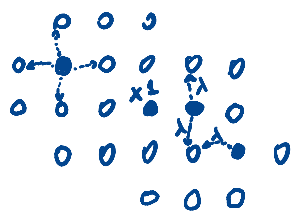
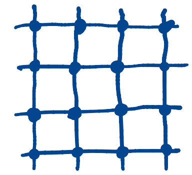
Contact Process

Definition

$G=(V,E)$ connected, bounded degree





At time t : $\eta_t : V \to \{0,1\}$

$\eta_t(x) = 1 \leftrightsquigarrow$ "x is infected"

$\eta_t(x) = 0$ "x is healthy"

Configuration space

$$\Sigma = \{0,1\}^V \quad (\text{or } \mathcal{P}(V)) \qquad \eta_t \in \Sigma$$

Process

$(\eta_t)_{t\geq 0}$ in $D([0,+\infty), \Sigma)$ càd lag

Transition rates: $\eta_t(x) \begin{cases} 1 \to 0 & \text{rate } 1 \\ 0 \to 1 & \text{rate } \lambda \cdot \sum_{y : y \sim x} \eta_t(y) \end{cases}$ recovery / infection

Generator

We want to describe the evolution rule by writing,

for $f \in C(\Sigma)$ depending on finitely many vertices,

$$\lim_{t \downarrow 0} \frac{\mathbb{E}^{\zeta}[f(\eta_t)] - f(\zeta)}{t} = \mathcal{L} f(\zeta) = \sum_x \mathbb{1}_{\zeta(x)=1} \left(f(\zeta^{x,0}) - f(\zeta) \right) + \mathbb{1}_{\zeta(x)=0} \cdot \lambda \left(\sum_{y : y \sim x} \mathbb{1}_{\eta(y)=1} \right) \cdot \left(f(\zeta^{x,1}) - f(\zeta) \right)$$

where $\zeta^{x,1}(y) = \begin{cases} \zeta(y), & y \neq x \\ 1, & y = x \end{cases}$ and $\zeta^{x,0}$ similar.

$C(\Sigma) = \{$ bounded continuous real functions $\}$

Markov semigroup

$\mu \, S_t \, f$

Questions

Invariant measures? $(\mu S_t = \mu)$

Survival? In what sense?

If it dies out a.s., how fast? What does it look like as it dies?

If it survives wpp, what does it look like when it survives?

Speed of propagation?

Some answers

Depends on the graph and λ! Certainly $\delta_\emptyset$ is invariant (it is an absorbing configuration!). Maybe only $\delta_\emptyset$, maybe $\delta_\emptyset \longleftrightarrow \nu^+$, maybe more!

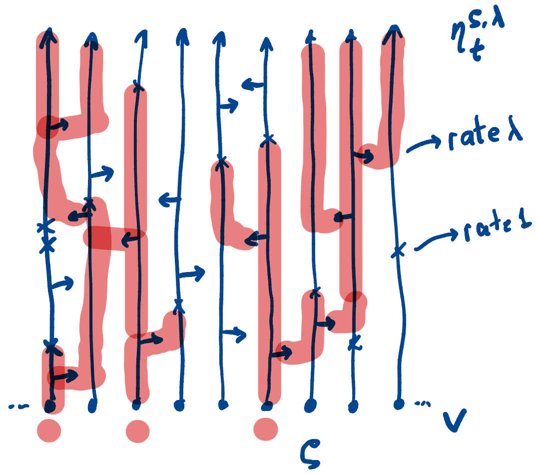
There are λ_c and λ_* so that

$$\mathbb{P}^{\delta_{\{x\}}}(\eta_t \neq \emptyset \ \forall t \geq 0) \begin{cases} = 0, & \lambda < \lambda_c \\ > 0, & \lambda > \lambda_c \end{cases} \qquad \mathbb{P}^{\delta_{\{x\}}}(\eta_t(y) \neq 0 \text{ i.o.}) \begin{cases} = 0, & \lambda < \lambda_* \\ > 0, & \lambda > \lambda_* \end{cases}$$

Depending on the graph, $\lambda_* > \lambda_c$ or $\lambda_* = \lambda_c$

Exponentially fast. We will see what it looks like...

Graphical Construction



Features:

* Multiple initial configurations (coupling)
* Attractive

  On Σ, partial order $\varsigma \geq \xi \iff \varsigma(x) \geq \xi(x)\ \forall x \in V$
  Say that $f \in C(\Sigma)$ is *increasing* if $f(\varsigma) \geq f(\xi)$ whenever $\varsigma \geq \xi$
  We say that μ *dominates* ν, $\mu \geq \nu$, if $\mu f \geq \nu f$ for all increasing f.
  A given process on Σ is *attractive* if $\mu S_t \geq \nu S_t$ whenever $\mu \geq \nu$
  (or, equivalently, if $S_t f$ is increasing whenever f is)

* Additive ($\therefore$ attractive)

$$\eta_t^{A \cup B} = \eta_t^A \cup \eta_t^B$$

* Monotone in λ

$$\eta_t^{\varsigma,\lambda} \geq \eta_t^{\varsigma,\lambda'} \quad \text{for } \lambda \geq \lambda'$$

Exercise!

* Duality:

$$\mathbb{P}(\eta_t^A \cap B \neq \emptyset) = \mathbb{P}(\eta_t^B \cap A \neq \emptyset)$$

* Restriction to subgraphs

Phase Transition

<u>Thm</u> Suppose G is infinite, connected, of bounded degree.

If λ is small, $\mathbb{P}($ weak $) = 0$
large, $\mathbb{P}($ strong $) > 0$

<u>Corollary</u>: $0 < \lambda_c \le \lambda_* < \infty$

Part I: Take $\lambda < \frac{1}{\text{max deg}}$

Compare the transition rates of $(|\eta_t|)_{t\ge 0}$

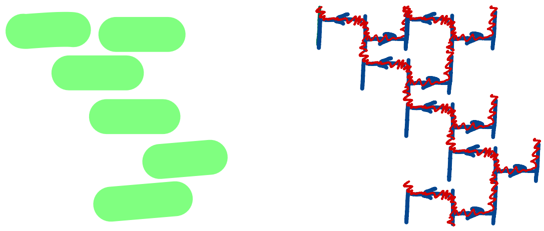
with those of a subcritical continuous-time branching process

Part II: Note that G contains a copy of $\mathbb{N}$

Suppose G contais $\mathbb{Z}$ Exercise If not!

Take δ small and λ large so that

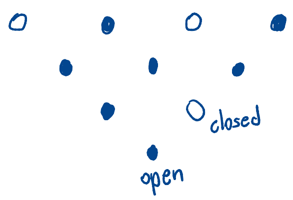
$\mathbb{P}($  $\updownarrow \delta)$ is large, compare it with
no recovery
<u>1-dependent</u> <u>oriented</u> <u>percolation</u>



Interlude: Oriented percolation

κ-dependent, highly supercritical

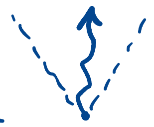
→ (every κ-separated collection of regions is indepent)

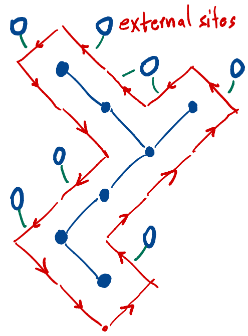
Model: 

<u>Theorem</u> If p is large enough (depending on κ), the configuration is κ-dependent, and $\mathbb{P}(x \text{ is open}) \geq p$ for every vertex x, then

$$\mathbb{P}\left(\text{“}(0,n) \text{ to } (0,0) \to \text{all open, for some upwards path”} \text{ for i.m. } n\right) > 0$$

Proof:

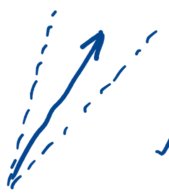
1) $\mathbb{P}\Big($  $\Big) > 1/2$

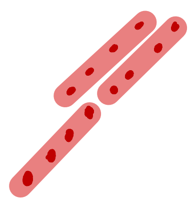
Proof: 

Circuit γ of length ℓ

- $\nearrow + \nwarrow + \swarrow + \searrow = \ell$
- $\nearrow = \swarrow$ and $\nwarrow = \searrow$ because the path ends where it starts
- Hence, $\nwarrow + \swarrow = \ell/2$
- Thus, at least $\ell/4$ "external sites"
- So there are at least $\ell/16\kappa^2$ external sites which are κ-separated

$$\sum_{\gamma} \mathbb{P}(\text{external sites of } \gamma \text{ closed}) \leq \sum_{\ell \geq 4} 3^{\ell} (1-p)^{\frac{\ell}{16\kappa^2}} \quad (\text{small if } p \text{ large})$$

2) $\mathbb{P}\Big($  $\Big) > \frac{1}{2}$

Proof: 

treat 4 sites as a singl site

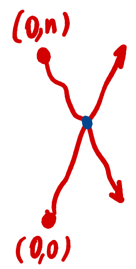
increase p further (to compensate)

reduce to previous case

3) $\inf_{n \in 2\mathbb{N}} \mathbb{P}\left((0,n) \text{ to } (0,0) \to \text{all open, for some upwards path} \right) > 0$

Proof:

Use (2), reflection symmetry and the fact that paths on the plane cannot cross without intersecting



Weak survival

<u>Thm</u> On $\mathbb{Z}^d$, $\lambda_c = \lambda_*$ and at $\lambda = \lambda_c$ the process dies out
(Proof below)

<u>Thm</u> On $\mathbb{T}^d$, $\lambda_c < \lambda_*$ ((d+1)-regular tree)

Proof for <u>large d</u> ($d \geq 17$)

Take $\lambda = \frac{2}{d}$

1) Weak survival

Compare with a restricted dynamics: no reinfection
Infected children, on average: $d \cdot \frac{\lambda}{1+\lambda} > 1$, compare with branching process.

2) No strong survival.
Self-infection path of length ℓ

$$\# \leq \underbrace{\binom{\ell}{\ell/2}}_{\leq 2^\ell} (d+1)^{\ell/2}$$

$$\text{prob} \leq \left(\frac{\lambda}{1+\lambda}\right)^\ell$$

sum over paths longer than K

$$\leq \sum_{\ell \geq K} (\underbrace{2\sqrt{d+1}\,\lambda}_{= 4\frac{\sqrt{d+1}}{d} < 1})^\ell$$

In both steps λ could have been perturbed so this really gives $\lambda_c < \lambda_*$

RmK: The proof also works for $d \geq 6$ and $\lambda = \frac{1+\varepsilon}{d-1}$

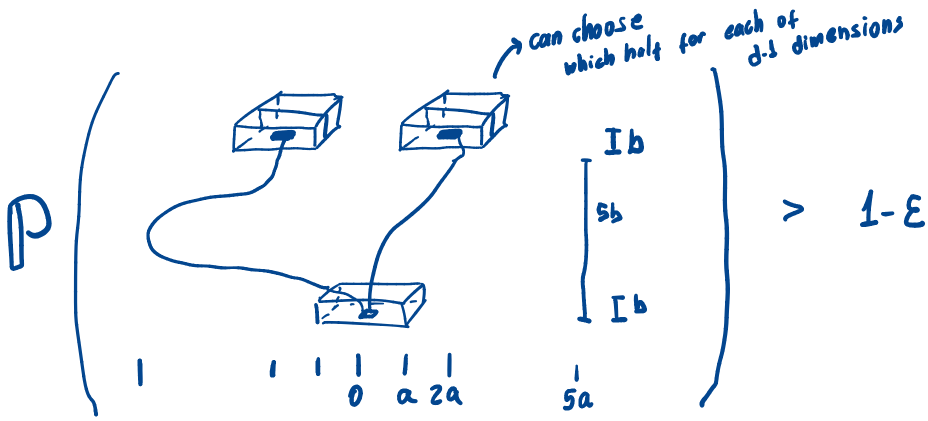
The critical point ($G = \mathbb{Z}^d$)

Main structure:

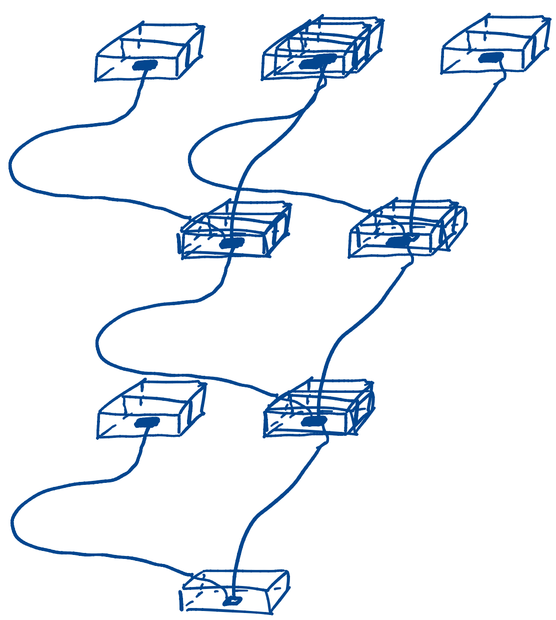
weak survival for λ $\Rightarrow$ finite condition $\Rightarrow$ strong survival
for some $\lambda' < \lambda$! for some $\lambda' < \lambda$!!!

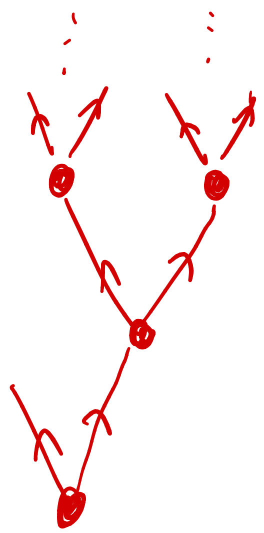
This gives: * weak survival $\Leftrightarrow$ strong survival
* $\{\lambda : \text{weak surv.}\}$ is an open interval
Hence no survival at $\lambda = \lambda_c$!

Finite condition:



Strong survival:





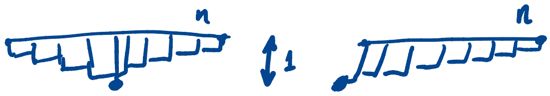
- Explore dynamically
- Steer in the (d-1) other dimensions
- Compare with highly supercritical 5-dependent oriented percolation

Ref: Bezudenhout & Grimmett (1990)

Building up finite condition:

* Choose n so that $\mathbb{P}\Big($  $\Big)$ is very large

* Consider the probabilities of



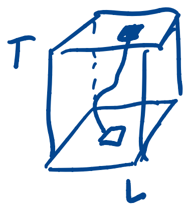
* Let M be so large that after $\frac{M}{(2n+1)^d}$ independent attempts, the probability of these events occurring is very large

* Argue that $\mathbb{P}(1 \leq |\eta_t^0| \leq M \text{ i.o.}) = 0$

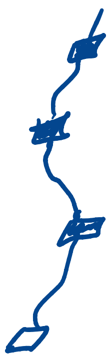
Each time $|\eta_t| \leq M$, the chance that $\eta_{t+1} = \emptyset$ is at least $\left(\frac{1}{2d\lambda+1}\right)^M > 0$

But $\mathbb{P}(G \mid \mathcal{F}_t) \xrightarrow[t\to\infty]{a.s.} \mathbb{1}_G$, so either this is eventually zero or converges to 1!

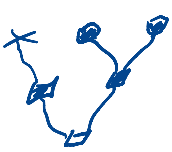
→ Conclude that $\mathbb{P}(|\eta_t^0| > M)$ is very large for large t

→ Taking L large enough, $\mathbb{P}\Big($  $\Big)$ is very large.

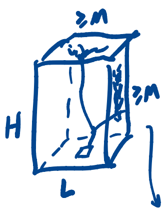
Note: this is a type of "finite condition", but:

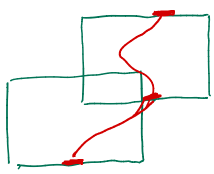


→ simple chain won't work!

Solution: we also want to move <u>sideways</u> in order to do 

* A more elaborate version of the previous argument and "fine tuning" gives

$\mathbb{P}\Big($  $\Big)$



↳ ensure both vertical and horizontal displacement!