonetheless, one can prove that the contracts ξ of the form $\xi^{V_0,Z,\Gamma}$ are "dense" in an appropriate class of admissible contracts.
- No loss of generality in restricting to them!

Outline

1. A primer on moral hazard
   - Motivation
   - Some more examples
   - Agent's problem
   - Principal's problem
   - And now?
2. Control and 2BSDEs
   - Intuition and verification
   - 2BSDEs
3. Back to Principal's problem
   - Control reinterpretation
   - HJB equation
   - Numerics

Principal's problem

- Why is the class $\xi_T^{v,Z,\Gamma}$ useful? $\longrightarrow$ Principal's problem becomes

$$V_0^P = \sup_{v \geq R} V_0^P(v),$$

$$V_0^P(v) := \sup_{Z,\Gamma} \mathbb{E}^{\mathbb{P}^\star(Z,\Gamma)}\left[U\left(-\xi_T^{v,Z,\Gamma} - \int_0^T g(X_s)ds - \frac{h}{2}\langle X\rangle_T\right)\right],$$

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- $V_0^P(v)$ is value function of a standard stochastic control problem, with state variables X and $\xi^{v,Z,\Gamma}$, and with controls Z and Γ.

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- $V_0^P(v)$ is value function of a standard stochastic control problem, with state variables X and $\xi^{v,Z,\Gamma}$, and with controls Z and Γ.
- Continuation utility of Agent is a performance index for Principal.
- HJB equation associated $\longrightarrow$ **explicit solutions or efficient numerical schemes!**

HJB equation

Take $U(x) := -\exp(-px)$. Then $V_0^P = -e^{pR+u(0,X_0)}$ where u solves

$$\begin{cases} 0 = -u_t(t,x) - p(g(x) - f(x)) + \dfrac{\mu\cdot\mathbf{1}}{2p}(u_x^+(t,x))^2 \\ \quad + \dfrac{p}{2}\displaystyle\sum_{j=1}^{d}\frac{\sigma_j^2}{\eta_j\lambda_j}\Big[\mathbf{1}_{\overline{\lambda}\mathcal{L}[u](t,x)\leq 1}\left(1+\eta_j\lambda_j\mathcal{L}[u](t,x)\right) \\ \qquad + \mathbf{1}_{\overline{\lambda}\mathcal{L}[u](t,x)>1}\mathbf{1}_{\lambda_j\mathcal{L}[u](t,x)\geq 1}(1+\eta_j)\left(\lambda_j\mathcal{L}[u](t,x)\right)^{\frac{\eta_j}{1+\eta_j}}\Big], \\ u(T,x) = 0,\ x\in\mathbb{R}. \end{cases}$$

where

$$\mathcal{L}[u](t,x) := \frac{u_{xx}(t,x)}{p} + \frac{u_x^2(t,x)}{2p} + h.$$

Calibration

Usage	d	2
Consumer	Cost on drift μ	$(0.1, 0.5)$
	Nominal volatilities σ	$(5, 2)$
	Cost on volatilities λ	$(10, 50)$
	Coefficient η	$(1, 1)$
	$f(x) = f_0(1 - e^{-f_1 x})$	$6(1 - e^{-0.01x})$
Producer	Risk aversion p	0.5
	Generation cost function $g(x)$	$50\left(e^{0.05x} - 1\right)$
	Volatility cost function h	40

Table: Summary of the model parameters in the reference case.

Consumption

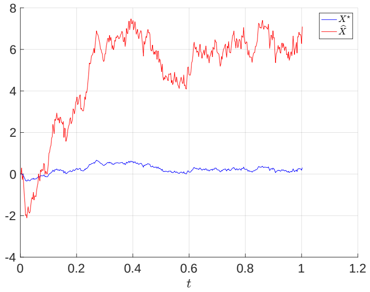

Figure: One day consumption trajectory: rational agent, passive agent

Rational consumer decreases her consumption and volatility compared to passive consumer.

Drifts

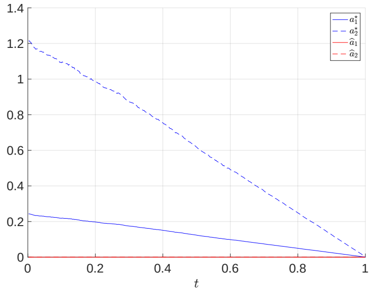

Figure: Efforts on the drift: rational agent, passive agent

Rational consumer makes decreasing effort with time and more efforts on the less costly usage.

Volatilities

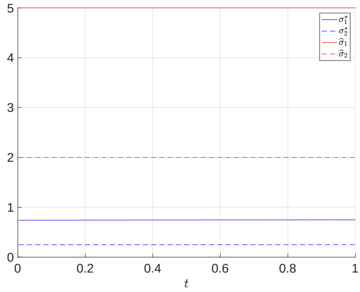

Figure: Effort on volatilities: rational agent, passive agent

Rational consumer makes decreasing effort with time and more efforts on the less costly usage.

Payment

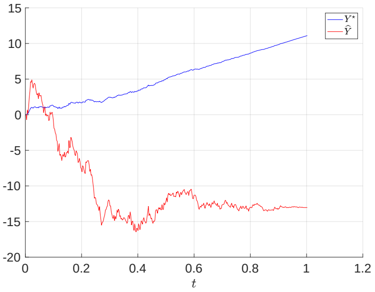

Figure: Payment: rational agent, passive agent

On this trajectory, rational consumer receives a positive and higher payment than the passive consumer.

Mean consumption

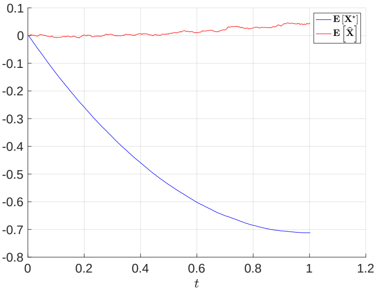

Figure: Mean daily consumption: rational agent, passive agent

Rational consumer decreases largely her consumption by 2.5 kW on average.

Mean drifts

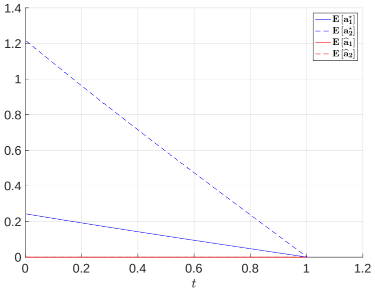

Figure: Mean efforts on drift: rational agent, passive agent

Rational consumer makes decreasing effort with time and more efforts on the less costly usage.

Mean volatilities

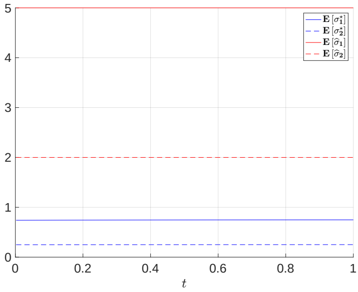

Figure: Mean efforts on volatilities: rational agent, passive agent

Rational consumer makes decreasing effort with time and more efforts on the less costly usage.

Payment distribution

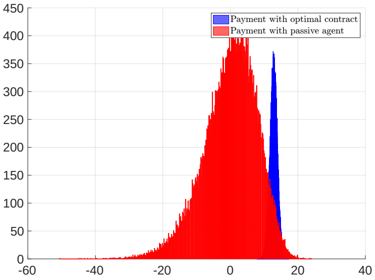

Figure: Payment distribution:rational agent, passive agent

In this situation, rational consumer receives on average 90 per day while the passive consumer receives 12 per day.

Principal utility distribution

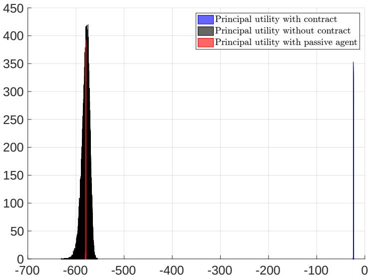

Figure: Principal's utility: rational agent, passive agent

In this situation, the Principal utility dealing with a rational agent is always better off than without contract or with a passive agent.

Markovian approximation of the optimal contract

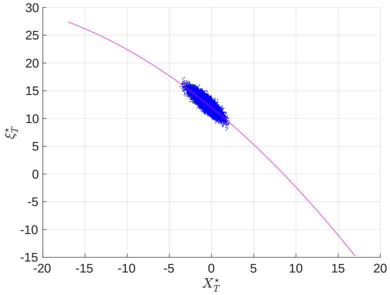

Figure: Quadratic regression of the optimal payment against X_T

A quadratic regression of Y_T explains 70% de la variance.

Markovian contract - Mean drifts

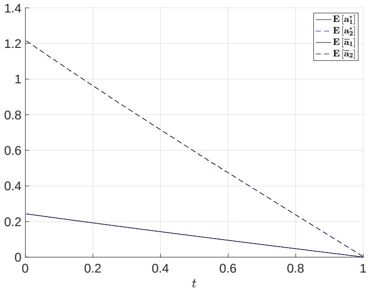

Figure: Mean efforts on drift: optimal contract, Markovian contract

Markovian contracts induces an increasing effort with time and more efforts on the less costly usage.

Markovian contract - Mean volatilities

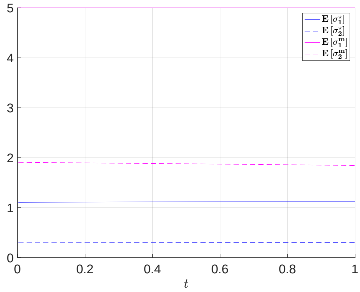

Figure: Mean efforts on volatilities: optimal contract, Markovian contract

Rational consumer is making nearly no effort on volatilities.

Principal utility distribution

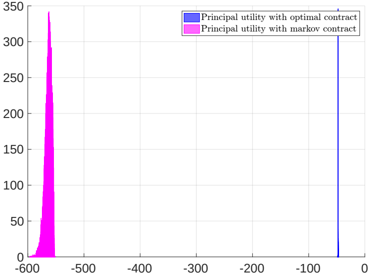

Figure: Principal's utility: optimal contract, Markovian contract

In this situation, the Principal's utility is far lower when proposing a Markovian approximation of the optimal contract.

Thank you for your attention!